

Maximum Interval of Stability and Convergence of Solution of a Forced Mathieu's Equation

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Abstract

This paper investigates the maximum interval of stability and convergence of solution of a forced Mathieu's equation, using a combination of Frobenius method and Eigenvalue approach. The results indicated that the equilibrium point was found to be unstable and maximum bounds were found on the derivative of the restoring force showing sharp condition for the existence of periodic solution. Furthermore, the solution to Mathieu's equation converges which extends and improves some results in literature.

Keywords

Frobenius Method, Eigenvalue Approach, Stability, Mathieu's Equation

1. Introduction

Consider a harmonically forced Mathieu's equation defined by

$$\frac{d^2x}{dt^2} + (w + \varepsilon \cos t)x = f \cos \lambda t \quad (1)$$

For small $|\varepsilon|$, this equation describes a simple harmonic oscillator whose frequency is a periodic function of time with the boundary condition as;

$$\begin{aligned} x(0) &= x(2\pi) \\ \dot{x}(0) &= \dot{x}(2\pi) \end{aligned} \quad (2)$$

where $\frac{d^2x}{dt^2}$ is the second derivative with respect to time, f is the amplitude of a periodic driving force, w and ε are the Mathieu's parameters and λ is the angular frequency of the periodic driving force.

Mathieu's equation is a special case of a linear second order homogenous equation [1]. In [2], Equation (1) was discussed in connection with problem of vibrations in elliptical membrane and developed the leading terms of the series known as Mathieu's function. Mathieu's function was further investigated by a number of researchers who found a considerable amount of results. [3] [4] [5] [6] wrote that Mathieu's differential equation occurs in two main categories involving elliptical geometrics, such as analysis of vibrating modes, elliptical membrane, the propagation modes of elliptic pipes and the oscillation of water in a lake of elliptic shape. Mathieu's equation arises after separating the wave equation using elliptic coordinates. Secondly, problems involving periodic motion are the trajectory of an electron in a periodic array of atoms [7].

Stability is an important concept in linear and nonlinear analysis. For instance, roughly speaking, a physical system is stable if small changes at sometimes cause only a small change in the behavior of the system in future [8]. Analytically, stability is determined by the interval placed on the total derivative of the system form by the given differential equation. Such interval is restricted only on the equilibrium point of the system. Other intervals of interest which have been studied by others researchers in these recent times can be seen in [9] [10]. For linear system, stability and boundedness are equivalent; thus, the search for convergence of solution and maximum interval of stability becomes an issue. Emphasis on stability and convergence has been discussed by many authors. For instance see [11] [12] [13] [14] [15] and their references therein. For other researchers who have worked on stability of non-linear systems, see [16] [17] [18] [19].

Motivated by the above literature and ongoing research in this direction, the objectives of this paper are to investigate the maximum interval of stability and convergence of solution of forced Mathieu's equation. We further prove that the solution converges in that interval of interest and that if all solutions are bounded, then the corresponding point in the w and ε parameter plane is said to be stable.

2. Preliminaries

Definition 2.1. Frobenius method which was named after Ferdinand Geog Frobenius is a method for finding an infinite series solution for a second-order ordinary differential equation of the form;

$$x^2 y'' + p(x)y' + q(x)y = 0 \quad (3)$$

with $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

In the vicinity of the regular singular point $x = 0$, we can divide (3) by x^2 to obtain a differential equation of the form;

$$y'' + \frac{p(x)}{x^2}y' + \frac{q(x)}{x^2}y = 0 \quad (4)$$

which will not be solvable with regular power series method if either $\frac{p(x)}{x^2}$ or $\frac{q(x)}{x^2}$ are not analytic at $x = 0$. The Frobenius method enables us to make use of powers series solution to solve a different equation, given that $p(x)$ and $q(x)$ are themselves analytic at 0 or being analytic elsewhere. In this case both limit exist at 0 and are finite [20].

Definition 2.2. Stability is a qualitative property of behavior of the solutions of differential equations by which given a reference solution $x^*(t, t_0^*, x_0^*)$ of $\dot{x} = x(t, t_0^*, x_0^*) \in \mathbb{R}^n$, $t \geq t_0^*$, $t_0^* \geq 0$, any other solution $x(t, t_0, x_0)$ starting close to $x^*(t, t_0^*, x_0^*)$ i.e. $t_0^* \sim t_0$ and $x_0^* \sim x_0$ remains close to $x^*(t, t_0^*, x_0^*)$ for later times.

Theorem 2.3. Suppose

- (i) x_0 is a regular singular point and
- (ii) $(x_0 - x)p(x)$ and $(x_0 - x)^2 q(x)$ are analytic at x_0

Then Frobenius method is effective at regular singular point of the form.

$$(x - x_0) \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Theorem 2.4. Assume that $x_0 = 0$ otherwise the point $x = x_0$ is a singular point of

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \tag{5}$$

and that $Q_1(x) = \frac{xQ(x)}{P(x)}$ and $Q_2(x) = \frac{x^2R(x)}{P(x)}$ are analytic at $x = 0$, then they will have Maclaurin series expansion

$$f(x) = f(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \tag{6}$$

with radius of convergence $r_1 > 0$ and $r_2 > 0$ respectively. That is

$Q_1(x) = \frac{xQ(x)}{P(x)} = \sum_{n=0}^{\infty} P_n X^n$ which converges for $|X| < r_2$. Then the point $x_0 = 0$ is called a regular singular point of (5).

Theorem 2.5. Consider a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{7}$$

with radius of convergence R , then term by term differential and integration of the power series is permitted and does not change the radius of convergence that is;

$$\frac{df}{dx} = \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}, |x| < R \tag{8}$$

$$\int f(x) dx = \int dx \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \int dx a_n x^n = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}, |x| < R \tag{9}$$

Theorem 2.6. Let A be an $n * n$ matrix and let the eigenvalue of A be de-

noted by $\lambda(A)$ and consider the linear system of differential equation

$$\dot{x}(t) = Ax(t), \quad x \in R^n \quad (10)$$

(i) If $Re(\lambda(A)) \leq 0$ and all the eigenvalues of A with real part zero are simple, then zero is a stable fixed point of (10)

(ii) If $Re\lambda(A) < 0$, then zero is a globally asymptotically stable solution of (10)

(iii) If there is an eigenvalue of A with positive real part, then zero is unstable

3. Results and Discussion

3.1. Stability Analysis of Mathieu Equation

We consider the equation

$$\frac{d^2x}{dt^2} + px = 0 \quad (11)$$

where p is a positive constant. Equation (11) can be written as

$$\ddot{x} + px = 0 \quad (12)$$

Let $x_1 = x, \dot{x}_1 = x_2, \ddot{x}_1 = \dot{x}_2$

then equation (12) can be written as

$$\dot{x}_2 + px_1 = 0 \quad (13)$$

The equivalent system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -px_1 \end{aligned} \quad (14)$$

(14) can be reduced in matrix form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (15)$$

(15) can further be written as

$$\dot{x} = Ax$$

where, $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$

For the eigenvalue of A we compute

$$|A - I\lambda| = 0 \quad (16)$$

$$|A - I\lambda| \Rightarrow \left| \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \quad (17)$$

$$\Rightarrow \left| \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -p & -\lambda \end{vmatrix} = 0 \quad (18)$$

$$\lambda^2 + p = 0 \quad (19)$$

Evaluating λ in (19) we have

$$\lambda = \pm ip^{\frac{1}{2}} \quad (20)$$

Since the eigenvalue is in complex form with real part equals to zero, then the equilibrium point is unstable.

3.2. Convergence of Mathieu Equation

We consider the Mathieu equation of the form;

$$\frac{d^2x}{dt^2} + px = 0 \quad (21)$$

From (21) we assume a power series solution of the form;

$$x = \sum_{n=0}^{\infty} a_n t^{n+k} \quad (22)$$

Differentiating (21) term by term we have

$$\dot{x} = \sum_{n=0}^{\infty} (n+k) a_n t^{n+k-1} \quad (23)$$

$$\ddot{x} = \sum_{n=0}^{\infty} (n+k)(n+k-1) a_n t^{n+k-2} \quad (24)$$

Substituting (23) and (24) into (21) we have

$$\sum_{n=0}^{\infty} (n+k)(n+k-1) a_n t^{n+k-2} + p \sum_{n=0}^{\infty} a_n t^{n+k} = 0 \quad (25)$$

The indicial equation is obtained by looking at the coefficient of the lowest power x^{k-2} . Since this only occur in the first sum (for $n = 0$), it must vanish. This reduces (25) to

$$k(k-1) a_0 t^{k-2} = 0 \quad (26)$$

There are therefore two possible values of the index $k = 0, k = 1$ and this is quite typical for a second order equation. Changing the dummy index in the first sum by $n \rightarrow n+2$ we have

$$\sum_{n=-2}^{\infty} (n+k+2)(n+k+1) a_{n+2} t^{n+k} + p \sum_{n=0}^{\infty} a_n t^{n+k} = 0 \quad (27)$$

All the powers now look alike and we can now compare coefficient to obtain the recurrence relation given by

$$(n+k+2)(n+k+1) a_{n+2} + p a_n = 0 \quad (28)$$

which reduces to

$$a_{n+2} = \frac{-p}{(n+k+2)(n+k+1)} a_n \quad (29)$$

Given the value of a_0 , we can evaluate a_2, a_4 , etc. The odd a_n are completely independent and as far as getting a solution is concerned, we can put them all to zero. This independent of the odd and even a_n is a consequence of the fact that odd and even solution of the differential equation are possible.

In order to generate these odd/even solution, it is easiest to put $a_1 = 0$ in order not to create extra solution by merely using some of the key solution into that of $k = 0$.

The recurrence relation for $k = 0$ given by

$$a_{n+2} = \frac{-p}{(n+2)(n+1)} a_n \quad (30)$$

has the solution

$$x(t) = a_0 \cos t + a_1 \sin t \quad (31)$$

The recurrence relation for $k = 1$ is

$$a_{n+2} = \frac{-p}{(n+3)(n+2)} a_n \quad (32)$$

with the solution

$$x(t) = a_1 \sin t + a_0 \cos t \quad (33)$$

Recall that

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \quad (34)$$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \quad (35)$$

Combining equations (31) and (33) we have

$$x(t) = a_0 \cos t + a_1 \sin t$$

where a_0 and a_1 are arbitrary parameter

Using Ratio Test on $\cos t$, we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{t^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{t^{2n}} \right| \quad (36)$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left| \frac{t^2}{(2n+2)!} \cdot \frac{(2n)!}{(2n+1)(2n)!} \right| = \lim_{n \rightarrow \infty} \frac{1}{4n^2 - 6n + 2} |t^2| \\ &= \lim_{n \rightarrow \infty} \frac{1}{4n^2 - 6n + 2} |t^2| = 0 \end{aligned} \quad (37)$$

Similarly for $\sin x = \frac{\sum_{n=0}^{\infty} (-1)^n t^{2n+1}}{2n+1}$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{t^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{t^{2n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{t^2}{(2n+3)(2n+2)} \right| = 0 \end{aligned} \quad (38)$$

Thus the interval of convergence is $(-\infty, \infty)$, radius = infinity

Therefore

$$\begin{aligned} x(t) &= a_0 \cos t + a_1 \sin t \\ &= a_0 \cdot 0 + a_1 \cdot 0 \\ &= 0 \end{aligned} \quad (39)$$

Since the limit of the odd and even function is equal to zero we conclude that the solution of Mathieu equation converges. Applying the boundary condition we have that

$$x(0) = x(2\pi) = a_1 \quad (40)$$

3.3. Numerical Simulation of Forced Mathieu's Equation

$$p = 0.01$$

Define a function that determines a vector of derivatives values at any solution point (t, X)

$$D(t, X) := \begin{bmatrix} X_1 \\ -100X_0 - p(X_0)^2 \end{bmatrix}$$

Define an additional argument for solving the ODE

$t_0 := 0$ Initial value of independent variable

$t_1 := 10$ Final value of independent variable

$X_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Vector of initial function values

$N := 1500$ Numbers of solution values on $[t_0, t_1]$

$$S := (X_0, t_0, t_1, N, D)$$

$t := S^{(0)}$ Independent variables values

$X_1 := S^{(1)}$ First solution function values

$X_2 := S^{(2)}$ Second solution function values

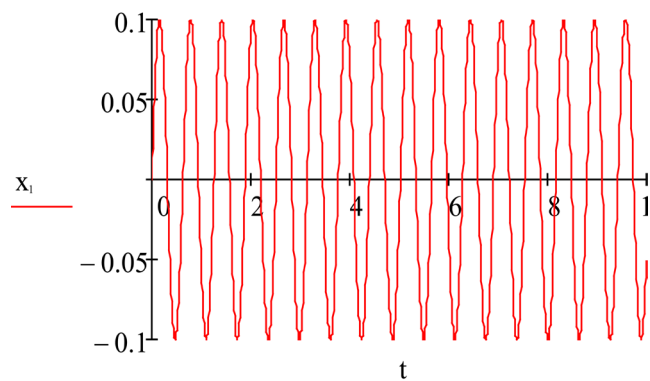
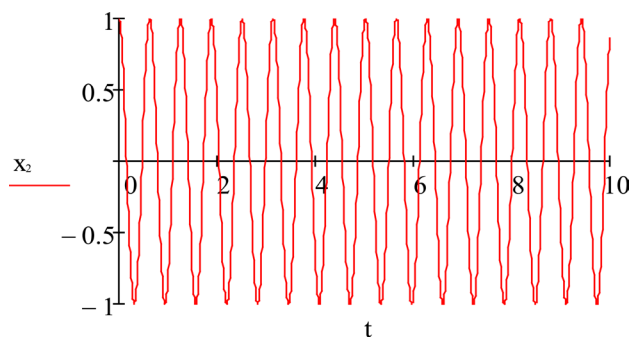
4. Conclusions

From our result, we observed that the solution existed using the Frobenius method and also periodic. The solution converged at the equilibrium point but unfortunately this convergence did not imply asymptotic stability, and the converse is true. The solution was observed to be unbounded for the given parameters $w-\varepsilon$. Since the solution is unbounded, we concluded that the corresponding equilibrium point in $w-\varepsilon$ plane is unstable. This can be seen in **Figure 3** where the phase portrait is far away from the equilibrium point. Due to the periodic nature of Mathieu's equation, its application can be seen in propagation of waves in radios and television. MATHCAD software was also used to demonstrate the numerical behavior of the solution of Mathieu's equation. The numerical behavior is explained as follows:

In **Figure 1**, periodic solution of Mathieu equation was shown by the relationship between the first solution function values and the independent variable values as seen in **Table 1**. The variable values were generated using the vector of initial function values X_0 and the constant p . The periodic nature shown is unstable since the trajectory did not start from the equilibrium point. The maximum displacement reached by the trajectory is at ± 0.1 representing a value along the first solution function values.

Table 1. The solution matrix table for solution function values.

| | 0 | 1 | 2 |
|----|-------|-------|-------|
| 0 | 0 | 0 | 1 |
| 1 | 6.667 | 6.662 | 0.998 |
| 2 | 0.013 | 0.013 | 0.991 |
| 3 | 0.02 | 0.02 | 0.98 |
| 4 | 0.027 | 0.026 | 0.965 |
| 5 | 0.033 | 0.033 | 0.945 |
| 6 | 0.04 | 0.039 | 0.921 |
| 7 | 0.047 | 0.045 | 0.893 |
| 8 | 0.053 | 0.051 | 0.861 |
| 9 | 0.06 | 0.056 | 0.825 |
| 10 | 0.067 | 0.062 | 0.786 |
| 11 | 0.073 | 0.067 | 0.743 |
| 12 | 0.08 | 0.072 | 0.697 |
| 13 | 0.087 | 0.076 | 0.647 |
| 14 | 0.093 | 0.08 | 0.595 |
| 15 | 0.1 | 0.084 | ... |

**Figure 1.** Periodic profile of Mathieu's equation obtained by plotting X_1 against t for $N = 1500$.**Figure 2.** Periodic profile of Mathieu's equation obtained by plotting X_2 against t for $N = 1500$.

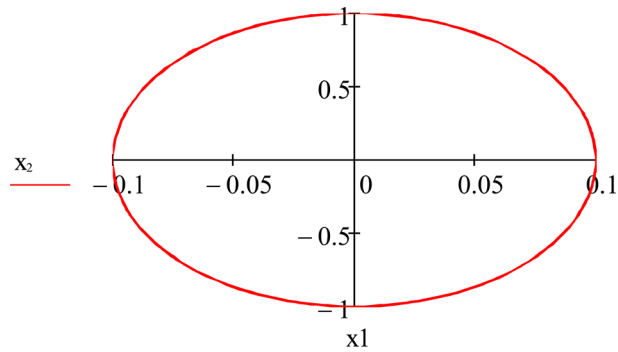


Figure 3. Phase portrait showing instability of solution of Mathieu's equation for $N = 1500$.

In **Figure 2**, the solution was also periodic using the second solution function values and independent variable values. The starting point of the trajectory was far from the origin hence showing instability of the system.

In **Figure 3**, phase portrait of Mathieu's equation was obtained by relating X_2 and X_1 . The phase portrait was seen to be far away from the equilibrium point. This shows that the solution of Mathieu's equation is unstable with the maximum displacement coinciding with the maximum displacement of X_1 in **Figure 1**. These values represent the maximum interval of stability of the system.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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