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The Substitution Process for Conventional Energy. The Logistic Map and Some Specific Fractional Aspects

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Abstract

In this Article, we observe the Logistic--Map Ansatz, which is a popular forecasting Model to estimate the Market Penetration of new technologies in Time evolution. Especially we focus on the Substitution Process of regenerative resources for electro-energy in B.R.D. as a Case Study using real available Data. The Aim of this Article is to develop some specific Models that could represent Logistic Growth implying explicitly the Fractality as the Substitution Dynamics is characterized by high Complexity and fractal Characteristics. According to this Target, we consider a specific Fokker-Planck Ansatz, which could represent the time-fractional Evolution of the Substitution Grade. Further, we implement a relaxation Model, which focuses on the time Evolution of the Expected Value of the Substitution Grade. Additionally, a time-discrete Hybrid model is proposed and a concrete Application of Homotopy Methode delivers interesting Results.

Keywords

Conventional Energy, Regenerative Resources, Logistic Map, Mittag-Leffler Relaxation, Fokker-Planck Ansatz, Homotopy Methode

1. Introduction

Energy is a very important Input for the production Processes and for Services. Its accessibility and the corresponding Prices are Dynamics of high complexity. During our Days the Discussion about the Substitution of Renewable Energy Resources for conventionally produced Energy is more than actual. A lot of Factors have a strong Impact on the Substitution Process and its Progress. We mention at this stage political Decisions and Strategies, Availability, Technology Progress, Price Differences in the Market, and Investment Interest are only a few of the parameters that affect the Dynamics of the Substitution Process. However, some factors are restrictive and cannot be manipulated only through political willingness. For example, when the government is interested in accelerating the Substitution Process, could subsidize the Market of new technologies making Investing in these economically more attractive within a free Market Economic Model. The subsidizing of new technologies in order to get an aspired substitution grade represents political willingness. However, the existing energy supply structure, accessibility of a minimum of conventional energy resources, and Capital, are restrictive factors that influence the energy transformation process. A favored Method (ansatz) to forecast the market penetration of new technologies, new products, or the usage of alternative energy resources like in our case, is that of the Logistic differential Equation in the classical Form. However, this Ansatz is applicable with specific terms, which in the praxis change in an unexpected manner, caused by political decisions, availability of financial resources, and so on. The Process usually results in dynamics with fractal characteristics. In this article, we will analyze the Role of Fractality in the Logistic Equation comparing the classical Model with a fractional version of it. Further, we provide some models that respect explicitly fractal characteristics in predicting dynamics in the sense of logistic Growth. Thus implying fractality in the modeling could be a corrective Faktor within the classical prediction frame of the penetration of new Technologies in the time evolution.

2. Classical Logistic Equation Ansatz

2.1. The Time-Continuous Case (Logistic Function or Verhulst Function)

For a trend that corresponds to its actual Value and to distance to a certain level, the Ansatz $\frac{\mathrm{d}y}{\mathrm{d}t} = ky(a-y)$ known as the "Logistic Equation" seems to be the most appropriate mathematical formula. The above Ansatz means that the shape of the Logistic growth consists of weak progress at the beginning of the process followed by a transition to a progressive phase and tends to a degressive growth (asymptotic Behavior), as the term $(a-y)$ tends to zero. The real Factor "k" accelerates the Growth and $(a-y)$ delays it. In the mathematical sense, the above classical Differential Equation can be solved with the help of Variable Separation by the Relation

$$\ln\left(\frac{y}{a-y}\right) = akt + d \tag{1}$$

Setting $c = ak$, and $b = \mathrm{e}^{-d}$, we get the final Solution

$$y = \frac{a}{1+b\mathrm{e}^{-ct}} \tag{2}$$

(Baumann et al, 1975). We consider a Substitution Process, where y should represent the Substitution grade of a Process in the time evolution.

2.2. Estimation of the Real Value "*d*"

The real value "*d*" can be gained by setting $t = 0$ in the Relation $\ln\left(\frac{y}{a-y}\right) = akt + d$, and assuming an existing certain value $y_0(t=0)$ of the process at the time $t = 0$. The Term "*a*" represents the asymptotic Limit of the process at the time $t \to \infty$.

2.3. Estimating the Real Value "*c*"

Assuming that the Variable y should have a certain value y_C at a certain time t_C, we can with the help of the relation $y_C = a/\left(1 + b * e^{-t_C * C}\right)$ estimate "*c*" as b is already known from $b = e^{-d}$.

The Logistic Function can be represented after algebraic manipulations in the following Form too:

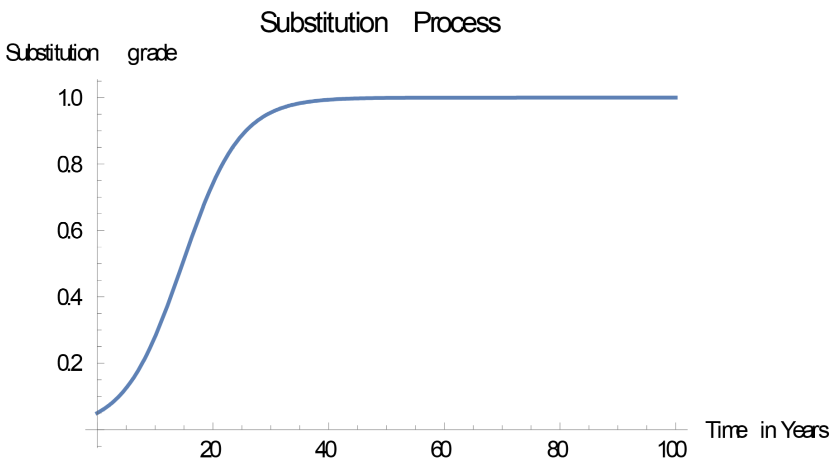
$$f(t) = y = \frac{A}{1 + e^{-kt} * \frac{A - f_0}{f_0}} \tag{1}$$

Following Notations counts in the above Relation (1): A = Current Capacity of the Process or the Asymptotic Limit of the image (fixed Point of the dynamics). This is the maximum value that can be reached by the Logistic Function. f_0 = Value of y at the time $t = 0 = y(t = 0)$. k = Growth Rate of the Process. In the time-continuous Version of the Logistic Function, the Factor k represents a Constant with a certain value.

2.4. An Example

We consider the Substitution process $y(t)$ of renewable energy resources for Electro-Power. We wish (plan) to substitute 50% of renewable energy resources for conventional Elektro Energy for 15 Years. We have extracted with the Help of real Data that an aggregated Investment of 77 Mrd. € should be needed to realize it. This fact allows us to define a constant Growth rate of 15.4 Mrd. € pro 0.1 Growth Rate and Time (In Years). In the dimensional Form holds then k = 0.2. The following Graph 1 expresses the image of the Logistic Function by certain Values of $y(t = 0)$, A, and f_0.

However, the calculation (plan) has a chance of realization if we can ensure the needed investment in regenerative Energy technologies in order to reach the desirable Substitution grade. In a further step, we must assess whether the needed investment can be realized and under which macroeconomic conditions. At this point, we will emphasize that intensive Investment in new technologies decreases financial resources planned for other economic policies and activities. As a short case study, we examined the actual production of Electric Energy in



Graph 1. The Logistic Map $y=\dfrac{A}{1+e^{-kt}*\dfrac{A-f_0}{f_0}}$ with $\dfrac{A-f_0}{f_0}=19$, $k=0.2$.

B.R.D. and its Limits and Options if it should be produced by renewable Resources. For this problem, we use Data from the renowned Institution "Fraunhofer Institut für Solartechnik BRD" (Kost et al., 2021). The extrapolated Data are related to the actual Demand for electric energy in BRD and their future projection. Additionally the arising Costs (€/KWH) from alternative Energies and the corresponding Investment (€ (KW) have been estimated. A favorable Mix of alternative Energy to produce elektro-power consists of Wind Power, Photovoltaic Systems, Batterie Power, Electrolysis, Biogas, Fortress Biomass, Methanation, as well as Electricity from Gas and Combustion Turbines. However, we notice here that the optimal Mix of the above resources could be estimated with the Help of linear programming if certain Cost Data are available. After elaboration on the above Data, we estimated an aggregated Investment Volume of 77 Mrd, which would be needed to reach a substitution Grade of regenerative Energy Sources of 50% of conventional Energy within 15 Years. The above-described classical Logistic Map constitutes only a theoretical Model. The above proceeding is based on an ex-ante Prediction modeling. By an ex-post analysis of given Data (Time Series), we could estimate the Values of A, and k of the Logistic Function by extracting a Logistic Curve, which feels at best within the given Data. Proceeding on this way an important question arises at this point: is the classical Logistic Graph in the time-continuous Version an idealized averaged Curve of fractal distributed Points at which the Fractality grade is not implied explicitly? If the above Assumption is true, then the Logistic Map gained by experimental Data is only an Approximation of a Trend (averaged Values) on a fractal Structure. But how could we proceed to get a better image of Dynamics with Sigmoid Characteristics considering the existing fractality? However, the Fractality of the Process could be caught at best, examining the Behavior of Investment in the Dynamics being considered as the Impuls Factor k generates solely the Fractality in the observed dynamics.

Two possible ways amongst others to handle this problem are described in the following intercepts A and B.

2.5. The Logistic Map and the Fractional Aspect

The Factor "k" (Process Accelerator) in the logistic map, is that Factor, which determines the speed of the process. Its changes in Value are unpredictable emphasizing the generated Fractality in Space and Time. The reasons that cause this behavior are changes in political Decisions in a short time, changes in the prices of conventional Energy Resources, and Availability of resources that are needed to realize the Transformation Process. The above reasons are the result of global Conflicts, Marketing Strategies of global Players (Big Firms and Organizations) in the energy field, and so on. The Process degenerates into dynamics with distinctive fractal characteristics. Fractality has a strong impact on the process. There are numerous models developed, which deliver Solutions to this problem. (See Metzler & Klafter, 2000; Mainardi et al., 2001, and references therein). A basic Tool for analyzing fractal Dynamics in the sense of Probability Density Functions is the continuous time Random walk Model (CTRW) of Montrol and Weiss (Montrol & Weiss, 1965). The discret Aspect of the Logistic Equation points out the relation between Logistic Growth and CTRW.

2.6. The Discrete-Time Version of the Logistic Growth

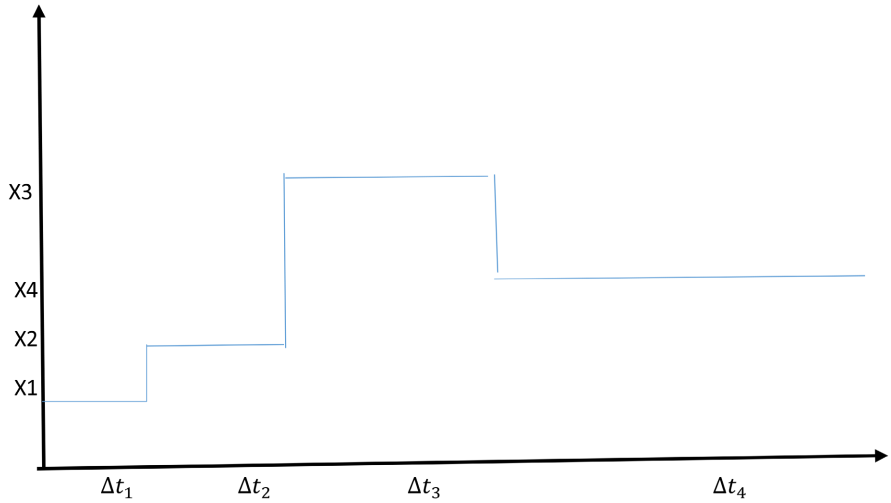
In this case, the Logistic Growth can be represented by the following formula:

$$y[n+1] = ky[n](1-y[n]) \tag{1}$$

n notes the n-th step (time interval) of the Process and Capacity is equal to 1. However, the Impuls Factor (Accelerator) is no longer a Constant Factor. The Value of k changes in Space and Time in an unpredictable manner for every step. The Changes of k are typically in the range between 0 and 1. Assuming that k could reach values higher than 1 it is von Interest to observe the behavior of the Logistic Growth in the extended range. The Curve exhibits under specific conditions chaotic behavior. Indeed observing and analyzing the CTRW Model of Montrol and Weiss can we get Solutions in the form of Probability Density Functions. The following Graph clarifies these relations:

For the Probability $W(x,t)$ of the Walker to be in Position x at the time t are responsible the Probability Density Functions of Δt and Δx. If both PDFs have limited expected value and limited Variance the Diffusion is of Gauss Character. If the Δt are Gauss-distributed and the Δk are characterized by a Power law Distribution we handle with Levy Flights. In the opposite case of Power Law distributed Δt and Gauss distributed Δk we handle with the fractional Brownian Motion (BFM). The last case is the most relevant for the Logistic Dynamics in the Praxis. (See for Details…)

Hybridization to solve the discrete-time Ansatz for Logistic Growth. The



Graph 2. The Shema of the CTRW Model.

discrete-time Logistic Ansatz looks like $y[n+1] = xy[n](1-y[n])$. In order to gain the Logistic Image, we extract the time-fractionality of the Times Series of Investment. Then by every Step of the discrete-time simulation, we obtain an "*x*" value stemming from the Distribution $W(x,t) = \frac{1}{2\pi} * \int Exp[-ikx*] E_\beta[-k^2 t^\beta] \mathrm{d}k$, with E_β = Mittag-Leffler Function of order β, and β = time-fractality of the Investment Time Series Data. The gained "*x*" value can be used as Input in the Ansatz $y[n+1] = xy[n](1-y[n])$.

2.7. Fokker-Planck Ansatz that can be Solved with the Help of a Time-Fractional Discrete Model

From the physical Point of view, the fractional "Fokker-Planck" Ansatz could be a suitable Model to interpret the Logistic Map assuming fractal Characteristics in the Process. (See Abdel, 2006)

The Fokker-Planck Ansatz looks like

$$D_{t*}^{\beta} u(x,t) = \gamma \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial}{\partial x}\left(F(x) u(x,t)\right) \quad (1)$$

The above Ansatz consists of a space-fractional Term in the form $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ and of the Drift Term $\frac{\partial}{\partial x}\left(F(x) u(x,t)\right)$. The time-fractional Derivative operator $D_{t*}^{\beta} u(x,t)$ is of the "Caputo" Form. We use in this case the following Assumption for the Drift Term. The Logistic Curve Ansatz

$y = \frac{a}{1+be^{-ct}}$ can also be observed as a Drift "*b*" of the Potential bx^2 representing the Cost Difference between conventional and alternative Energy Resources, and also the technical Maturity of the new technology in the time Evolution. The progress of renewal energies and in general of new technologies exhibit such Cost characteristics resulting in a slow initial stage Phase followed by progressive growth and tending to an asymptotic behavior over the course of time. Using a discrete Simulation Model developed by R.Gorenflo und Abdel Rehim, we can get a solution by observing the Diffusion-Convection equation

$$D_{t*}^{\beta}u(x,t)=\gamma\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}-\frac{ax}{1+\mathrm{b}e^{-ct}}u(x,t) \tag{2}$$

Assuming the Diffusions-Laplacian $\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}$ in the Form $\frac{\partial^{2}u(x,t)}{\partial x^{2}}$ (Gauss Behavior) we get the Relation $D_{t*}^{\beta}u(x,t)=\gamma\frac{\partial^{2}u(x,t)}{\partial x^{2}}-\frac{ax}{1+be^{-ct}}u(x,t)$ can be observed as time-fractional Ansatz with the fractional Derivative Operator of Caputo Typ. The formula (2) can be manipulated as follows: Defining $\gamma\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}-\frac{ax}{1+be^{-ct}}u(x,t)=L_{FP}u(x,t)$, and substituting on the left sides of (2) Caputo Derivative by the Riemann-Liouvill Derivative Operator D_{t}^{β} we obtain the Identity

$$D_{t}^{\beta}\left(u(x,t)-u(x,0)\right)=L_{FP}u(x,t) \tag{3}$$

Integrating both sides of (3) with the Riemann-Liouville Integral Operator J^{β} we gain the formula

$$u(x,t)-u(x,0)=J^{\beta}\left(L_{FP}u(x,t)\right) \tag{4}$$

Differentiating both sides of (4) with respect to time we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x,t)=\frac{\mathrm{d}}{\mathrm{d}t}J^{\beta}\left(L_{FP}u(x,t)\right) \tag{5}$$

The Relation (5) means the following Statement: The Logistic Growth $\frac{\mathrm{d}}{\mathrm{d}t}u(x,t)=ku(x,t)*\left(1-u(x,t)\right)$ can be approximated by appropriate Transformation of $u(x,t)$ with the Help of the Fokker-Planck Ansatz differentiating the Riemann-Liouville fractional Operator assuming that the Drift Term has Sigmoid Character (See Abdel, 2006; Gorenflo & Abdel-Rehim, 2005).

2.8. The Bird Perspective

To understand and imply the following Model, we start from the idealized classical Logistic Map Ansatz $\frac{a}{1+be^{-ct}}$ with certain Values of the Factors "*a*", "*b*", and "*c*". The classical Ansatz can provide a Solution in the Form of averaged Values, without respecting Fractality Characteristics. Our aim consists now in to compare the Results of the classical Model with the Results of an Ansatz, which implies explicitly the Fractality order. Due to this scope, we observe the fractional Differential Equation that is known in the Literature as the fractional Oscillator. It reads as

$$D_{t*}^{\beta}u(x,t)=-\lambda u(x,t) \tag{1}$$

and has the Solution

$$u(x,t) = E_\beta\left(-\lambda t^\beta\right) \text{ with } E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+1)}, \text{ with } 0<\beta<1,\ z\in\mathbb{C} \quad (2)$$

(See Gorenflo & Mainardi 1991; Kilbas et al., 2006). The Function $E_\beta(z)$ is known in the Literature as the "Mittag-Leffler" Function with time-fractality of order "β". After algebraic Manipulations of the Relation $D_{t*}^{\beta}u(x,t) = \gamma\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right)$ we get the initial Value Problem $D_{t*}^{\beta}\langle x(t)\rangle = -\frac{a}{1+be^{-ct}} * \langle x(t)\rangle$, which after taking the LaplacTransform to both sides leads in our case to the Solution

$$\langle x(t)\rangle = E_\beta\left(-\frac{a}{1+be^{-ct}} * t^\beta\right) \text{ for } 0<\beta<1. \quad (3)$$

(See Abdel, 2006). The Operator "$\langle . \rangle$" denotes here the Expected Value of a stochastic Variable. Plotting the above Relation for Example for $a = 0.95$, $b = 87$ and $c = 0.3$ we obtain the following Graph

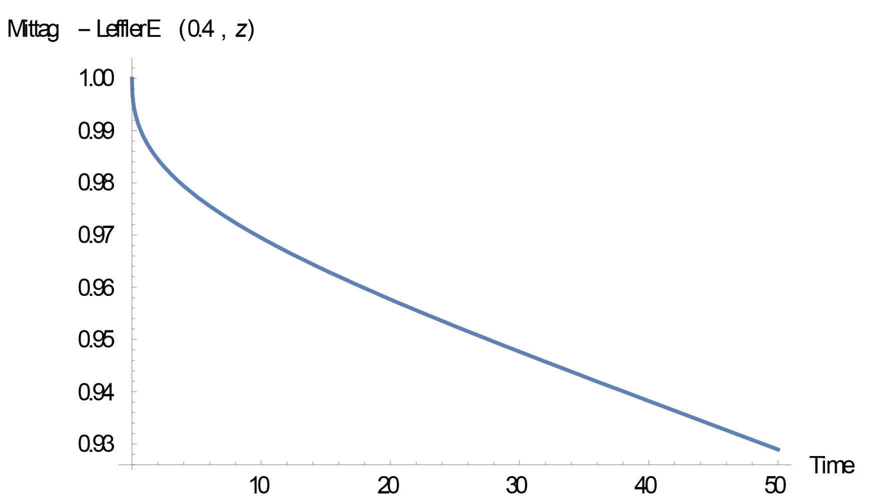
In order to match the classical Logistic Map with the Mittag-Leffler Ansatz we must consider both in the stochastic Sense as the Mittag-Lefler Solution is of stochastic structure. Thus we use the cumulative density Functions. The following Graph represents the expected Value in the Form

$$\langle x(t)\rangle = 1 - E_\beta\left(-\frac{a}{1+be^{-ct}} * t^\beta\right)$$

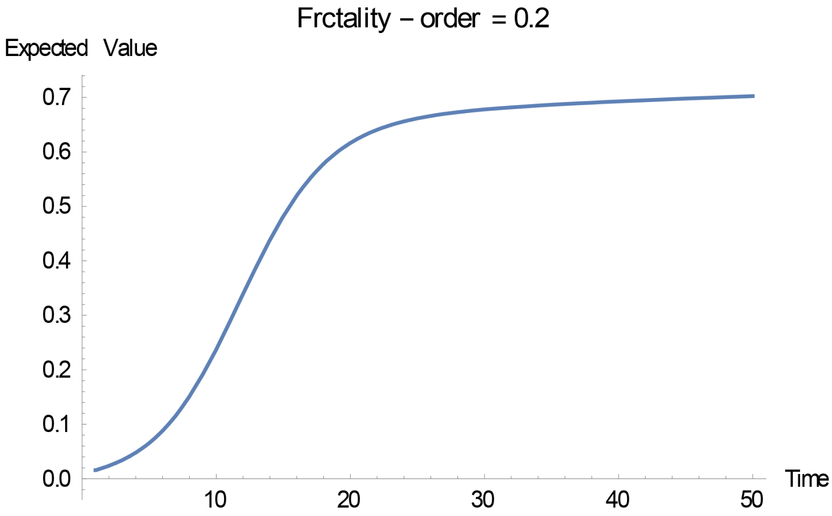
In order to compare in a straight manner the Results between the classical Logistic Map and the appropriate fractional Ansatz we use by the classical Map the Values $a = 1$, $b = 87$, $c = 0.3$, whereat we reduce the fractionality Order in the fractional Ansatz from lower (high fractality) to higher Values (lower fractality).

2.9. Homotopy Pertubation Method

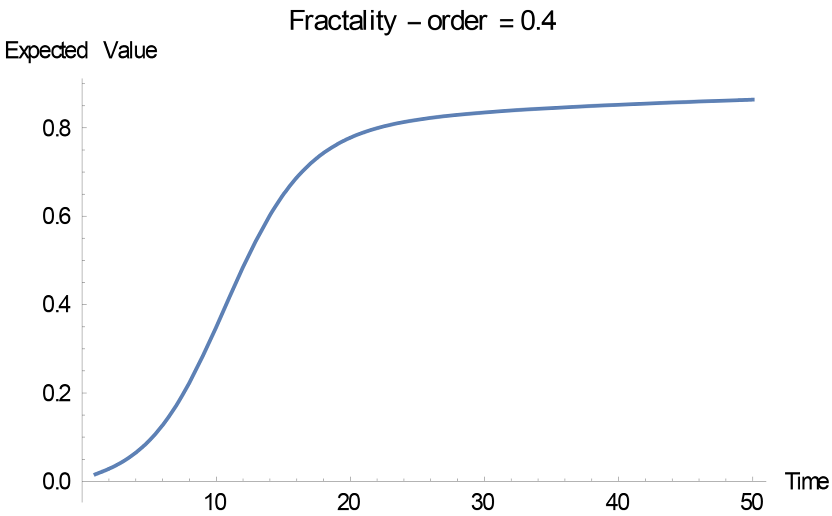
The Homotopy perturbation Method (Algorithm) for nonlinear partial differential



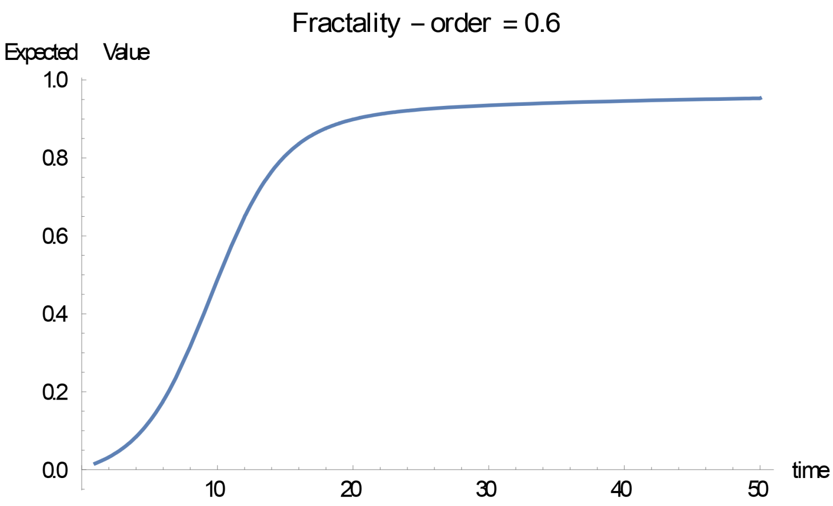
Graph 3. The Relaxation of the classical Logistic Map in Time with $a = 0.95$, $b = 87$, $c = 0.3$.



Graph 4. Expected Value for the Mittag-Leffler Ansatz in the time evolution with $\beta = 0.2$.

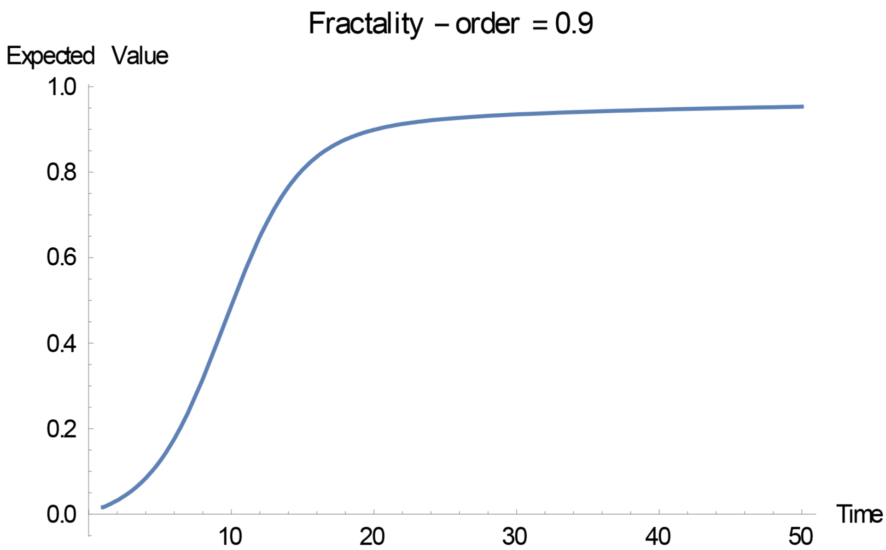


Graph 5. Expected Value for the Mittag-Leffler Ansatz in the time evolution with $\beta = 0.4$.

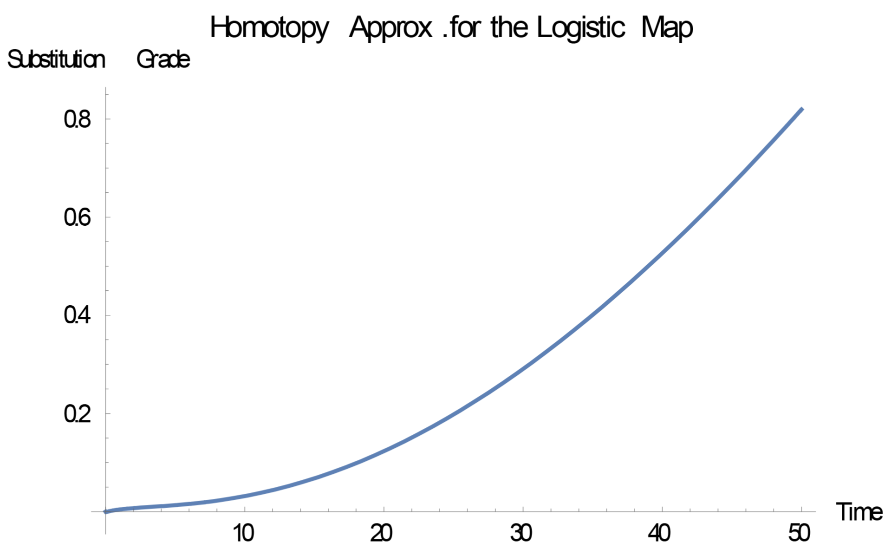


Graph 6. Expected Value for the Mittag-Leffler Ansatz in the time evolution with $\beta = 0.6$.

equations of fractional order is an efficient method to solve differential equations of fractional order represented with the Caputo derivative Operator. The approximate solutions are represented in the form of convergent series. This Ansatz corresponds generally to time-fractional dynamics. (See Momani & Odibat, 2007). The Homotopy, which we provide in this Article is of the form $\frac{\partial u}{\partial t}=P\left[\frac{\partial u}{\partial t}+u_{xx}+ku(1-u)-D_{*t}^{\beta}u\right]$, $\beta=0.4$. We focus on the nonlinear time-fractional Fisher's Equation that corresponds to the nonlinear kernel $ku(1-u)$, which generates the Logistic Map in the classical Sense. After numerous mathematical Manipulations in the fractional sense, we get the final solution in two Term-approximation:



Graph 7. Expected Value for the Mittag-Leffler Ansatz in the time evolution with $\beta=0.6$.



Graph 8. The fractional Logistic Map approximated by the Homotopy Method with time-fractality of order 0.4.

$$
\begin{aligned}
u(t) &= u_1(t) + u_2(t) \\
&= -0.0123t^{0.4} + 0.0123t^{0.6} + 0.003916t - 0.0123t^{0.4} + 0.0123t^{0.6} \\
&\quad + 0.0116t^{0.8} - 0.00707t - 0.00428t^{1.4} - 0.0026357t^{1.4} + 0.0023t^{1.6} \\
&\quad - 0.000025214t^{1.8} + 0.000587t^{2} + 0.00004538t^{2} - 0.00002063t^{2.2} \\
&\quad + 0.000012t^{2.4} - 0.00001111t^{2.6} - 0.0000015335t^{3}
\end{aligned}
$$

The following Graph is a Plot of the above Equation

Remark: The Homotopy Algorithm for the Logistic Map in the time-fractional Sense provides the same Results like in B. We use here a fractionality Order of 0.4 and for the Term k the same Value as in A.

3. Results

When we plan_to predict the penetration of new technologyies (ex-ante Point of view) as we are interested in calculating subsidizing Capital using the Logistic growth Ansatz, often we start this focusing on the Invest, which should be needed to realize a certain desirable result in a certain time. This Ansatz does not imply explicitly the Impact of probably generated Fractality on the Process. A corrective way to get better Solutions consists of applying Models that imply Fractality Dimension within the analytical Frame explicitly. Due to our Research, the Fractality can slow down the expected Results using the classical Logistic Ansatz. Applying fractional Models we can notice that the Results of the classical Calculation seem to be escorted by a time-and Space Lag.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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