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Transformation Matrix for Combined Loads Applied to Thin-Walled Structures

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Abstract

This paper transforms combined loads, applied at an arbitrary point of a thin-walled open section beam, to the shear centre of the cross-section of the beam. Therein, a generalized transformation matrix for loads with respect to the shear centre is derived, this accounting for the bimoments that develop due to the way the combined loads are applied. This and the authors' earlier paper (World Journal of Mechanics 2021, 11, 205-236) provide a full solution to the theory of thin-walled, open-section structures bearing combined loading. The earlier work identified arbitrary loading with the section's area properties that are necessary to axial and shear stress calculations within the structure's thin walls. In the previous paper attention is paid to the relevant axes of loading and to the transformations of loading required between axes for stress calculations arising from tension/compression, bending, torsion and shear. The derivation of the general transformation matrix applies to all types of loadings including, axial tensile and compression forces, transverse shear, longitudinal bending. One application, representing all these load cases, is given of a simple channel cantilever with an eccentrically located end load.

Keywords

Thin-Walled Structure, Open Sections, Transformation Matrix, Load Transformation, Combined Load Transformation, Shear Centre, Warping, Bimoment, Sectorial Area Properties

1. Introduction

Beams constructed with thin rectangular plates are popular with designers, the advantages being that they are easy to produce and assemble, their performance under different force systems giving high efficiency in terms of their weight versus load.

The essential analysis of thin-walled beams of open sections involves the theory of warping, developed in the 1930's by Wagner [1], then later by Vlasov [2]. However, both considered mainly the elastic deformation problem in reviewing the assumptions of thin-walled beams. Both concluded that, the Bernoulli's hypothesis of plane cross sections remaining plane under pure torsion, is no longer true. In fact, the cross-section of thin-walled beam may undergo longitudinal extension, at different rates and signs, because of torsion, which was later called warping of a cross section. Consequently, longitudinal normal stresses to these warping strains are created [1] and [2].

Another important phenomenon was found that longitudinal force applied at an arbitrary position will cause warping on cross section planes depending on the position and the boundary conditions, since this force may not be replaced by a statically equivalent longitudinal force. Thus, the beam will be subjected to a self-balancing set of longitudinal forces. This is of course different from the elementary theory of beam bending, which assumes that shear strains and stresses are zero [2].

Although a general stiffness matrix for an element of open section thin-walled beam, was derived in 1965 by Marten [3], and then by Przemieniecki [4], Livesley [5] and many others, it appears that this matrix applies only to the beam for pure bending.

Relatively few papers dealt with the torsion matrix also to be combined with the bending matrix. In 1967 Krahula [6], depending on Vlasov's theory of thin-walled structures [2], used the coupling twist mode and the warping mode, defined by φ, and φ', respectively. The solution to the homogeneous differential equation (for zero moment), was used to define both modes, as well as the corresponding torque and bimoment applied to the element. In 1969, Kracinovic, also derived a similar stiffness matrix using Galerkin's method to find the work done by external and internal forces [7].

Barsoum and Gallagher [8], in 1970 derived the stiffness and geometric matrices using the principle of potential energy which the strain energy expression used, depended upon hypotheses made by Bleich [9]. They also assumed a displacement function to derive their matrix. The solution found was approximate, but it remains useful for engineering applications under specific loadings.

Rajasekaran [10] in 1977 employed "element-wise approximations", by replacing the physical beam with an assembly of discrete elements. Stiffness and geometric matrices were derived using an assumed shape of mode deflections. By considering the second order terms in the strain expression, he was able to analyse the structure in the elastic and plastic regions. The most important feature of Rajasekaran's method is that he endorsed the assumptions of the thin-walled beam theory.

Baigent and Hancock (1982) [11], presented a matrix method for the analysis of thin-walled beams, the non-uniform torsion effect was included in the matrix displacement analysis. The study included the eccentricity of the load system from the shear centre.

Both, Rajasekaran [10], and Baigent and Hancock [11], derived separately a transformation matrix for loads applied at arbitrary points on a cross-section of a thin-walled beam. However, it appears that neither derivation admitted the presence of the bimoments that arise when longitudinal loadings are offset from the shear centre.

Alsheikh [12], in 1985, studied the thin-walled theory stated by Vlasov [2], and by using a pure mathematical derivation introduced a new form to the theory, which enabled him to present the following:

1) Normal and tangential stress formulae were derived.

2) Proof that the centre of rotation of an open cross section thin-walled beam, and the principal sectorial poles (shear centres) do coincide.

3) A general differential equation for a thin-walled beam was derived and solved. Consequently, an overall stiffness matrix accounting for the action of a set of combined loads, was derived.

4) A load transformation matrix was derived to account for the forces applied apart from the shear centre.

In 1988 Chen and Hu [13], reviewed a torsional stiffness matrix and a transformation matrix for a thin-walled beam application under combined loading. Given that their study was an application of earlier work, it served to clarify the state of knowledge upon the subject at that time.

Eduardo N. Dvorkin, *et al.* [14], have presented in 1989, analysis of a thin-walled open section beam, using the same differential equation of Vlasov. This study also seems to rely upon older works. They applied these equations to a curved beam, but the numerical results were not compared with any of other results.

A study appeared in 1999 by Musat and Epureanu [15] on thin-walled beams with open cross-section under complex load using the concept of a strip-plate as a macro-element. In addition to adopting the hypotheses of Vlasov's theory, they claimed an approach which delivered better results than the classical theory could provide.

A general solution was developed by, R. Emre Erkmen, in 2006 [16], based on a non-orthogonal coordinate system, this yielded nodal values in shearing stresses, and included the shear deformation effects due to torsional warping, useful for design purposes.

Wang, *et al.* [17], in 2011 investigated restrained torsion of open thin-walled beam to include the effects of shear deformation. By using a first-order torsion theory based on Vlasov's theory [2], they studied the relationship between overall rotation and free warping. This, they claimed provided more accurate results compared with older methods.

Mohammad Ferradi and Xavier Cespedes in 2014, [18], used an iterative equilibrium scheme to determine the transformation modes by decomposing the cross section of a thin-walled structure into a 1 D element. It was claimed that their simplification had provided an exact solution to the differential equation derived in the section that follows. Most analyses mentioned above assume, reasonably, that open cross-sections do not vary with beam length and that they are uniformly thin. A higher order analysis [19], required to account for such variations, are not considered here.

It is seen from the literature that there are a multitude of aspects to the analysis if thin-walled structures. Crucial among these is the nature of the applied loading and the way loading should be transformed to the appropriate longitudinal axis. In the case of a cantilever beam three axes apply which are located arbitrarily or placed to coincide with the shear centre or the centroid of the beam section area. Transformation as it appears in this paper refers a loading along any arbitrary axis to either of the other two axes mentioned as is necessary for a stress conversion.

2. Theoretical Development

The displacements and internal forces developed due to the theory of thin-walled structures, are related to the principal sectorial pole, see [2] [12], and Alsheikh and Rees [20], also Alsheikh and Sharman [21]. In other words, in a complete stiffness matrix, which includes bending, shear and twisting modes, the terms from the bending modes are relative to transverse forces through the shear centre, and those due to axial modes are relative to forces through the centroid. In the simpler theory of beam bending an axial force applied away from the centroid, will in fact produce bending moments on the structure. The load transformation from an arbitrary point on the structure, must consider therefore the two modes, transverse, and axial loading, separately. This requires first to study the effect of each load on the bending moments and the bimoments.

2.1. Transformation of Transverse Forces

According to the hypothesis which considers the beam section as rigid, then, the stresses within that section will not change when an external transverse load is replaced by another set of forces statically equivalent to the first. Thus, in general, a transverse force applied apart from the shear centres (SC), $T = Pe$, may cause a combined of torsional and flexural moments to the beam (see **Figure 1**).

2.2. Transformation of Longitudinal Force

When a longitudinal force, applied at an arbitrary point, is transferred to the shear centre, it will produce an additional bimoment B proportional to the value of the sectorial coordinate at this point as follows:

$$B = P\omega_D \tag{1}$$

where in **Figure 2**:

P is a longitudinal force applied at point D.

ω_D is the sectorial coordinate of point D with respect to the shear centre (principal sectorial pole).

If a thin-walled beam of open section is subjected to a longitudinal force N, bending moments M_y and M_Z and a bimoment B_D act at an arbitrary point D on the structure (**Figure 2**).

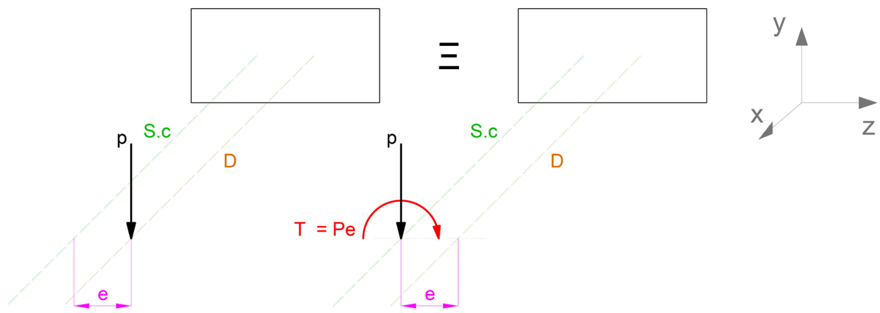

Figure 1. Transformation of a transverse force.

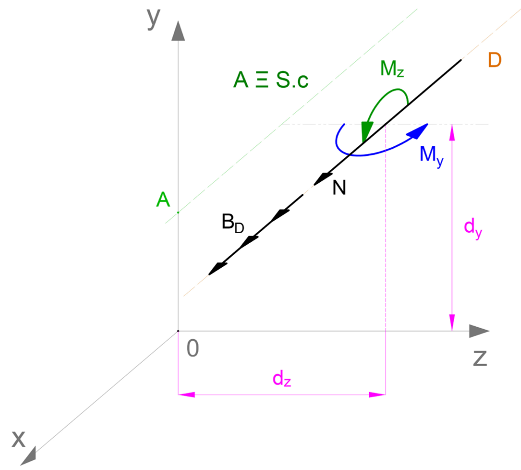

Figure 2. A load system applied at D in centroid co-ordinates *x*, *y*, *z*.

The axial stress at an arbitrary point on the open profile will be according to the following equation, Alsheikh [12], p28

$$\sigma = E\left(u_0' - v''y - w''z - \varphi''\omega_D\right) \tag{2}$$

where primed u, v and w are displacement derivatives and ω_D is the sectorial coordinate with respect to the sectorial pole D. The generalized forces on the structure at point, D will be:

$$N = \int_A \sigma \mathrm{d}A \tag{3}$$

$$\bar{M}_y = -\int_A \sigma z \mathrm{d}A \tag{4}$$

$$\bar{M}_z = \int_A \sigma y \mathrm{d}A \tag{5}$$

and the bimoment as references [2] and [12], and [20] show, is:

$$B_D = \int_A \sigma \omega_D \mathrm{d}A \tag{6}$$

Substituting Equation (2) into Equations (3)-(6) and re-writing we have:

$$Eu_0' = N/A + E\varphi'' S_{WD}/A \tag{7}$$

$$Ev'' = -\left[\left(\bar{M}_y I_{yz} + \bar{M}_z I_y\right) + E\left(S_{\omega yD} I_y - S_{\omega zD} I_{yz}\right)\right] / \left(I_y I_z - I_{yz}^2\right) \tag{8}$$

$$Ew'' = \left[\left(\bar{M}_y I_z + \bar{M}_z I_{yz}\right) - E\left(S_{\omega zD} I_z - S_{\omega yD} I_{yz}\right)\right] / \left(I_y I_z - I_{yz}^2\right) \tag{9}$$

$$B_D = Eu_0' S_{\omega D} - Ev'' S_{\omega yD} - Ew'' S_{\omega zD} - E\Gamma_D \phi'' \tag{10}$$

where,

$\int_A \mathrm{d}A = A$ *total area of the cross section.*
$\int_A y\mathrm{d}A = S_z$ *first moment of area of the cross-section about axis-z.*
$\int_A z\mathrm{d}A = S_y$ *first moment of-area of the cross-section about axis-y.*
$\int_A y^2\mathrm{d}A = I_z$ *second moment of area of the cross-section about axis-z.*
$\int_A z^2\mathrm{d}A = I_y$ *second moment of area of the cross-section about axis-y.*
$\int_A yz\mathrm{d}A = I_{yz}$ *product moment of area of the cross-section.*

In addition to these expressions, which are well known from the strength of materials, there are new expressions to be considered, which are related to the law of sectorial area referred to point D. These are:

$\int_A \omega\mathrm{d}A = S_\omega$ *first moment of sectorial area.*
$\int_A \omega z\mathrm{d}A = S_{\omega z}$ *product moment of sectorial area about y-axis.*
$\int_A \omega y\mathrm{d}A = S_{\omega y}$ *product moment of sectorial area about z-axis.*
$\int_A \omega^2\mathrm{d}A = \Gamma$ *second moment of sectorial area.*

Substituting Equations (7), (8) and (9) into Equation (10) we have:

$$\begin{aligned} B_D &= \frac{N}{A} S_{\omega D} + \frac{\bar{M}_z I_y + \bar{M}_y I_{yz}}{I_y I_z - I_{yz}^2} S_{\omega yD} - \frac{\bar{M}_y I_z + \bar{M}_z I_{yz}}{I_y I_z - I_{yz}^2} S_{\omega zD} \\ &\quad - \left(\Gamma_D - \frac{S_{\omega yD} I_y - S_{\omega zD} I_{yz}}{I_y I_z - I_{yz}^2} S_{\omega yD} - \frac{S_{\omega zD} I_z - S_{\omega yD} I_{yz}}{I_y I_z - I_{yz}^2} S_{\omega zD} - \frac{S_{\omega D}^2}{A} \right) E\phi'' \end{aligned} \tag{11}$$

Recalling Equations (32)-(34) from Alsheikh and Rees [20]

$$a_y - d_y = -\frac{S_{\omega zD} I_z - S_{\omega yD} I_{yz}}{I_y I_z - I_{yz}^2} \tag{12}$$

$$a_z - d_z = \frac{S_{\omega yD} I_y - S_{\omega zD} I_{yz}}{I_y I_z - I_{yz}^2} \tag{13}$$

$$\Gamma = \Gamma_D + \left(a_y - d_y\right) S_{\omega zD} - \left(a_z - d_z\right) S_{\omega yD} - \frac{S_{\omega D}^2}{A} \tag{14}$$

where a_z and a_y are the co-ordinates of the shear centre relative to the centroid. Substituting Equations (12), (13), and (14) into (11) we have,

$$B_D = \frac{N}{A} S_{\omega D} + \left(a_y - d_y\right)\bar{M}_y + \left(a_z - d_z\right)\bar{M}_z - E\Gamma\varphi'' \tag{15}$$

Also, from Equation (47) Alsheikh and Rees [20]:

$$B = -E\Gamma\varphi'' \tag{16}$$

By substituting (16), into (15) we get,

$$B = B_D - \frac{N}{A} S_{\omega D} - \left(a_y - d_y\right)\bar{M}_y - \left(a_z - d_z\right)\bar{M}_z \tag{17}$$

In Equation (17) both the bending moments $\bar{M}_y$ and $\bar{M}_z$ represent the total generalized bending moment on the structure at point D, and these are represented by the applied bending moments M_y and M_z and the bending moment due to the generalized longitudinal force N, positioned at point D, as **Figure 3**.

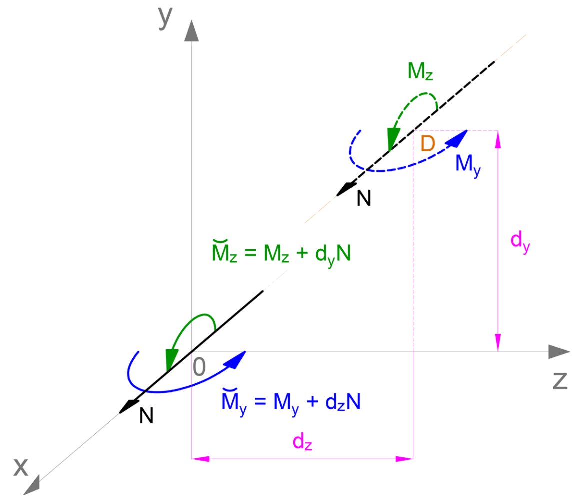

Figure 3. Transformation of a load system from point D to centroidal axes.

$$\bar{M}_y = M_y - d_z N \quad (18)$$

$$\bar{M}_z = M_z + d_y N \quad (19)$$

Substituting Equations (18) and (19) into (17) we have:

$$B = B_D - \frac{N}{A} S_{\omega D} - \left(a_y - d_y\right)\left(M_y - d_z N\right) - \left(a_z - d_z\right)\left(M_z + d_y N\right) \quad (20)$$

It is obvious, in Equation (20) that the first term is a result of multiplication of the longitudinal force N by the sectorial coordinate of the point D with respect to the shear centre A, as we can see from the following.

Recall from [20] the following equations:

$$\omega_A = \left(z_o - a_z\right)\left(y - y_o\right) - \left(y_o - a_y\right)\left(z - z_o\right) \quad (21)$$

$$\frac{S_{\omega D}}{A} = \left(a_y - d_y\right) z_o - \left(a_z - d_z\right) y_o \quad (22)$$

Solving the above two equations shows,

$$\omega_A\left(D\right) = a_y d_z - a_z d_y - \frac{S_{wD}}{A} \quad (23)$$

Now substituting Equation (23) into (20) reveals final equation of force transformation

$$B = N\omega\left(D\right) - \left(a_y - d_y\right) M_y - \left(a_z - d_z\right) M_z + B_D \quad (24)$$

Equation (24) does not only give the effect of a longitudinal load applied at an arbitrary point D, to thebimoment on the structure, but also gives the effect of the bending moment on the bimoment, in terms of the values of the coordinates of the point D and the coordinates of the shear centre, multiplied by the value of the bending moment. This is as proved in the earlier work by AI-Sheikh [12].

2.3. Transformation of Bending Moments

The bimoment due to an applied bending moment is given by Me, where M is the bending moment applied at an arbitrary point D, and e is the distance between the plane of the moment and the shear centre (see Figure 4). The bimoment produced by a bending moment offset from the shear centre is a self-balancing longitudinal load, whether the bending moment consists of a transverse or longitudinal couple with a distance ($\Delta s \to 0$).

The distribution of the bimoment along the x axis is always similar to that of a longitudinal force. This is also possible if a bending moment is not applied to the cross-section directly but to an arbitrary point connected to the section by a rigid bracket.

2.4. Transformation of the Stiffness Matrix

The transformation for the stiffness matrix of a thin-walled beam of open section, comprises the coordinate transformation of local systems of each node to the global system. This includes the transformation of node actions acting at an arbitrary point D on the cross-section to the centroid O and shear centre A, as applicable. The former (centroid) can be found in many texts, see Beaufait *et al.* [22], ZIENKIEWICZ, O. C [23], and WEAVER, W. and GERE, J.M. the latter (shear centre) will be defined as in Figure 5 and by means of Equation (24) as follows:

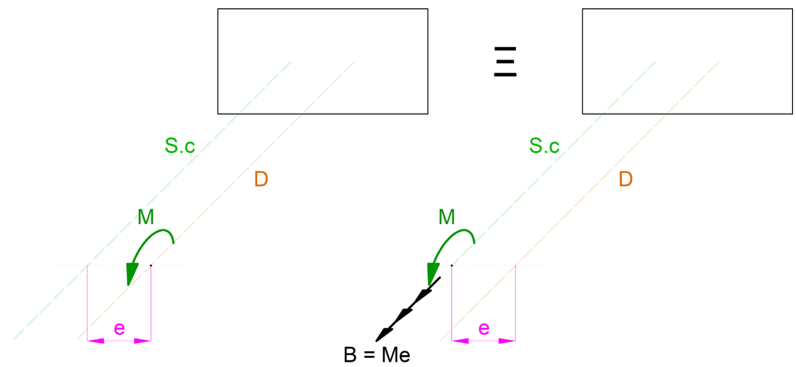

Figure 4. Transformation of loading to shear centre.

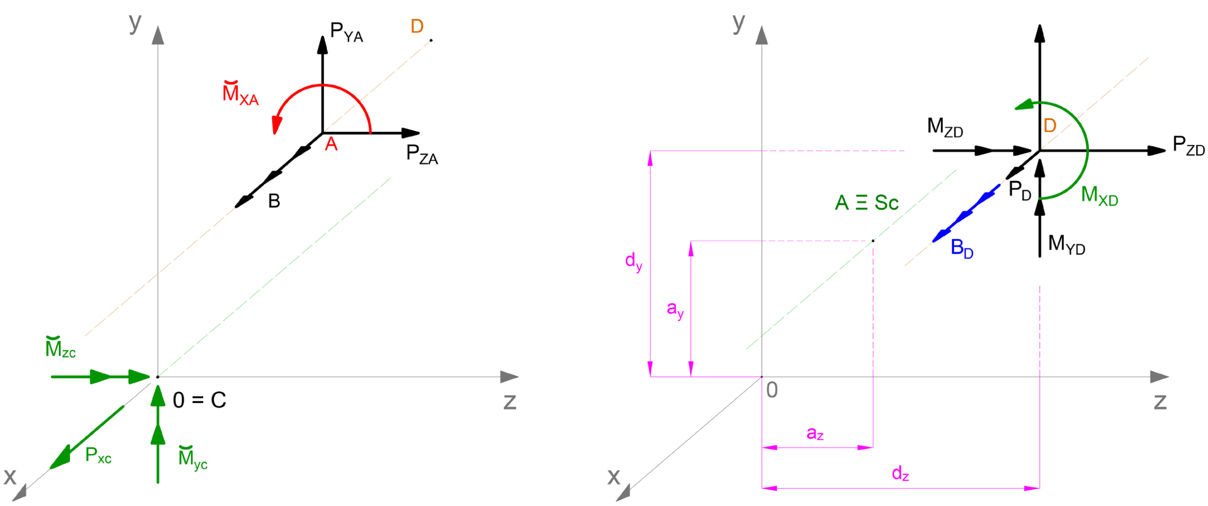

Figure 5. Transformation of a load system applied at D to the centroid 0 and to the shear centre A.

$$\begin{aligned}
&P_{xc} = P_{xD} \\
&P_{yA} = P_{yD} \\
&P_{zA} = P_{zD} \\
&\bar{M}_{xA} = P_{yD}\left(d_z - a_z\right) + P_{zD}\left(a_y - d_y\right) + M_{xD} \\
&\bar{M}_{yC} = -P_{xD}\cdot d_z + M_{yD} \\
&\bar{M}_{zC} = P_{xD}\cdot d_y + M_{zD} \\
&B_A = P_{xD}\omega_A\left(D\right) - \left(a_y - d_y\right)M_{yD} - \left(a_z - d_z\right)M_{zD} + B_D
\end{aligned} \tag{25}$$

Writing Equations (25) in matrix form we have:

$$\begin{bmatrix} P_{xc} \\ P_{ya} \\ P_{za} \\ \bar{M}_{xa} \\ \bar{M}_{yc} \\ \bar{M}_{zc} \\ B_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\left(a_z - d_z\right) & \left(a_y - d_y\right) & 1 & 0 & 0 & 0 \\ -d_z & 0 & 0 & 0 & 1 & 0 & 0 \\ d_y & 0 & 0 & 0 & 0 & 1 & 0 \\ \omega\left(D\right) & 0 & 0 & 0 & -\left(a_y - d_y\right) & -\left(a_z - d_z\right) & 1 \end{bmatrix} \times \begin{bmatrix} P_{xd} \\ P_{yd} \\ P_{zd} \\ \bar{M}_{xd} \\ \bar{M}_{yd} \\ \bar{M}_{zd} \\ B_d \end{bmatrix} \tag{26}$$

In Equations (26) we see that the second and third terms of fourth row are the contribution factors froma transverse force applied at point D to the twisting moment $\bar{M}_{xa}$. The first term, in the fifth row, is the contribution factor of the longitudinal force to the bending moment $\bar{M}_{yc}$ about the y-axis. The first term of the sixth row is the contribution factor of the longitudinal force to the bending moment $\bar{M}_{zc}$ about the z-axis. In row seven, the first term is the contribution factor of the axial force P_x applied at point D to the total bimoment. The fifth and sixth terms in the seventh row are the contribution factors of the two bending moments $\bar{M}_{yd}$ and $\bar{M}_{zd}$, at point D, to the total bimoment B_A.

Equations (26) may be written in a symbolic matrix form for the actions upon a single nodal point i:

$$\left[P_i\right] = \left[T_i\right]\left[P_{Di}\right]$$

where:

$$\left[P_i\right] = \left[P_{xc}P_{yA}P_{zA}\bar{M}_{xa}\bar{M}_{yc}\bar{M}_{zc}B_a\right]_i^{\mathrm{T}} \tag{27}$$

$$\left[T_i\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\left(a_z - d_z\right) & \left(a_y - d_y\right) & 1 & 0 & 0 & 0 \\ -d_z & 0 & 0 & 0 & 1 & 0 & 0 \\ d_y & 0 & 0 & 0 & 0 & 1 & 0 \\ \omega\left(D\right) & 0 & 0 & 0 & -\left(a_y - d_y\right) & -\left(a_z - d_z\right) & 1 \end{bmatrix} \tag{28}$$

$$\left[P_D\right] = \left[P_{xd}P_{yd}P_{zd}\bar{M}_{xd}\bar{M}_{yd}\bar{M}_{zd}B_d\right]_i^{\mathrm{T}} \tag{29}$$

The equation for transforming the stiffness matrix from local to global co-ordinates involves $[T_i]$ as follows:

$$[K_i] = [T_{ij}]^{\mathrm{T}} [K_{ij}][T_{ij}] \tag{30}$$

where K_{ii} is the 12 × 12 element stiffness matrix which is the combination, of bending modes as given in many texts, see Zienkiewicz [23], Weaver and Gere [24], and the St Venant torsional mode, given in reference [20] Table 1. That combination has been assembled in the 14 × 14 matrix given by Alsheikh and Rees [20] Table 3. The 7 × 7 transformation matrix $[T_{ij}]$ in Equation (27), when applied between adjacent nodes i and j, appears with the element stiffness matrix $[K_{ij}]$ in Equation (30) as follows:

$$[T_{ij}] = \begin{bmatrix} [T_i] & 0 \\ 0 & [T_j] \end{bmatrix} \tag{31}$$

where,

$$[T_j] = [T_i] \tag{32}$$

and

$$[T_{ij}^{\mathrm{T}}] = \begin{bmatrix} [T_i^{\mathrm{T}}] & 0 \\ 0 & [T_j^{\mathrm{T}}] \end{bmatrix} \tag{33}$$

2.5. Rotation of Principal Directions

Equations (26) may be applied only for translations of node actions, *i.e.*, when the principal axes coincide with the global axis. When failing to secure this condition, a matrix for rotation of the principal directions is needed.

The rotation matrix r may be derived from **Figure 6** as follows:

$$r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \tag{34}$$

This rotation matrix applies to actions associated with the translated displacements u, v and w. It is also applicable to the actions associated with twist rotations φ_x, φ_y and φ_z. Specifically, the action associated with the rate of twist, *i.e.*, the bimoment, remains the same after rotation. Here it can be seen from **Figure 6** and **Figure 7** that if a symmetric I-beam of web depth d is subjected to a bimoment B, this bimoment may be represented by two equal and opposite bending moments in the y direction as shown:

$$B = M_y b \tag{35}$$

$$B = M_y d \cos\beta \tag{36}$$

where:

$$b = d \cos\beta \tag{37}$$

The projections of M_y on the principal axes y_m and z_m are,

$$M_{ym} = M_Y \cos\beta \tag{38}$$

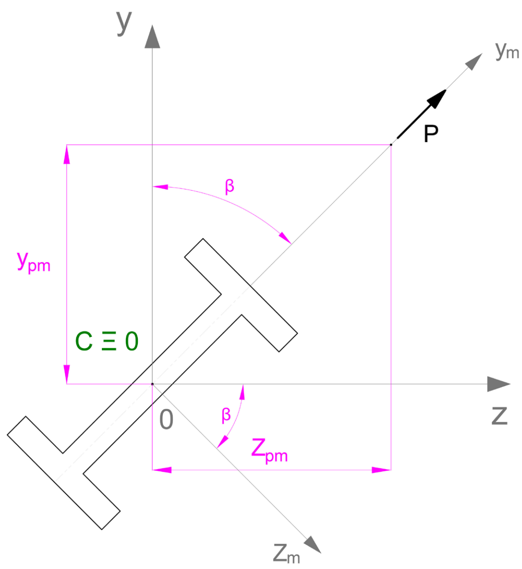

Figure 6. Local and global coordinates of an I-Beam.

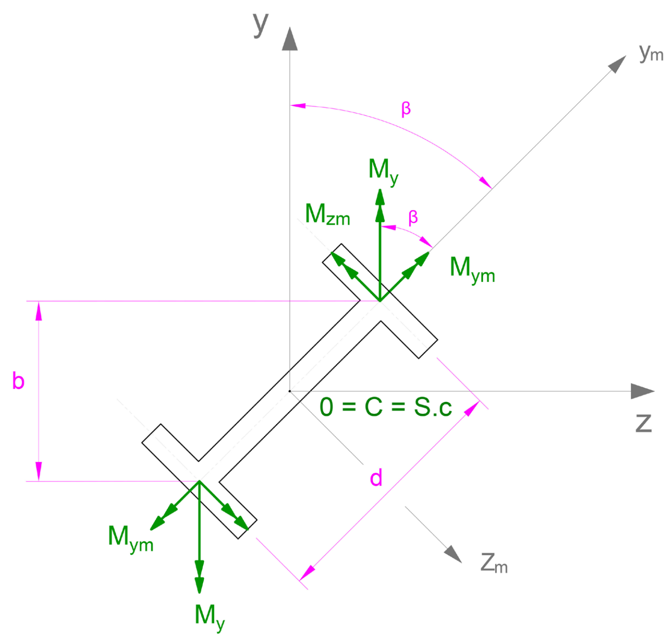

Figure 7. Transformation of bending moment from global to local coordinates.

$$M_{zm} = M_Y \sin\beta \tag{39}$$

It is also seen from the **Figure 7** that the bimoment due to M_z vanishes because its plane (x_m, y_m) passes through the shear centre, while the bimoment due to M_{ym} is

$$B_m = M_{ym} d$$

Note that M_{ym} acts in the plane of the flange.

Thus

$$B_m = M_y d \cos\beta \tag{40}$$

Identical Equations (36) and (40) means that, the term in the rotation matrix, associated with the bimoment becomes unity, $R_{33} = 1$. Thus, a rotation matrix for actions at node i will be,

$$[R_i] = \begin{bmatrix} [r] & 0 & 0 \\ 0 & [r] & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

Hence the rotation for beam element, will be,

$$[R_{ij}] = \begin{bmatrix} [R_i] & 0 \\ 0 & [R_j] \end{bmatrix} \tag{42}$$

where:

$$[R_i] = [R_j] \tag{43}$$

Combining Equations (30), (41) and (42) provides a comprehensive transformation for the beam element stiffness matrix:

$$[K_{ij}]_R = [R_{ij}^{\mathrm{T}}][T_{ij}^{\mathrm{T}}][K_{ij}][T_{ij}][R_{ij}] \tag{44}$$

3. Conclusions

This beam element transformation amends earlier forms [6] [8] [13] [15] in allowing for bimoment terms within $[T_{ij}^{\mathrm{T}}]$. As a sequel to the authors' previous paper [20], the combined work provided a complete solution to the response of a thin-walled beam structure under any combination of applied loading including axial and transverse forces, flexural bending, and axial torsion. The physical form of the structure may appear as a beam or as a long torsional section having ends that are free or position fixed.

Stress calculations arising from specific combined loadings are to appear in future papers. Thus, in consideration of structural safety, the probability of failure is to be assessed from plastic collapse and/or local buckling, which will be dealt with in future work.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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