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A Note about Characterization of Calendar Spread Arbitrage in eSSVI Surfaces

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Abstract

This paper provides a little correction to a proposition about calendar spread arbitrage in eSSVI volatility surfaces and gives exact conditions under which two eSSVI slices have tangency points without crossing over each other. The original proposition was stated in the paper where Hendriks and Martini (2019) introduced the eSSVI surface model. However the original statement (and the one given in a preprint version which is slightly different) is wrong and from the original proofs (which are slightly different in the preprint and final article) it is not obvious to infer the correct statement. The proof given in this paper is based on the main ideas of the original proof, but it fills in several details which eventually lead to a sharper result.

Keywords

Volatility Model, eSSVI Surface, Arbitrage Bounds

1. Introduction

Volatility surface models are vital tools for derivative pricing and risk hedging in financial markets. Among these models, the eSSVI proposed by Hendriks and Martini (2019) has attracted quite a bit of attention in the research community, and not only given that a financial data and software company like Factset provides an implementation of the algorithm described in Corbetta, Cohort, Laachir & Martini (2019) which calibrates eSSVI surfaces to market data. (see Akhundzadeh, Huber, Hyatt, & Schneider, 2020)

The eSSVI surface is an offspring of SVI model for the implied variance smile at a fixed maturity. According to Gatheral & Jacquier (2014a), the latter was devised at Merrill Lynch in 1999 and was subsequently publicly disseminated at the Global Derivatives & Risk Management conference in Madrid (Gatheral, 2004). In order to motivate the introduction of the eSSVI model and to explain the contribution of the present paper, it will be convenient to review the main properties of the SVI first. According to the latter, for a given fixed maturity date the implied variance skew $w(k)$ can be represented through the function

$$w(k) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}$$

where $k = \ln K/F$ is the natural logarithm of the ratio between the strike and the forward price of the underlying, and where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$ and $\sigma > 0$ are parameters governing the shape and position of the smile. The following appealing features of the SVI parametrization are well known:

1) the SVI smiles are asymptotically linear in k as $|k| \to \infty$ and therefore consistent with Roger Lee's moment formula given in Lee (2004).

2) and the large maturity limit of the implied variance smile of a Heston model with correlation parameter ρ is SVI with the same value of ρ (Gatheral & Jacquier, 2014b).

However, it is also well known that the SVI parametrization, in its full generality, is not arbitrage free. For example, it is easy to see that $\inf_k w(k) = a + b\sigma\sqrt{1-\rho^2}$ and hence it should always be required that $a + b\sigma\sqrt{1-\rho^2} \geq 0$. However, the latter condition is not enough to rule out butterfly arbitrage as can be seen from a well known counterexample of Axel Vogt (see Gatheral & Jacquier (2014a)). Moreover, fitting the SVI parametrization to more than a single maturity date may produce slices that cross over each other which is equivalent to the existence of calendar spread arbitrage opportunities. To overcome these issues, Gatheral & Jacquier (2014a) introduced the SSVI parametrization which is a global parametrization for the whole implied total variance *surface* where the fixed maturity slices are restricted to a subfamily of the SVI parametrization (the first "S" in front of SSVI stands for "*surface*"). In the SSVI parametrization, the implied total variance surface is given by

$$w_t(k) = \frac{\theta_t}{2}\left\{1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1-\rho^2)}\right\} \tag{1}$$

where θ_t is the ATM implied total variance at maturity t, $|\rho| < 1$ and $\varphi(\theta_t)$ is a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$ such that the limit $\lim_{t\to\infty} \theta_t\varphi(\theta_t)$ exists in $\mathbb{R}$. According to Theorems 4.1 and 4.2 in Gatheral & Jacquier (2014a), the SSVI surface (1) is free of calendar spread arbitrage if and only if

C1) $\partial_t\theta_t \geq 0$ for all $t \geq 0$

C2) and $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1 + \sqrt{1-\rho^2}\right)\varphi(\theta)$ for all $\theta > 0$ (the upper bound is infinite when $\rho = 0$)

and it is free of butterfly arbitrage *if* for all $\theta > 0$ the following two conditions are both satisfied:

B1) $\theta\varphi(\theta)(1+|\rho|) < 4$,

B2') $\theta\varphi(\theta)^2(1+|\rho|) \leq 4$.

Mingone (2022), using results given in Martini & Mingone (2022), strengthens this result by showing that absence of butterfly arbitrage is *equivalent* to

B1 and

B2) $\theta\varphi(\theta)^2 \le f_{MM}(\theta,|\rho|)$, where

$$f_{MM}(\theta,|\rho|) = \inf_{l>l_2(|\rho|)} \frac{4\theta\sqrt{1-\rho^2}\,h^2(l;|\rho|)}{\theta\sqrt{1-\rho^2}\,g^2(l;|\rho|) - g_2(l;|\rho|)}$$

with

$$l_2(|\rho|) = \tan\left(\frac{1}{3}\arccos(-|\rho|)\right)$$

$$g(l,\rho) = \frac{\partial_l N(l,\rho)}{4},$$

$$h(l,\rho) = 1 - \left(l - \frac{\rho}{\sqrt{1-\rho^2}}\right)\frac{\partial_l N(l,\rho)}{2N(l,\rho)},$$

$$g_2(l,\rho) = \partial_l^2 N(l,\rho) - \frac{\left[\partial_l N(l,\rho)\right]^2}{2N(l,\rho)},$$

$$N(l,\rho) = \sqrt{1-\rho^2} + \rho l + \sqrt{l^2+1}.$$

Now we are finally ready to introduce the eSSVI. In order to make the SSVI surface more flexible, Hendriks and Martini (2019) made the ρ-parameter maturity dependent as well and called the resulting implied total variance surface model *eSSVI surface* (the "e" in front of SSVI stands for "*extended*"). Proposition 3.1 in Hendriks & Martini (2019) provides necessary and sufficient conditions for the absence of calendar spread arbitrage between two time slices. In order to state these conditions, we indicate the parameters of two slices with θ_i, $\varphi_i = \varphi(\theta_i)$ and ρ_i, where the subscript i takes on the value 1 or 2 according to whether the closer ($i=1$) or farther ($i=2$) maturity date is referred to. Proposition 3.1 in Hendriks & Martini (2019) says that two eSSVI slices do not cross over each other only if

N') $\frac{\theta_2}{\theta_1} \ge 1$ and $\left(\frac{\theta_2\varphi_2}{\theta_1\varphi_1}\rho_2 - \rho_1\right)^2 \le \left(\frac{\theta_2\varphi_2}{\theta_1\varphi_1} - 1\right)^2$

and that condition N along with condition S below is sufficient to rule out the existence of crossing points:

S) $\frac{\varphi_2}{\varphi_1} \le 1$ or $\left(\frac{\theta_2\varphi_2}{\theta_1\varphi_1}\rho_2 - \rho_1\right)^2 \le \left(\frac{\theta_2}{\theta_1} - 1\right)\left(\frac{\theta_2\varphi_2^2}{\theta_1\varphi_1^2} - 1\right)$

However, the statement about sufficiency is wrong and this is where the contribution of the present paper comes in. In fact, as can be seen from Proposition 13 in Section 2 below,

- when $\frac{\theta_2}{\theta_1} = 1$ there are no crossing points if and only if either (i) $\rho_1 = \rho_2 = 0$ and $\varphi_2/\varphi_1 \ge 1$ or (ii) $\varphi_2/\varphi_1 = \rho_1/\rho_2$ and $\rho_1^2 \ge \rho_2^2$.
- and when $\frac{\theta_2}{\theta_1} \ne 1$ there are no crossing points if and only if condition S

holds jointly with condition

N) $\frac{\theta_2}{\theta_1} > 1$ and $1 - \frac{\theta_2\varphi_2}{\theta_1\varphi_1} \le \frac{\theta_2\varphi_2}{\theta_1\varphi_1}\rho_2 - \rho_1 \le \frac{\theta_2\varphi_2}{\theta_1\varphi_1} - 1$.

Almost all of the proof of Proposition 13 in Section 2 is built on the main ideas from the proof of Proposition 3.1 in Hendriks & Martini (2019). As far as I know the only novelty are the result about tangency points in Lemma 10 and the two final Lemmas 11 and 12.

2. The Correct and Sharper Statement of the Hendriks-Martini Proposition and Its Proof

Consider two eSSVI slices which we shall denote by

$$w_i(k) = \frac{\theta_i}{2}\left\{1 + \rho_i\varphi_i k + \sqrt{\varphi_i^2 k^2 + 2\varphi_i\rho_i k + 1}\right\}, \quad i = 1, 2.$$

As in the previous section, assume that the subscript $i = 1$ refers to the closer maturity date. Then there is absence of calendar spread arbitrage if and only if $w_1(k) \le w_2(k)$ for all $k \in \mathbb{R}$.

Note that

$$w_i'(k) = \frac{1}{2}\theta_i\varphi_i\left(\frac{k\varphi_i + \rho_i}{\sqrt{\varphi_i^2 k^2 + 2\varphi_i\rho_i k + 1}} + \rho_i\right)$$

$$w_i''(k) = \frac{\theta_i\left(1-\rho_i^2\right)\varphi_i^2}{2\left(\varphi_i^2 k^2 + 2\varphi_i\rho_i k + 1\right)^{3/2}}$$

so that $w_i''(k) > 0$ for all $k \in \mathbb{R}$. Since $w_i'(k) = 0$ if and only if $k = k_i^* := -\frac{2\rho_i}{\varphi_i}$, we conclude that

$$\inf_k w_i(k) = w_i(k_i^*) = \theta_i\left(1-\rho_i^2\right).$$

By combining this result with the fact that $w_i(0) = \theta_i$, we see that absence of calendar spread arbitrage implies

$$\Theta := \frac{\theta_2}{\theta_1} \ge \max\left\{1, \frac{1-\rho_1^2}{1-\rho_2^2}\right\}. \tag{2}$$

Another necessary condition may be obtained by considering the asymptotes of the two slices. Since

$$2w_i(k) \sim \begin{cases} \theta_i\varphi_i(1+\rho_i)k & \text{if } k \to \infty, \\ \theta_i\varphi_i(1-\rho_i)k & \text{if } k \to -\infty, \end{cases}$$

We conclude that absence of calendar spread arbitrage also implies

$$\Theta\Phi := \frac{\theta_2\varphi_2}{\theta_1\varphi_1} \ge \max\left\{\frac{1+\rho_1}{1+\rho_2}, \frac{1-\rho_1}{1-\rho_2}\right\} \tag{3}$$

The latter condition is satisfied if and only if

$$\Theta\Phi \geq 1 \quad \text{and} \quad \left(\Theta\Phi\rho_2 - \rho_1\right)^2 \leq \left(\Theta\Phi - 1\right)^2. \tag{4}$$

Of course, in the argument leading to the necessary condition (2) we are tacitly assuming that φ_1, φ_2 and θ_1 are all strictly positive and in this case it follows from (2) that $\theta_2 \geq \theta_1$, i.e. that $\Theta \geq 1$. Note that if $\varphi_1 = 0$ and/or $\theta_1 = 0$, then $w_1(k) = \theta_1$ for all $k \in \mathbb{R}$, and in this case we have absence of calendar spread arbitrage if and only if $\theta_2 \geq \theta_1$ or $\theta_2\left(1-\rho_2^2\right) \geq \theta_1$ according to whether φ_2 is also zero or not. On the other hand, if $\varphi_2 = 0$, then we have $w_2(k) = \theta_2$ for all $k \in \mathbb{R}$, and in this case it follows from the asymptotic behavior of $w_1(k)$ that we have absence of calendar spread arbitrage if and only if $\varphi_1 = 0$ and $\theta_1 \leq \theta_2$. In what follows we rule out these trivial cases by assuming that $\Phi := \varphi_2/\varphi_1$ and $\Theta := \theta_2/\theta_1$ are well defined (i.e. that their denominators are strictly positive) and that $\Phi > 0$ and $\Theta \geq 1$.

Lemma 1. If θ_1, φ_1 and φ_2 are all strictly positive, then there is absence of calendar spread arbitrage only if conditions (2) and (3) are both satisfied.

Now it raises the question whether the conditions (2) and (3) are sufficient as well. To answer this question we look for conditions under which the graphs of $w_1(k)$ and $w_2(k)$ have at least one point in common. We will proceed as in Hendriks & Martini (2019), but we will expand on some details. So let $x := \varphi_1 k$,

$$\alpha := \alpha(x) = \Theta - 1 + \left(\Theta\Phi\rho_2 - \rho_1\right)x, \quad z_1 := z_1(x) = \sqrt{x^2 + 2\rho_1 x + 1},$$

$$z_2 := z_2(x) = \sqrt{\Phi^2 x^2 + 2\rho_2 \Phi x + 1}$$

and note that the two eSSVI slices do have points in common if and only if the equation

$$\alpha(x) + \Theta z_2(x) = z_1(x)$$

has real solutions. Squaring twice yields the quartic polynomial

$$P(x) := 4\alpha^2\Theta^2 z_2^2 - \left(z_1^2 - \alpha^2 - \Theta^2 z_2^2\right)^2$$

where we have omitted the independent variable x on the RHS. Note that every root of $P(x)$ must satisfy one (and only one) of the following conditions:

$$\begin{aligned}
&\text{a)} \quad 2\alpha\Theta z_2 = -\left(z_1^2 - \alpha^2 - \Theta^2 z_2^2\right) \quad \text{and} \quad \alpha - \Theta z_2 = \pm z_1, \\
&\text{b)} \quad 2\alpha\Theta z_2 = z_1^2 - \alpha^2 - \Theta^2 z_2^2 \quad \text{and} \quad \alpha + \Theta z_2 = -z_1, \\
&\text{c)} \quad 2\alpha\Theta z_2 = z_1^2 - \alpha^2 - \Theta^2 z_2^2 \quad \text{and} \quad \alpha + \Theta z_2 = z_1.
\end{aligned} \tag{5}$$

Of course, a root of $P(x)$ is a point where the two slices intersect if and only if it satisfies condition c). To explore the existence of such roots we first observe that $P(x) = x^2 Q(x)$, where

$$\begin{aligned}
Q(x) := &\left[\left(\Theta\Phi\rho_2 - \rho_1\right)^2 - \left(\Theta\Phi - 1\right)^2\right]\left[\left(\Theta\Phi + 1\right)^2 - \left(\Theta\Phi\rho_2 - \rho_1\right)^2\right]x^2 \\
&+ 4\Theta\Big[\rho_1\left(-\Theta^2\Phi^2 + (\Theta - 2)\Theta\rho_2^2\Phi^2 + 2\Theta\Phi^2 - 1\right) \\
&+ \rho_2\Phi\left(\Theta^2\rho_2^2\Phi^2 - \Theta^2\Phi^2 + 2\Theta - 1\right) + \left(1 - 2\Theta\right)\rho_2\rho_1^2\Phi + \rho_1^3\Big]x \\
&+ 4(\Theta - 1)\Theta\left(\Theta\Phi^2\rho_2^2 - \Theta\Phi^2 - \rho_1^2 + 1\right).
\end{aligned} \tag{6}$$

Note that $x=0$ (which is a root of $P(x)$) is an intersection point if and only if $\theta_1=\theta_2$, i.e. if and only if $\Theta=1$ (in fact, $w_1(0)=\theta_1=\theta_2=w_2(0)$ if and only if $\Theta=1$). Assuming that this is the case, we will now find conditions under which the two slices do cross over each other. To this aim we consider their derivatives. With $\theta_1=\theta_2=\theta$ (i.e. with $\Theta=1$) we obtain

$$w_i'(0)=\theta\varphi_i\rho_i \quad \text{and} \quad w_i''(0)=\frac{1}{2}\theta\varphi_i^2\left(1-\rho_i^2\right).$$

To rule out the possibility that the two slices cross over in $x=0$, we must therefore impose

$$w_1'(0)=w_2'(0) \quad \text{and} \quad w_1''(0)\le w_2''(0). \tag{7}$$

If either one of these conditions fails, the two slices cross over in $x=0$. Since we are assuming that the θ_i's and φ_i's are all strictly positive, the conditions (7) can be jointly satisfied only if

- either $\rho_1=\rho_2=0$ and $\varphi_2\ge\varphi_1$, in which case it is easy to verify that $w_2(k)\ge w_1(k)$ for all $k\in\mathbb{R}$;
- or $\Phi=\rho_1/\rho_2$ and $\rho_1^2\ge\rho_2^2$, in which case the constant term and the coefficient of *x* in the polynomial $Q(x)$ do both vanish, and hence the two slices have no intersection points other than $x=0$.

These considerations prove the following lemma:

Lemma 2. Assume that Φ and Θ are well defined and that $\Phi>0$. If $\Theta=1$, there is no calendar spread arbitrage if and only if either (i) $\rho_1=\rho_2=0$ and $\Phi\ge1$ or (ii) $\Phi=\rho_1/\rho_2$ and $\rho_1^2\ge\rho_2^2$.

Note that conditions (2) and (3) do not imply condition (i) or (ii) of the previous lemma (take for example $\Phi=1.2$, $\rho_1=0.9$ and $\rho_2=0.81$) and the former are therefore not strong enough to rule out calendar spread arbitrage even if we restrict to the case where $\Theta=1$.

Consider now what happens when $\Theta>1$. In this case $w_1(0)=\theta_1<\theta_2=w_2(0)$ and $x=0$ is therefore not an intersection point. To investigate the existence of intersection points we analyze the polynomial $Q(x)$. We begin with the following lemma:

Lemma 3. Assume that $\Phi>0$ and $\Theta>1$. Then $Q(x)$ is of second degree if and only if

$$\left(\Theta\Phi\rho_2-\rho_1\right)^2\ne\left(\Theta\Phi-1\right)^2. \tag{8}$$

and in this case its discriminant is given by

$$D:=16\Theta\left(\rho_1^2-\Theta^2\Phi^2\rho_2^2+\Theta^2\Phi^2-1\right)^2\left[\left(\Theta\Phi\rho_2-\rho_1\right)^2-\left(\Theta-1\right)\left(\Theta\Phi^2-1\right)\right] \tag{9}$$

Proof. The coefficient of x^2 in $Q(x)$ vanishes if and only if either

$$\left(\Theta\Phi\rho_2-\rho_1\right)^2=\left(\Theta\Phi-1\right)^2 \quad \text{or} \quad \left(\Theta\Phi\rho_2-\rho_1\right)^2=\left(\Theta\Phi+1\right)^2.$$

The second condition implies $\Theta\Phi<0$, and hence we conclude that $Q(x)$ is of second degree if and only if condition (8) holds. In this case the discriminant of $Q(x)$ can be written as in expression (9).□

From the previous lemma we know that $Q(x)$ must have real roots if condition (8) holds jointly with

$$\rho_1^2 - \Theta^2\Phi^2\rho_2^2 + \Theta^2\Phi^2 - 1 = 0 \quad \text{or} \quad (\Theta-1)(\Theta\Phi^2-1) \le (\Theta\Phi\rho_2 - \rho_1)^2 \qquad (10)$$

Since we are only concerned with the case where the necessary conditions (2) and (4) hold, we must consider the set of (ρ_1, ρ_2)-pairs where the equality in (10) holds and the set (ρ_1, ρ_2)-pairs where

$$(\Theta-1)(\Theta\Phi^2-1) \le (\Theta\Phi\rho_2 - \rho_1)^2 \le (\Theta\Phi-1)^2.$$

We denote these two sets by $H_{\Theta,\Phi}$ and $R_{\Theta,\Phi}$, respectively:

$$H_{\Theta,\Phi} := \left\{(\rho_1,\rho_2) \in (-1,1)^2 : \rho_1^2 - \Theta^2\Phi^2\rho_2^2 + \Theta^2\Phi^2 - 1 = 0\right\},$$

$$R_{\Theta,\Phi} := \left\{(\rho_1,\rho_2) \in (-1,1)^2 : (\Theta-1)(\Theta\Phi^2-1) \le (\Theta\Phi\rho_2 - \rho_1)^2 \le (\Theta\Phi-1)^2\right\}.$$

It will be useful to visualize these two sets. Since we have already dealt with the case $\Theta = 1$, and since we are assuming the necessary condition (4), we need to consider only the case where $\Theta > 1$ and $\Theta\Phi \ge 1$. **Figure 1** shows all possible shapes of $H_{\Theta,\Phi}$ and $R_{\Theta,\Phi}$. The set $H_{\Theta,\Phi}$ is the graph of a hyperbola. It is always symmetric with respect to both axes of the (ρ_1, ρ_2)-plane and its prolongation always goes through the four vertices of the square $[-1,1]^2 \subset \mathbb{R}^2$. Moreover,

- if $\Theta\Phi > 1$, $H_{\Theta,\Phi}$ does not intersect the ρ_1 axis and intersects the ρ_2 axis in $\rho_2 = \pm\sqrt{\dfrac{\Theta^2\Phi^2 - 1}{\Theta^2\Phi^2}}$;
- if $\Theta\Phi = 1$, $H_{\Theta,\Phi}$ reduces to the straight lines $\rho_1 = \pm\rho_2$.

Also for the set $R_{\Theta,\Phi}$ there are essentially only two possible shapes:

- If $\Theta\Phi^2 - 1 \le 0$ (since we are assuming $\Theta > 1$, this implies $\Phi < 1$), the set $R_{\Theta,\Phi}$ is given by the stripe

$$S := \left\{(\rho_1,\rho_2) \in (-1,1)^2 : \Theta\Phi\rho_2 - \Theta\Phi + 1 \le \rho_1 \le \Theta\Phi\rho_2 + \Theta\Phi - 1\right\}.$$

The stripe reduces to the line $\rho_1 = \rho_2$ if $\Theta\Phi = 1$.

- If $\Theta\Phi^2 - 1 > 0$ (since we are assuming $\Theta > 1$, this implies $\Theta\Phi > 1$), the set $R_{\Theta,\Phi}$ is the union of the two parallel and disjoint stripes

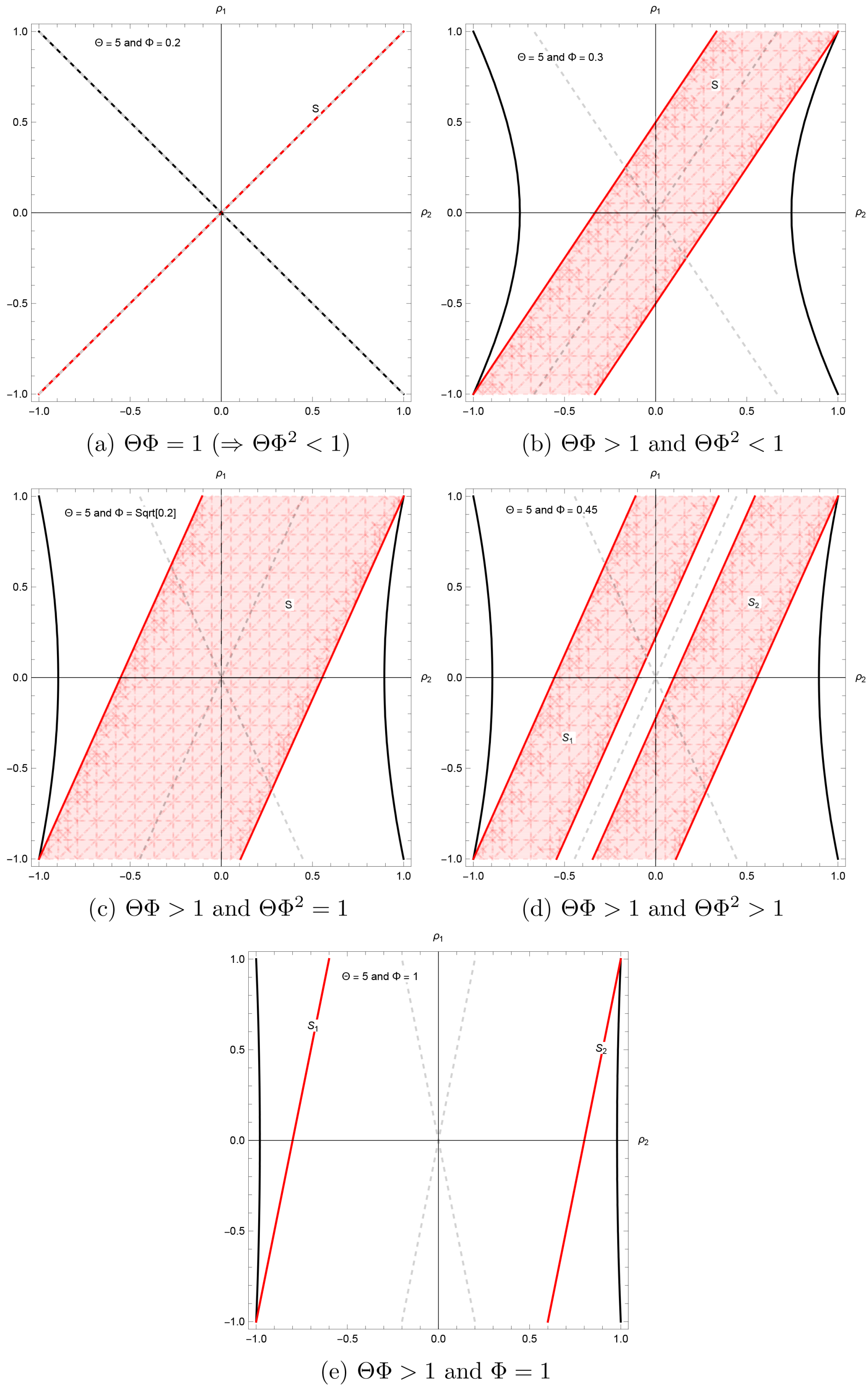
$$S_1 := \left\{(\rho_1,\rho_2) \in (-1,1)^2 : \Theta\Phi\rho_2 + \sqrt{(\Theta-1)(\Theta\Phi^2-1)} \le \rho_1 \le \Theta\Phi\rho_2 + \Theta\Phi - 1\right\}$$

and

$$S_2 := \left\{(\rho_1,\rho_2) \in (-1,1)^2 : \Theta\Phi\rho_2 - \Theta\Phi + 1 \le \rho_1 \le \Theta\Phi\rho_2 - \sqrt{(\Theta-1)(\Theta\Phi^2-1)}\right\}.$$

The two stripes reduce to the lines $\rho_1 = \Theta\rho_2 \pm (\Theta - 1)$ if $\Phi = 1$.

From the description of the sets $H_{\Theta,\Phi}$ and $R_{\Theta,\Phi}$ we see immediately that $H_{\Theta,\Phi} \cap R_{\Theta,\Phi} = \left\{(\rho_1,\rho_2) \in (-1,1)^2 : \rho_1 = \rho_2\right\}$ when $\Theta\Phi = 1$. Next we prove that



(a) $\Theta\Phi = 1$ ($\Rightarrow \Theta\Phi^2 < 1$) (b) $\Theta\Phi > 1$ and $\Theta\Phi^2 < 1$

(c) $\Theta\Phi > 1$ and $\Theta\Phi^2 = 1$ (d) $\Theta\Phi > 1$ and $\Theta\Phi^2 > 1$

(e) $\Theta\Phi > 1$ and $\Phi = 1$

Figure 1. The red areas show all possible shapes of the $R_{\Theta,\Phi}$ when $\Theta > 1$ and $\Theta\Phi \geq 1$. The black line is the graph of the hyperbola $H_{\Theta,\Phi}$. The two dashed lines are the asymptotes $\rho_1 = \pm\Theta\Phi\rho_2$ of $H_{\Theta,\Phi}$.

except for this special case the intersection is empty.

Lemma 4. If $\Theta\Phi \neq 1$, then it follows that $S \cap H_{\Theta,\Phi} = \varnothing$.

Proof. If $(\rho_1, \rho_2) \in H_{\Theta,\Phi}$ we must have

$$\Theta\Phi\left(1-\rho_2^2\right) = 1-\rho_1^2 \quad \Rightarrow \quad \Theta\Phi = \sqrt{\frac{1-\rho_1^2}{1-\rho_2^2}}.$$

From the latter equality we obtain

$$\left(\Theta\Phi\rho_2 - \rho_1\right)^2 - \left(\Theta\Phi - 1\right)^2 = 2\sqrt{\frac{1-\rho_1^2}{1-\rho_2^2}}\left(1-\rho_1\rho_2\right) - 2\left(1-\rho_1^2\right).$$

The quantity on the RHS is positive because

$$\frac{1-\rho_1^2}{1-\rho_2^2}\left(1-\rho_1\rho_2\right)^2 > \left(1-\rho_1^2\right)^2 \quad \Leftrightarrow \quad \left(\rho_1 - \rho_2\right)^2 > 0$$

and because we are assuming that $\Theta\Phi = \sqrt{\frac{1-\rho_1^2}{1-\rho_2^2}} \neq 1$. Hence we conclude that $(\rho_1, \rho_2) \notin S$, because otherwise we should have

$$\left(\Theta\Phi\rho_2 - \rho_1\right)^2 - \left(\Theta\Phi - 1\right)^2 \leq 0,$$

□

From now on we consider roots of $Q(x)$ as functions of ρ_1 and ρ_2. Any root of $Q(x)$ will be denoted by $x_{(\rho_1,\rho_2)}$. Of course the subset of the (ρ_1, ρ_2)-plane where such a root exists depends on Θ and Φ. We denote this set with $Q_{\Theta,\Phi}$. Since we are only interested in ρ_i values within the interval $(-1,1)$, we consider $Q_{\Theta,\Phi}$ as a subset of the open square $(-1,1)^2$. Note that $x_{(\rho_1,\rho_2)}$ must be a continuous function of $(\rho_1, \rho_2) \in Q_{\Theta,\Phi}$ and that $(\rho_1, \rho_2) \in Q_{\Theta,\Phi}$ only if condition (10) holds.

As we have already seen, a root $x_{(\rho_1,\rho_2)}$ must be of one of the three types in (5) and only roots of type c) are intersection points. In order to determine which type applies, we will need the following functions:

$$(\rho_1, \rho_2) \mapsto \alpha\left(x_{(\rho_1,\rho_2)}\right) := \Theta - 1 + \left(\Theta\Phi\rho_2 - \rho_1\right)x_{(\rho_1,\rho_2)}$$

and

$$\begin{aligned}(\rho_1, \rho_2) \mapsto Z\left(x_{(\rho_1,\rho_2)}\right) &:= z_1^2\left(x_{(\rho_1,\rho_2)}\right) - \Theta^2 z_2^2\left(x_{(\rho_1,\rho_2)}\right) - \alpha^2\left(x_{(\rho_1,\rho_2)}\right) \\ &= x_{(\rho_1,\rho_2)}^2 + 2\rho_1 x_{(\rho_1,\rho_2)} + 1 - \Theta^2\left(\Phi^2 x_{(\rho_1,\rho_2)}^2 + 2\Phi\rho_2 x_{(\rho_1,\rho_2)} + 1\right) \\ &\quad - \left[\Theta - 1 + \left(\Theta\Phi\rho_2 - \rho_1\right)x_{(\rho_1,\rho_2)}\right]^2.\end{aligned}$$

Note that these functions must all be continuous functions of $(\rho_1, \rho_2) \in Q_{\Theta,\Phi}$.

Lemma 5. Assume that $\Phi > 0$ and $\Theta > 1$. Then $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right) = 0$ only if $(\rho_1, \rho_2) \in H_{\Theta,\Phi}$.

Proof. The proof is essentially the same as the proof of Lemma A.2 in Hendriks & Martini (2019). Since $x_{(\rho_1,\rho_2)}$ must satisfy one of the two equalities

$$2\alpha\left(x_{(\rho_1,\rho_2)}\right)\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)=\pm Z\left(x_{(\rho_1,\rho_2)}\right)$$

(otherwise $x_{(\rho_1,\rho_2)}$ is not a root of $P(x)$ and hence neither a root of $Q(x)$) and since $z_2(x)>0$ for all $x\in\mathbb{R}$, we conclude that $\alpha\left(x_{(\rho_1,\rho_2)}\right)Z\left(x_{(\rho_1,\rho_2)}\right)=0$ if and only if $\alpha\left(x_{(\rho_1,\rho_2)}\right)$ and $Z\left(x_{(\rho_1,\rho_2)}\right)$ do both vanish. In this case we must have

$$\begin{aligned}&\alpha\left(x_{(\rho_1,\rho_2)}\right)=0 \quad\Rightarrow\quad \Theta\left(1+\rho_2\Phi x_{(\rho_1,\rho_2)}\right)=1+\rho_1 x_{(\rho_1,\rho_2)}\\&\Rightarrow\quad \Theta^2\left(1+2\Phi\rho_2 x_{(\rho_1,\rho_2)}+\Phi^2\rho_2^2x_{(\rho_1,\rho_2)}^2\right)=1+2\rho_1 x_{(\rho_1,\rho_2)}+\rho_1^2x_{(\rho_1,\rho_2)}^2\end{aligned}\tag{11}$$

and we must also have

$$\begin{aligned}&\Theta^2 z_2^2\left(x_{(\rho_1,\rho_2)}\right)=z_1^2\left(x_{(\rho_1,\rho_2)}\right)\\&\Rightarrow\quad \Theta^2\left(\Phi^2x_{(\rho_1,\rho_2)}^2+2\Phi\rho_2 x_{(\rho_1,\rho_2)}+1\right)=x_{(\rho_1,\rho_2)}^2+2\rho_1 x_{(\rho_1,\rho_2)}+1,\end{aligned}\tag{12}$$

Subtracting Equation (11) from Equation (12) yields

$$x_{(\rho_1,\rho_2)}^2\left(\Theta^2\Phi^2-\Theta^2\Phi^2\rho_2^2-1+\rho_1^2\right)=0$$

Since $x_{(\rho_1,\rho_2)}$ must be different from zero (otherwise we would have $\alpha\left(x_{(\rho_1,\rho_2)}\right)=\Theta-1>0$ contrary to our assumption), this implies that $\Theta^2\Phi^2-\Theta^2\Phi^2\rho_2^2-1+\rho_1^2=0$ which is equivalent to $(\rho_1,\rho_2)\in H_{\Theta,\Phi}$.☐

Corollary 1. Assume that $\Phi>0$, $\Theta>1$ and that A is a connected subset of $Q_{\Theta,\Phi}$ which does not intersect the set

$$H_{\Theta,\Phi}:=\left\{(\rho_1,\rho_2)\in(-1,1)^2:\Theta^2\Phi^2\left(1-\rho_2^2\right)=\left(1-\rho_1^2\right)\right\}.$$

Then it follows that function $(\rho_1,\rho_2)\mapsto\alpha\left(x_{(\rho_1,\rho_2)}\right)Z\left(x_{(\rho_1,\rho_2)}\right)$ does not change sign on A.

The previous corollary will be useful to distinguish whether a given root $x_{(\rho_1,\rho_2)}$ is of type a) rather than of type b) or c). Once we know that it is not of type a), we will apply the next lemma in order to find out whether it is of type b) or c).

Lemma 6. Let A be a connected subset of $Q_{\Theta,\Phi}$ such that $\alpha\left(x_{(\rho_1,\rho_2)}\right)Z\left(x_{(\rho_1,\rho_2)}\right)>0$ for all $(\rho_1,\rho_2)\in A$. Then it follows that the function $(\rho_1,\rho_2)\mapsto\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)$ does not change sign on A.

Proof. Under the assumptions of the lemma $x_{(\rho_1,\rho_2)}$ must be a root of either type b) or c) in (5). Hence we must have either

$$\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)=z_1\left(x_{(\rho_1,\rho_2)}\right)$$

or

$$\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)=-z_1\left(x_{(\rho_1,\rho_2)}\right).$$

The conclusion of the lemma follows now from the fact that $\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)$ is continuous and that $z_1(x)>0$ for all $x\in\mathbb{R}$.☐

Now we are finally ready to investigate about the existence of intersection points. We start from the special cases where $\Phi = 1$ or $\Theta\Phi = 1$.

Lemma 7. Assume that $\Theta > 1$ and that either $\Phi = 1$ or $\Theta\Phi = 1$. Then $(\rho_1, \rho_2) \in R_{\Theta,\Phi}$ implies $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$.

Proof. If $\Theta > 1$, we must have $w_1(0) < w_2(0)$ and thus there exist intersection points only if the polynomial $Q(x)$ has real roots different from $x = 0$. Now, consider first the case $\Phi = 1$. Since we are assuming that $(\rho_1, \rho_2) \in R_{\Theta,\Phi}$, it follows that $\rho_1 = \Theta\rho_2 \pm (\Theta - 1)$ (see the description of the set $R_{\Theta,\Phi}$ for the special case where $\Phi = 1$). However, it can be verified that in this case we must have

$$Q(x) = -4\Theta^2 (\Theta - 1)^2 (\rho_2 \pm 1)^2 < 0$$

which has no roots at all.

Next, consider the case $\Theta\Phi = 1$. In this case we must have $\rho_1 = \rho_2$ (see the description of the set $R_{\Theta,\Phi}$ for the special case where $\Theta\Phi = 1$). Substituting $\rho_1 = \rho_2 = \rho$ and $\Phi = 1/\Theta$ in the coefficients of $Q(x)$ shows that

$$Q(x) = 4(\Theta - 1)^2 (1 - \rho^2) > 0$$

which has no roots at all.□

Next, we deal with the case where Φ is strictly smaller than 1 and different from $1/\Theta$ (i.e. $\Theta\Phi \neq 1$). The inequality $\Theta\Phi \geq 1$, which is necessary by condition (4), allows then only for values of Φ in the range $1/\Theta < \Phi < 1$. Note that for $\Phi \leq 1$ the necessary condition (2) is already implied by condition (4) and therefore we do not need to assume condition (2) explicitly.

Lemma 8. Assume that $\Theta > 1$, $\Theta\Phi > 1$ and $\Theta\Phi^2 \leq 1$ (this implies $\Phi < 1$). Then $(\rho_1, \rho_2) \in R_{\Theta,\Phi}$ implies $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$.

Proof. Once again, if $\Theta > 1$ we must have $w_1(0) < w_2(0)$ and there exist points $k \in \mathbb{R}$ where $w_1(k) \geq w_2(k)$ if and only if intersection points exist, i.e. if and only if the polynomial $Q(x)$ has at least one real root which satisfies condition c) in (5). From Lemma 3 we know that $Q(x)$ must have roots if $\Theta > 1$, $\Theta\Phi > 1$, $\Theta\Phi^2 \leq 1$ and if (ρ_1, ρ_2) belongs to the interior of $R_{\Theta,\Phi}$. Since under the present conditions $R_{\Theta,\Phi}$ is connected and does not intersect $H_{\Theta,\Phi}$ (see Lemma 4), we may apply Corollary 1 to check whether the roots in $R_{\Theta,\Phi}$ are of type a). This will be the case if there exists a single $(\rho_1, \rho_2) \in int(R_{\Theta,\Phi})$ such that $\alpha(x_{(\rho_1,\rho_2)}) Z(x_{(\rho_1,\rho_2)}) < 0$. Under the present conditions the origin belongs to $int(R_{\Theta,\Phi})$. Hence we use $(\rho_1, \rho_2) = (0,0)$ as test point. Of course, $\alpha(x_{(0,0)}) = \Theta - 1 > 0$. Moreover, it is easy to check that

$$x_{(0,0)} = 2\frac{\sqrt{\Theta(\Theta - 1)(1 - \Theta\Phi^2)}}{\Theta^2\Phi^2 - 1}$$

so that

$$Z(x_{(0,0)}) = -\frac{2(\Theta - 1)\Theta\left[\Theta^2\Phi^2 - 1 + 2(1 - \Theta\Phi^2)\right]}{\Theta^2\Phi^2 - 1} < 0.$$

We conclude that $x_{(\rho_1,\rho_2)}$ must be of type a) whenever $(\rho_1, \rho_2) \in R_{\Theta,\Phi}$.

In the previous lemma we have assumed that $\Theta\Phi^2 \le 1$ which forces $\Phi < 1$. To apply the same method of proof for the case where $\Theta\Phi^2 > 1$ we must however *assume* that $\Phi < 1$.

Lemma 9. Assume $\Theta > 1$, $\Theta\Phi^2 > 1$ and $\Phi < 1$ (note that $\Theta > 1$ and $\Theta\Phi^2 > 1$ implies $\Theta\Phi > 1$). Then $(\rho_1, \rho_2) \in R_{\Theta,\Phi}$ implies $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$.

Proof. The proof is similar to the proof of the previous lemma. However, in the present case we must deal with the fact that the set $R_{\Theta,\Phi}$ is not connected but only the union of the two connected sets S_1 and S_2. In each one of these two sets we must therefore find a point (ρ_1, ρ_2) such that $\alpha\left(x_{(\rho_1,\rho_2)}\right)$ and $Z\left(x_{(\rho_1,\rho_2)}\right)$ are of opposite sign. To locate these points, note that the ρ_2-axis intersects both sets and hence we choose the (ρ_1, ρ_2)-points with $\rho_1 = 0$ and

$$\rho_2 = \rho_2^{\pm} := \pm \frac{\sqrt{(\Theta - 1)(\Theta\Phi^2 - 1)}}{\Theta\Phi}.$$

This choice is convenient because it makes the discriminant of $Q(x)$ vanish. According to the sign in $\rho_2^{\pm}$, it gives rise to the roots

$$x_{(0,\rho_2^{\pm})} = \pm \frac{2\sqrt{(\Theta - 1)(\Theta\Phi^2 - 1)}}{\Theta(1 - \Phi^2)} \tag{13}$$

which, regardless of the sign, yields

$$\alpha\left(x_{(0,\rho_2^{\pm})}\right) = \frac{(\Theta - 1)(\Theta\Phi^2 + \Theta - 2)}{\Theta(1 - \Phi^2)} \tag{14}$$

and

$$Z\left(x_{(0,\rho_2^{\pm})}\right) = -\frac{2(\Theta - 1)(\Theta\Phi^2 + \Theta - 2)^2}{\Theta(\Phi^2 - 1)^2}. \tag{15}$$

Note that $Z\left(x_{(0,\rho_2^{\pm})}\right) < 0$ regardless of the value of Φ (provided that $\Phi \ne 1$), but to make sure that $\alpha\left(x_{(0,\rho_2^{\pm})}\right) > 0$ we need to assume $\Phi < 1$.□

Lemma 7, Lemma 8 and Lemma 9 show that the necessary condition (4) along with $\Theta > 1$ and $\Phi \le 1$ are jointly sufficient to rule out calendar spread arbitrage. The next lemma deals with the condition

$$(\Theta - 1)(\Theta\Phi^2 - 1) \ge (\Theta\Phi\rho_2 - \rho_1)^2 \tag{16}$$

which allows for values of Φ larger than 1.

Lemma 10. Assume that $\Theta > 1$ and that condition (16) holds (note that these conditions jointly imply the necessary condition (4)). Then it follows that $w_1(k) \le w_2(k)$ for all $k \in \mathbb{R}$. Under the assumptions of this lemma there exist tangency points (i.e. values of k where $w_1(k) = w_2(k)$) if and only if $\Phi > 1$ and condition (16) holds with equality sign. In that case there must exist exactly one tangency point.

Proof. If $\Theta > 1$ and condition (16) holds, we must have $\Theta\Phi^2 \ge 1$ and hence $\Theta\Phi > 1$ (otherwise there would not exist any (ρ_1, ρ_2)-pair for which (16) holds). Consider first what happens when $\Theta\Phi^2 = 1$. In this case we must have $\Phi < 1$ and for $\Phi \le 1$ we have already proved that $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$.

Consider now what happens when $\Phi > 1$. Since we are assuming that $\Theta > 1$ (and hence we must have $\Theta\Phi^2 > 1$), we must have $R_{\Theta,\Phi} = S_1 \cup S_2$ and the assumed inequality (16) is satisfied if and only if the (ρ_1, ρ_2)-pair belongs to the area between the two stripes S_1 and S_2 or to one of the inner boundaries of S_1 or S_2. We indicate this set of (ρ_1, ρ_2)-pairs with S_3. Note that S_3 must be a proper subset of S since we are assuming that $\Phi > 1$. Since $\Phi > 1$ implies $\Theta\Phi > 1$, we can apply Lemma 4 and conclude that $S_3 \cap H_{\Theta,\Phi} \subset S \cap H_{\Theta,\Phi} = \varnothing$. Now it follows from Lemma 3 that $Q(x)$ has no real roots when (ρ_1, ρ_2) belongs to the interior of S_3, i.e. if the inequality (16) is strict. Since the two slices can intersect only if $Q(x)$ has real roots, we conclude that $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$ in this case. On the other hand, if the (ρ_1, ρ_2)-pair belongs to the boundary of S_3, then it must also belong to the inner boundary of one of the two stripes S_1 or S_2. In other words, there must be equality in (16) which means that

$$\rho_2 = \rho_2^{\pm}(\rho_1) := \frac{1}{\Theta\Phi}\left(\rho_1 \pm \sqrt{(\Theta - 1)(\Theta\Phi^2 - 1)}\right)$$

and that the discriminant of $Q(x)$ must be zero (see Lemma 3). Hence $Q(x)$ must have a root and this root must be unique. As usual we write $x_{(\rho_1,\rho_2)}$ to indicate the root. Since we are assuming that Θ and Φ are both larger than 1, we can apply Lemma 5 and conclude that $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right) \ne 0$ when the (ρ_1, ρ_2)-pair belongs to $R_{\Theta,\Phi} = S_1 \cup S_2$ and hence that $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right) \ne 0$ for all (ρ_1, ρ_2)-pairs which belong to the boundary of S_3 where a root $x_{(\rho_1,\rho_2)}$ must exist and must be unique. Since S_1 and S_2 are two disjoint and connected sets, $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right)$ does not change sign on each of these two sets. We will now show that the sign of $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right)$ is positive on both sets. This can be done by proving that the sign of $\alpha\left(x_{(\rho_1,\rho_2)}\right) Z\left(x_{(\rho_1,\rho_2)}\right)$ is positive at a single point in each of the two sets (see Corollary 1). As in the proof of Lemma (9) we use the (ρ_1, ρ_2)-pairs with $\rho_1 = 0$ and $\rho_2 = \rho_2^{\pm} := \pm\frac{1}{\Theta\Phi}\sqrt{(\Theta - 1)(\Theta\Phi^2 - 1)}$ as test points (note that these points also belong to the boundary of S_3). With this choice we still get the expressions in (13), (14) and (15) for $x_{(0,\rho_2^{\pm})}$, $\alpha\left(x_{(0,\rho_2^{\pm})}\right)$ and $Z\left(x_{(0,\rho_2^{\pm})}\right)$. However, since we are now assuming that $\Phi > 1$, we see that $\alpha\left(x_{(0,\rho_2^{\pm})}\right) < 0$ and not

$\alpha\left(x_{\left(0,\rho_2^{\pm}\right)}\right)>0$ as in the proof of Lemma 9 (of course, $Z\left(x_{\left(0,\rho_2^{\pm}\right)}\right)$ remains still negative). We conclude that the roots we are considering now must be either of type b) or c) in (5). Hence we must have $\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)=\pm z_1\left(x_{(\rho_1,\rho_2)}\right)$. In order to prove that the roots correspond to intersection points, we first note that the mapping $(\rho_1,\rho_2)\mapsto\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)$ does not change sign on each of the two sets S_1 and S_2 (use Lemma 6). However, as far as we know by now, the sign of $\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)$ might be different according to whether (ρ_1,ρ_2) belongs to S_1 or to S_2. Thus, if there exists a single point $(\rho_1,\rho_2)\in S_i$ such that the sign of $\alpha\left(x_{(\rho_1,\rho_2)}\right)+\Theta z_2\left(x_{(\rho_1,\rho_2)}\right)$ is positive, we can conclude that $x_{(\rho_1,\rho_2)}$ is of type c) and hence that $w_1\left(x_{(\rho_1,\rho_2)}/\varphi_1\right)=w_2\left(x_{(\rho_1,\rho_2)}/\varphi_1\right)$ for every $(\rho_1,\rho_2)\in S_i$ ($i=1,2$). Again, we use the two (ρ_1,ρ_2)-pairs with $\rho_1=0$ and $\rho_2=\rho_2^{\pm}:=\pm\frac{1}{\Theta\Phi}\sqrt{(\Theta-1)\left(\Theta\Phi^2-1\right)}$ as test points. For these points we get the same expressions of $x_{\left(0,\rho_2^{\pm}\right)}$ and $\alpha\left(x_{\left(0,\rho_2^{\pm}\right)}\right)$ as in equations (13) and (14), while for $z_2\left(x_{\left(0,\rho_2^{\pm}\right)}\right)$ we get the expression

$$z_2\left(x_{\left(0,\rho_2^{\pm}\right)}\right)=\frac{\Theta\Phi^2+\Theta-2}{\Theta\left(\Phi^2-1\right)}.$$

Hence we conclude that

$$\alpha\left(x_{\left(0,\rho_2^{\pm}\right)}\right)+\Theta z_2\left(x_{\left(0,\rho_2^{\pm}\right)}\right)=\frac{\Theta\Phi^2+\Theta-2}{\Theta\left(\Phi^2-1\right)}>0.$$

This argument shows that every root $x_{(\rho_1,\rho_2)}$ with $(\rho_1,\rho_2)\in S_1\cup S_2$ is an intersection point and that there must be a unique intersection point when $(\rho_1,\rho_2)\in(S_1\cup S_2)\cap S_3$. It is not difficult to show that in the latter case the intersection point must be a tangency point. In fact, if it was a crossing point, there should exist one further crossing point because under our present assumptions the left and right asymptotes of $w_2(k)$ are both steeper than those of $w_1(k)$ (recall that we are assuming $\Theta>1$ and $\Phi>1$: on $(S_1\cup S_2)\cap S_3$ condition (4) must therefore hold with strict inequality sign).

As far as I know, the results in the next two lemmas are new.

Lemma 11. If

$$\Theta>1,\quad \Phi>1 \quad \text{and} \quad (\Theta-1)\left(\Theta\Phi^2-1\right)<\left(\Theta\Phi\rho_2-\rho_1\right)^2<\left(\Theta\Phi-1\right)^2, \tag{17}$$

there must exist exactly two points where the slices $w_1(k)$ and $w_2(k)$ cross over each other.

Proof. From Lemma 3 and Lemma 4 we know that under condition (17) there must exist two roots $x_{(\rho_1,\rho_2)}$. Moreover, from the proof of Lemma 10 we know that both these roots must be intersection points. The chained inequality in (17) says that both asymptotes of $w_2(k)$ are steeper than those of $w_1(k)$. Therefore only two cases can occur: either (i) both intersection points are tangency points, or (ii) both intersection points are crossing points. It is not difficult to see that case (i) is impossibile. In fact, if both intersection points were tangency points, any increase of Θ should lead to $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$. However, given fixed values of Φ, ρ_1 and ρ_2, a small enough increase of Θ does not lead to a violation of condition (17) which implies the existence of two intersection points.□

Now, it remains to see what happens when

$$\Theta > 1, \quad \Phi > 1 \quad \text{and} \quad (\Theta\Phi\rho_2 - \rho_1)^2 = (\Theta\Phi - 1)^2 \tag{18}$$

i.e. when the left or right asymptote of $w_2(k)$ is the same as the corresponding asymptote of $w_1(k)$.

Lemma 12. Assume condition (18) holds. Then there must exist exactly one point where the slices $w_1(k)$ and $w_2(k)$ cross over each other.

Proof. Define $B_{\Theta,\Phi}$ as the subset of the (ρ_1,ρ_2)-plane where the equality in condition (18) holds. $B_{\Theta,\Phi}$ is then the boundary of *S*, i.e. the subset of the (ρ_1,ρ_2)-plane where

$$\rho_1 = \rho_1^+(\rho_2) := \Theta\Phi\rho_2 + \Theta\Phi - 1 \quad \text{and} \quad -1 < \rho_2 < \min\left\{\frac{2}{\Theta\Phi} - 1, 1\right\}$$

or

$$\rho_1 = \rho_1^-(\rho_2) := \Theta\Phi\rho_2 + 1 - \Theta\Phi \quad \text{and} \quad \max\left\{1 - \frac{2}{\Theta\Phi}, -1\right\} < \rho_2 < 1.$$

On $B_{\Theta,\Phi}$ the polynomial $Q(x)$ reduces to

$$Q^\pm(x) = -4\Theta^2(\rho_2 \pm 1)\Phi \times \left[(\Theta-1)^2\Phi\rho_2 \pm (\Theta-1)(\Theta\Phi+\Phi-2) - 2x(\Phi-1)(\Theta\Phi-1)\right]$$

and the only root of $Q^\pm(x)$ is given by

$$x_{(\rho_1^\pm(\rho_2),\rho_2)} = \frac{(\Theta-1)^2\Phi\rho_2 \pm (\Theta-1)(\Theta\Phi+\Phi-2)}{2(\Phi-1)(\Theta\Phi-1)}.$$

We will show that this root must be a crossing point. To this aim note that $B_{\Theta,\Phi}$ is the union of two disjoint connected sets which we denote with $B_{\Theta,\Phi}^\pm$. From Lemma 4 it follows that $B_{\Theta,\Phi} \cap H_{\Theta,\Phi} = \varnothing$. Hence we may apply Corollary 1 and conclude that $\alpha\left(x_{(\rho_1^\pm,\rho_2)}\right)Z\left(x_{(\rho_1^\pm,\rho_2)}\right)$ does not change sign on each one of the two connected components of $B_{\Theta,\Phi} = B_{\Theta,\Phi}^+ \cup B_{\Theta,\Phi}^-$. In order to show that $\alpha\left(x_{(\rho_1^\pm,\rho_2)}\right)$ and $Z\left(x_{(\rho_1^\pm,\rho_2)}\right)$ are of the same sign, it is therefore sufficient to find a single point (ρ_1, ρ_2) in each of the two sets $B^+_{\Theta,\Phi}$ and $B^-_{\Theta,\Phi}$ for which $\alpha\left(x_{(\rho_1^\pm, \rho_2)}\right)$ and $Z\left(x_{(\rho_1^\pm, \rho_2)}\right)$ are of the same sign. As test points we choose the points $(\rho_1, \rho_2) = (\rho_1^\pm(\rho_2), \rho_2)$ where $\rho_1^\pm(\rho_2) = 0$. It is easily seen that these points are $(0, \pm\rho_*)$ where $\rho_* = \frac{1}{\Theta\Phi} - 1$. Substituting in the formula for the root we get

$$x_{(0,\pm\rho_*)} = \pm\frac{2\Theta^2\Phi - \Theta^2 - 2\Theta\Phi + 1}{2\Theta(\Phi-1)(\Theta\Phi-1)}.$$

Regardless of the sign in $\rho_2 = \rho_*$, this root yields

$$\alpha\left(x_{(0,\pm\rho_*)}\right) = -\frac{(\Theta-1)^2}{2\Theta(\Phi-1)}$$

and

$$Z\left(x_{(0,\pm\rho_*)}\right) = -\frac{(\Theta-1)^2\left(\Theta^2\Phi + 2\Theta\Phi^2 - 4\Theta\Phi - \Phi + 2\right)}{2\Theta(\Phi-1)^2(\Theta\Phi-1)}.$$

Of course $\alpha\left(x_{(0,\pm\rho_*)}\right) < 0$. As for $Z\left(x_{(0,\pm\rho_*)}\right)$, its sign depends on the sign of

$$\Theta^2\Phi + 2\Theta\Phi^2 - 4\Theta\Phi - \Phi + 2$$

which is positive whenever $\Theta > 1$ and $\Phi > 1$ (we omit the details of the proof of this assertion). Hence also $Z\left(x_{(0,\pm\rho_*)}\right) < 0$ and thus we conclude that $x_{(0,\pm\rho_*)}$ is a root of either type b) or c) in (5). To find out which type applies, we must determine the sign of $\alpha\left(x_{(0,\pm\rho_*)}\right) + \Theta z_2\left(x_{(0,\pm\rho_*)}\right)$. It is not difficult to verify that

$$z_2\left(x_{(0,\pm\rho_*)}\right) = \frac{\Theta^2\Phi + 2\Theta\Phi^2 - 4\Theta\Phi - \Phi + 2}{2\Theta(\Phi-1)(\Theta\Phi-1)}$$

and hence we get

$$\alpha\left(x_{(0,\pm\rho_*)}\right) + \Theta z_2\left(x_{(0,\pm\rho_*)}\right) = \frac{2\Theta^2\Phi^2 - 2\Theta^2\Phi + \Theta^2 - 2\Theta\Phi + 1}{2\Theta(\Phi-1)(\Theta\Phi-1)}.$$

The numerator in this expression can be written as

$$\Theta^2(\Phi-1)^2 + (\Theta\Phi-1)^2$$

and therefore we must have $\alpha\left(x_{(0,\pm\rho_*)}\right) + \Theta z_2\left(x_{(0,\pm\rho_*)}\right) > 0$. By Lemma 6 we conclude that $x_{(\rho_1,\rho_2)}$ must be an intersection point for every $(\rho_1, \rho_2) \in B_{\Theta,\Phi}$.

To complete the proof it remains to show that for every $(\rho_1, \rho_2) \in B_{\Theta,\Phi}$ the corresponding root $x_{(\rho_1,\rho_2)}$ is a crossing point. To this aim we apply the argument in the proof of Lemma 11 once again: if $x_{(\rho_1,\rho_2)}$ was a tangency point, then by increasing Θ a little bit we should have no intersection points at all. However, if we increase Θ a little bit while leaving Φ, ρ_1 and ρ_2 unchanged, we pass from condition (18) to condition (17) which implies the existence of two crossing points.

Combining the statements in Lemma 1, Lemma 2, Lemma 7, Lemma 8, Lemma 9, Lemma 10, Lemma 11 and Lemma 12 yields a corrected and sharper version of Proposition 3.1 in Hendriks & Martini (2019). The little corrections concern

i) the special case where $\Theta = 1$ and

ii) the fact that Proposition 3.1 in Hendriks & Martini (2019) seems to imply that with $\Theta > 1$, $(\Theta\Phi\rho_2 - \rho_1)^2 \le (\Theta\Phi - 1)^2$ and $\Phi < 1$ it would be possible to have absence of calendar spread arbitrage even if $\Theta\Phi < 1$ which goes against the necessary condition (4). However, the preprint version of the article contains a slightly different version of the proposition which is not subject to this problem but where the two necessary conditions are a little too strong due to strong inequality signs instead of weak ones (see Proposition 3.5 in Hendriks & Martini, 2017).

The sharper (and corrected) statement of the Hendriks-Martini proposition is given below. To make it more concise, the necessary condition (3) will be stated as in (19).

Proposition 13. Assume that θ_1 and φ_1 are both strictly positive and let $\Theta := \theta_2/\theta_1$ and $\Phi := \varphi_2/\varphi_1$. Then, there is absence of calendar spread arbitrage (i.e. $w_1(k) \le w_2(k)$ for all $k \in \mathbb{R}$) only if $\Theta \ge 1$ and

$$1 - \Theta\Phi \le \Theta\Phi\rho_2 - \rho_1 \le \Theta\Phi - 1 \tag{19}$$

Moreover,

- when $\Theta = 1$ there is absence of calendar spread arbitrage if and only if either (i) $\rho_1 = \rho_2 = 0$ and $\Phi \ge 1$ or (ii) $\Phi = \rho_1/\rho_2$ and $\rho_1^2 \ge \rho_2^2$;
- when $\Theta > 1$ there is absence of calendar spread arbitrage if and only if condition (19) holds jointly with

$$\Phi \le 1 \quad \text{or} \quad (\Theta\Phi\rho_2 - \rho_1)^2 \le (\Theta - 1)(\Theta\Phi^2 - 1);$$

- when $\Theta > 1$ and condition (19) holds jointly with

$$\Phi \le 1 \quad \text{or} \quad (\Theta\Phi\rho_2 - \rho_1)^2 < (\Theta - 1)(\Theta\Phi^2 - 1)$$

  there are no intersection points (i.e. $w_1(k) < w_2(k)$ for all $k \in \mathbb{R}$)
- when $\Theta > 1$, $\Phi > 1$ and $(\Theta\Phi\rho_2 - \rho_1)^2 = (\Theta - 1)(\Theta\Phi^2 - 1)$ the two slices have exactly one intersection point which is a tangency point;
- when $\Theta > 1$, $\Phi > 1$ and $(\Theta - 1)(\Theta\Phi^2 - 1) < (\Theta\Phi\rho_2 - \rho_1)^2 < (\Theta\Phi - 1)^2$ there must exist exactly two points where the slices $w_1(k)$ and $w_2(k)$ cross over each other.
- when $\Theta > 1$, $\Phi > 1$ and $(\Theta\Phi\rho_2 - \rho_1)^2 = (\Theta\Phi - 1)^2$ there must exist exactly one point where the slices $w_1(k)$ and $w_2(k)$ cross over each other.

3. Conclusion

This paper provides a detailed proof of conditions which characterize calendar spread arbitrage in eSSVI volatility surfaces. Moreover, it gives a full characterization for the case where two eSSVI slices have tangency points without crossing over each other. The motivation for this paper stems from a little error in the statement of a proposition in Hendriks & Martini (2019) where Hendriks and Martini introduced the eSSVI model, and from the fact the correct statement cannot be easily deduced from their proof. From a practical point of view, the conditions given in this paper can be used to check for the presence of calendar spread arbitrage in calibrated eSSVI surfaces and/or they can be incorporated in a calibration algorithm in order to obtain fitted volatility surfaces which are free of calendar spread arbitrage.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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