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Modification of the Lagrange-Jacobi Equation and Its Application

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Abstract

The Lagrange-Jacobi equation is one of the significant tools for the qualitative analysis of the n-body problem. In this paper, we present the modified Lagrange-Jacobi equation by introducing a new formal parameter of n-body problem and propose its application to the dynamical study of clusters of galaxies which are large-scale structures of Universe. We put forward and study a new dynamical problem which is related to the stage of relaxation of observed stationary clusters of galaxies which are considered as a non-equilibrium systems of point masses. We also received the analytical form of the potential energy of such galaxy clusters. One of the applications of this analytical form is the analytical relation between the time τ of setting up the virial equilibrium in relaxing clusters of galaxies and the cosmological epoch T.

Keywords

Celestial Mechanics, Large-Scale Structure of Universe, Galaxies, Clusters, Cosmological Epoch

1. Modification of the Lagrange-Jacobi Equation

One of the fundamental equation of Celestial Mechanics which may have a plenty of applications is the Lagrange-Jacobi equation (LJE).

Let us consider a non-stationary system of gravitationally bound system ($h < 0$) of n gravitating bodies with constant masses m_i and the barycentric radii-vectors $\boldsymbol{r}_i \left(i = 1, 2, \cdots, n \right)$. Along with the energy integral

$$\frac{1}{2}\sum_i m_i \left(\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} \right)^2 + U = const \equiv h \tag{1}$$

we have the LJE [1]

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = -U + 2h \tag{2}$$

of such gravitationally bound system ($h < 0$), where I—is the barycentric momentum of inertia of the system and U—is the potential energy of the system:

$$I = \sum_i m_i r_i^2 , \tag{3}$$

$$U = -G\sum_{i>j} \frac{m_i m_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|} , \tag{4}$$

where G is the gravitational constant.

Let $\boldsymbol{r}_{i0}$ and $\dot{\boldsymbol{r}}_{i0}$ denotes the values of variables $\boldsymbol{r}_i$ and $\mathrm{d}\boldsymbol{r}_i/\mathrm{d}t$ in some initial epoch t_0, for which we have

$$-U_0 + 2h \neq 0 . \tag{5}$$

Now we introduce a new formal parameter of n body problem instead of momentum of inertia I in Equation (2) to obtain the modification of the LJE.

The most large-scale structures of Universe whose density is by a few orders higher than density δ_M of the gravitating background matter are clusters of galaxies [2].

Let assume the clusters of galaxies as subsystems of already formed n gravitating bodies [3] which are non-equilibrium initially. The time of setting up of the virial equilibrium is of order of $(G\delta_1)^{-1/2}$, where G—is the gravitational constant and δ_1—the density of cluster of galaxies and the relaxation of such systems occurs at their non-stationary state [4]. Therefore one may expect that at given view at the density δ_M as a function of the cosmological time t the following condition can play a constructive role for non-stationary dynamics of relaxing clusters of galaxies:

$$\delta_1 \gg \delta_M \tag{6}$$

where in case of a flat, matter-dominated universe (Einstein-de Sitter universe) [5] [6]

$$\delta_M(t) = \frac{1}{6\pi G t^2} . \tag{7}$$

The complex dynamics of n galaxies with many degrees of freedom is usually considered from the epoch of the virial equilibrium state [7].

We can expect that the analytical consequences of the generalized equations of motion of n body problem inside gravitating background of density $\delta_M(t) = 1/6\pi G t^2$ are convenient for taking directly into consideration the dynamical condition $\delta(t) \gg 1/6\pi G t^2$ where δ is the mean density of the system of bodies with their individual masses $m_i\ (i = 1, 2, \cdots, n)$. At the same time, an appropriate extreme formulas of such generalized problem shall reflect enough good the analytical properties of the ordinary n body problem for which the relation $\delta(t) \gg 1/6\pi G t^2$ is valid under the same initial conditions.

Let take into consideration the auxiliary system of n bodies inside of a gravitating matter of density $\delta(t)=\alpha/6\pi G t^2$, where α is a constant. Denoting the corresponding radii-vectors by $\boldsymbol{\rho}_i$, we prescribe to the system the initial conditions $\boldsymbol{\rho}_{i0}=\boldsymbol{r}_{i0}$, $\dot{\boldsymbol{\rho}}_{i0}=\dot{\boldsymbol{r}}_{i0}$ under $t=t_0$ $(i=1,2,\cdots,n)$. Then the analogs of Formulas (1) and (2) will take the form:

$$\frac{1}{2}\sum_i m_i\left(\frac{\mathrm{d}\boldsymbol{\rho}_i}{\mathrm{d}t}\right)^2=\Phi+\frac{2\alpha}{9}\int_{t_0}^{t}\frac{1}{t^2}\frac{\mathrm{d}J}{\mathrm{d}t}\mathrm{d}t, \tag{8}$$

$$\frac{1}{2}\frac{\mathrm{d}^2J}{\mathrm{d}t^2}-\frac{2\alpha}{9t^2}J+\frac{2\alpha}{9}\int_{t_0}^{t}\frac{1}{t^2}\frac{\mathrm{d}J}{\mathrm{d}t}\mathrm{d}t=-\Phi+2h, \tag{9}$$

where

$$J=\sum_i m_i\rho_i^2,\quad \Phi=-G\sum_{i>j}\frac{m_i m_j}{\left|\boldsymbol{\rho}_i-\boldsymbol{\rho}_j\right|}. \tag{10}$$

The differentiation of the formula (9) gives the following equation:

$$\frac{\mathrm{d}^3J}{\mathrm{d}t^3}+\frac{8\alpha}{9t^2}\frac{\mathrm{d}J}{\mathrm{d}t}-\frac{8\alpha}{9t^3}J=-2\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{11}$$

If in the right side of Equation (12) $\Phi=const$, the left side

$$\frac{\mathrm{d}^3J}{\mathrm{d}t^3}+\frac{8\alpha}{9t^2}\frac{\mathrm{d}J}{\mathrm{d}t}-\frac{8\alpha}{9t^3}J=0 \tag{12}$$

has a particular solution $J=t$.

Accordingly, it is easy to integrate the formal inhomogeneous Equation (11). Doing so we can put here $8\alpha/9=1$. Moreover, under $\alpha=9/8$ the density of gravitating background $\delta(t)$ and the cosmological density $\delta_M(t)=1/6\pi G t^2$ are the values of the same order. In this case we find

$$\begin{aligned}J=&\frac{1}{2}\left(\ddot{J}_0-\frac{\dot{J}_0}{t_0}+\frac{J_0}{t_0^2}+2\Phi_0\right)t_0 t\left(\ln\frac{t}{t_0}\right)^2+\left(\dot{J}_0-\frac{J_0}{t_0}\right)t\ln\frac{t}{t_0}+J_0\frac{t}{t_0}\\&+t\int_{t_0}^{t}\Phi\ln t\ln\left(e^2t\right)\mathrm{d}t-2t\ln t\int_{t_0}^{t}\Phi\ln(et)\mathrm{d}t+t\ln^2 t\int_{t_0}^{t}\Phi\mathrm{d}t\end{aligned} \tag{13}$$

where J_0, $\dot{J}_0$, $\ddot{J}_0$ and Φ_0 are the values of *J*, $\mathrm{d}J/\mathrm{d}t$, $\mathrm{d}^2J/\mathrm{d}t^2$ and Φ under $t=t_0$. From the initial conditions $\boldsymbol{\rho}_{i0}=\boldsymbol{r}_{i0}$, $\dot{\boldsymbol{\rho}}_{i0}=\dot{\boldsymbol{r}}_{i0}$ follows

$$J_0=I_0,\quad \dot{J}_0=\dot{I}_0,\quad \Phi_0=U_0 \tag{14}$$

where the values I_0, $\dot{I}_0$ and U_0 refers to the classical *n* body problem. From Equation (9) under $\alpha=9/8$ we have:

$$\ddot{J}_0=4h-2\Phi_0-\frac{1}{2}\frac{J_0}{t_0^2}. \tag{15}$$

If in the introduced generalized *n* body problem the impact of gravitating background is rather weak, then in formula (13) we may put $J\approx I$ and $\Phi\approx U$. Correspondingly, we will have the following approximate expression for the moment of inertia of the ordinary classical *n*-body problem, satisfying the condition $\delta(t)\gg 1/6\pi G t^2$:

$$I \approx \frac{1}{2}t_0\left(4h-\frac{\dot{I}_0}{t_0}+\frac{1}{2}\frac{I_0}{t_0^2}\right)t\ln\left(\frac{t}{t_0}\right)^2+\left(\dot{I}_0-\frac{I_0}{t_0}\right)t\ln\frac{t}{t_0}+I_0\frac{t}{t_0}$$
$$+t\int_{t_0}^{t}U\ln t\ln\left(e^2t\right)\mathrm{d}t-2t\ln t\int_{t_0}^{t}U\ln\left(et\right)\mathrm{d}t+t\ln^2 t\int_{t_0}^{t}U\mathrm{d}t. \tag{16}$$

On the other hand, integration of the LJE (2) gives us

$$I=I_0+\dot{I}_0\left(t-t_0\right)+2h\left(t-t_0\right)^2-2\int_{t_0}^{t}\int_{t_0}^{t}U\mathrm{d}t\mathrm{d}t\,. \tag{17}$$

Let us φ denotes a semi-difference of the right parts of Relations (16) and (17):

$$\varphi=-\int_{t_0}^{t}\int_{t_0}^{t}U\mathrm{d}t\mathrm{d}t-\frac{1}{2}t\int_{t_0}^{t}U\ln t\ln\left(e^2t\right)\mathrm{d}t+t\ln t\int_{t_0}^{t}U\ln\left(et\right)\mathrm{d}t-\frac{1}{2}t\ln^2 t\int_{t_0}^{t}U\mathrm{d}t$$
$$+h\left(t-t_0\right)^2+At\left(\ln\frac{t}{t_0}\right)^2+Bt\left(\ln\frac{t}{t_0}-1\right)+Bt_0 \tag{18}$$

where

$$A=-\frac{t_0}{4}\left(4h-\frac{\dot{I}_0}{t_0}+\frac{1}{2}\frac{I_0}{t_0^2}\right),\quad B=-\frac{1}{2}\left(\dot{I}_0-\frac{I_0}{t_0}\right). \tag{19}$$

An inverse differential relation can be found for Formula (18). Let us put

$$V=\int_{t_0}^{t}U\mathrm{d}t\,,\quad \int_{t_0}^{t}\int_{t_0}^{t}U\mathrm{d}t\mathrm{d}t=\int_{t_0}^{t}V\left(s\right)\mathrm{d}s \tag{20}$$

Using the equalities

$$t\int_{t_0}^{t}U\ln t\ln\left(e^2t\right)\mathrm{d}t=Vt\ln t\ln\left(e^2t\right)-2\int_{t_0}^{t}\frac{t\ln\left(es\right)}{s}V\left(s\right)\mathrm{d}s,$$
$$t\ln t\int_{t_0}^{t}U\ln\left(et\right)\mathrm{d}t=Vt\ln t\ln\left(et\right)-\int_{t_0}^{t}\frac{t\ln t}{s}V\left(s\right)\mathrm{d}s \tag{21}$$

Formula (18) can be rewritten in the form of the following integral equation:

$$-\int_{t_0}^{t}\left(1-\frac{t}{s}\ln\frac{es}{t}\right)V\left(s\right)\mathrm{d}s=-h\left(t-t_0\right)^2-At\left(\ln\frac{t}{t_0}\right)^2-Bt\left(\ln\frac{t}{t_0}-1\right)-Bt_0+\varphi\,. \tag{22}$$

Twice differentiation over t gives

$$-\int_{t_0}^{t}\frac{V\left(s\right)}{s}\mathrm{d}s=-2ht-2A\left(1+\ln\frac{t}{t_0}\right)-B+t\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} \tag{23}$$

hence we obtain

$$-V=2ht-2A+t^2\frac{\mathrm{d}^3\varphi}{\mathrm{d}t^3}+t\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\,, \tag{24}$$

so that under any constants A and B in Formula (18) we have the equation

$$-U+2h=t^2\frac{\mathrm{d}^4\varphi}{\mathrm{d}t^4}+3t\frac{\mathrm{d}^3\varphi}{\mathrm{d}t^3}+\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\,. \tag{25}$$

Taking into account now in Formula (18) the LJE (2) and the values (19) of constants A and B, we find

$$\varphi = -\frac{1}{8} I_0 \frac{t}{t_0}\left(\ln\frac{t}{t_0}\right)^2 + \frac{1}{2} t \ln t \int_{t_0}^{t} \frac{I}{t^2} \mathrm{d}t - \frac{1}{2} t \int_{t_0}^{t} \frac{I}{t^2} \ln t \mathrm{d}t . \tag{26}$$

From Formulas (25) and (26) we have the following modification of the LJE equation for the n-body problem:

$$-U + 2h = t^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + 3t \frac{\mathrm{d}\psi}{\mathrm{d}t} + \psi \tag{27}$$

where

$$\psi = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = -\frac{1}{4}\frac{I_0}{t_0}\frac{1}{t}\ln\frac{t}{t_0} - \frac{1}{4}\frac{I_0}{t_0}\frac{1}{t} + \frac{1}{2}\frac{I}{t^2} + \frac{1}{2t}\int_{t_0}^{t}\frac{I}{t^2}\mathrm{d}t . \tag{28}$$

In such equation it is no longer difficult to take into account the dynamical condition of the form $\delta \gg 1/6\pi G t^2$ for the relaxing gravitating system of density δ, what is useful in case of clusters of galaxies.

2. The Analytical Form of the Potential Energy and Its Application to the Relaxing Clusters of Galaxies

We assume now in Equation (27) time t_0 is the epoch of formation T of a large-scale structure of Universe as a non-equilibrium system of gravitating bodies. Let us expand the corresponding Function (28) into Tailor series over degrees of time $t - T > 0$ taking only that first terms which coefficients are completely determined by the initial conditions for gravitating system:

$$\Psi \approx \Psi_0 + \left(\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)_0 (t-T) + \frac{1}{2}\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2}\right)_0 (t-T)^2 + \frac{1}{6}\left(\frac{\mathrm{d}^3\Psi}{\mathrm{d}t^3}\right)_0 (t-T)^3 \tag{29}$$

where

$$\begin{gathered}\Psi_0 = \frac{1}{4}\frac{I_0}{t^2}, \left(\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)_0 = \frac{1}{2T^2}\left(\dot{I}_0 - \frac{I_0}{T}\right), \left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2}\right)_0 = \frac{-U_0 + 2h}{T^2} + \frac{5}{4}\frac{I_0}{T^4} - \frac{3}{2}\frac{\dot{I}_0}{T^3}, \\ \left(\frac{\mathrm{d}^3\Psi}{\mathrm{d}t^3}\right)_0 = -\frac{\dot{U}_0}{T^2} + \frac{5(-U_0 + 2h)}{T^3} - \frac{17}{4}\frac{I_0}{T^5} + \frac{11}{12}\frac{\dot{I}_0}{T^4}\end{gathered} \tag{30}$$

Then the modified LJE (27) gives us

$$\begin{aligned}-U + 2h \approx{}& \frac{4}{3}(-U_0 + 2h) + \frac{25}{12}\frac{I_0}{T^2} + \frac{1}{6}\dot{U}_0 T + \frac{26}{12}\frac{\dot{I}_0}{T} + 2\dot{U}_0 T - 19\frac{\dot{I}_0}{T}\Bigg]\frac{t}{T} \\ &+ \left[27(-U_0 + 2h) + \frac{198}{8}\frac{I_0}{T^2} + \frac{9}{2}\dot{U}_0 T - \frac{126}{4}\frac{\dot{I}_0}{T}\right]\left(\frac{t}{T}\right)^2 \\ &- \left[\frac{40}{3}(-U_0 + 2h) + \frac{34}{3}\frac{I_0}{T^2} + \frac{8}{3}\dot{U}_0 T - \frac{44}{3}\frac{\dot{I}_0}{T}\right]\left(\frac{t}{T}\right)^3 .\end{aligned} \tag{31}$$

Correspondingly, we have

$$\begin{aligned}-U + 2h \approx{}& \frac{4}{3}(-U_0 + 2h) + \frac{1}{6}\dot{U}_0 T - \left[14(-U_0 + 2h) + 2\dot{U}_0 T\right]\frac{t}{T} \\ &+ \left[27(-U_0 + 2h) - \frac{9}{2}\dot{U}_0 T\right]\left(\frac{t}{T}\right)^2 - \left[\frac{40}{3}(-U_0 + 2h) + \frac{8}{3}\dot{U}_0 T\right]\left(\frac{t}{T}\right)^3 ,\end{aligned} \tag{32}$$

if we can put

$$\frac{I_0}{T^2} \ll \left|-U_0 + 2h\right|, \quad \frac{\left|\dot{I}_0\right|}{T^2} \ll \frac{1}{10}\left|\dot{U}_0\right|. \tag{33}$$

Let us now make the following estimations regarding condition (33). Potential energy of the gravitating system with mean density δ is proportional to δ^2 under given characteristic radius R of the system:

$$U \approx -\frac{3}{5}\frac{G}{R}\left(\frac{4}{3}\pi R^3\delta\right)^2 = -\frac{48}{45}G\pi^2 R^5\delta^2. \tag{34}$$

The value I/t^2 is proportional to product of densities δ and $\delta_M = 1/6\pi G t^2$:

$$\frac{I}{t^2} \approx \frac{3}{5}\left(\frac{4}{3}\pi R^3\delta\right)\cdot\frac{R^2}{t^2} = \frac{24}{5}G\pi^2 R^5\delta\cdot\delta_M. \tag{35}$$

When $\delta/\delta_M \propto 5\times10^2$ it is possible to neglect the function I/t^2 and its derivative $\frac{\mathrm{d}\left(I/t^2\right)}{\mathrm{d}t}$ in comparison with U and $\mathrm{d}U/\mathrm{d}t$ while searching $U(t)$.

Relation (22) we may also simplify in the following way. If an effective mechanism of relaxation exists in a non-equilibrium gravitating system then the module of mean velocity of variation of the value $-U + 2h$ over the time $\tau \propto (G\delta)^{-\frac{1}{2}}$ that stands for time of setting up a virial equilibrium as a result of relaxation process, much exceeds the module of the initial value of its derivative:

$$\left|-U_0 + 2h\right| \gg \left|\dot{U}_0\right|\tau. \tag{36}$$

Excluding from consideration the case $\tau \ll T$, let us replace τ by T:

$$\left|-U_0 + 2h\right| \gg \left|\dot{U}_0\right|T. \tag{37}$$

Under condition (37) relation (32) can be given the form

$$-U + 2h \approx \left(-U_0 + 2h\right)\left[\frac{4}{3} - 14\frac{t}{T} + 27\left(\frac{t}{T}\right)^2 - \frac{40}{3}\left(\frac{t}{T}\right)^3\right]. \tag{38}$$

For the root $\frac{t}{T} > 1$ of equation

$$\frac{4}{3} - 14\frac{t}{T} + 27\left(\frac{t}{T}\right)^2 - \frac{40}{3}\left(\frac{t}{T}\right)^3 = 0 \tag{39}$$

we find $1.24 < \frac{t}{T} < 1.25$.

Thus, for the non-equilibrium system of gravitating bodies reaching the virial equilibrium over the time τ as a result of relaxation process, we have: either $\tau \ll T$, or under condition of applicability of Formula (32) the relation

$$\tau \approx \frac{1}{4}T, \tag{40}$$

where T is the epoch of formation of such system.

Assuming the variation of the density δ_1 is approximately by one order over the time of existing the galaxies as a systems of gravitating bodies and decreasing of the cosmological density δ_M also is approximately by one order for the same time, we obtain the ratio δ_1/δ_M ranging from 10^3 to 10^5. To the orders of these magnitudes the order of extreme ratio (from the left) $\delta/\delta_M \approx 5\times 10^3$ of the large scale gravitating subsystem's density to the cosmological density itself is very close, when the use of the formula (32) is still possible. Accordingly, there is a reason to refer the relations (38) and (40) to the relaxing clusters of galaxies.

Thus, the above considered approach to the clusters of galaxies gives us an analytical form of their potential energy at the stage of relaxation as the function of time and two constants—the energy integral h and the epoch T of formation of these objects as a non-equilibrium systems of gravitating bodies:

$$U \approx 2h - \left(2h - U_0\right)\left[\frac{4}{3} - 14\frac{t}{T} + 27\left(\frac{t}{T}\right)^2 - \frac{40}{3}\left(\frac{t}{T}\right)^3\right]. \qquad (41)$$

3. Conclusions and Discussion

Summarizing results of the present paper, we conclude that:

We have introduced a new formal parameter of n-body problem instead of momentum of inertia I in Equation (2) to obtain the modification of the LJE;

We received the analytical form of the potential energy of clusters of galaxies at the stage of relaxation as the function of time and two constants—the energy integral h and the epoch T of formation of these objects as a non-equilibrium systems of gravitating bodies.

One of the applications of the Formula (41) is the analytical Relation (40) between the time τ of setting up the virial equilibrium in relaxing clusters of galaxies and the cosmological epoch T.

We would like to highlight that while deducing the Equation (40) we did not used the analytical Property (7) of the Einstein-de Sitter cosmological model, but the corresponding Relation (6) for the clusters of galaxies as a large-scale structure of Universe.

It can be anticipated that the relation of order $\delta \gg 1/6\pi G t^2$ admits some deviations in properties of the real Universe from the Einstein-de Sitter model. Therefore, we can consider Formulas (40) and (41) as the results written in the real cosmological time scale.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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