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An Overview of Fractal Modeling—Some Specific Applications on the Dynamics of Employment and Personal Income: Entropic Criticism and Steady States

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Abstract

In this article, we investigate the criticality of the employment and personal-income dynamics using Shannon's Entropy Concept. Y. L. Klimontovich published an article in the nineties in which the importance of Entropy as a Criterion of degradation and self-organization in evolution has been emphasized. Thus, we would like in this article to use the term "Criticism" in place of criticality, especially in economic dynamics. From the physical point of view, economic dynamics are convolutions of probabilities in space and time arising commonly in fractal structures characterized by Ergodicity break. This fact enforces analyzing those dynamics in the fractional image and stochastic sense to get quantitative results, which are defensible. The considerable work of R. Gorenflo, F. Mainardi, V.V. Tarasova, V. Tarasov, R. Metzler, and others is an operable basis to bother this scientific area. We focus in this article, especially on concrete economic dynamics, as they point out the possibilities and limits of analyzing economic complexity in a paradigmatic way.

Keywords

Economic Complexity

1. Introduction

The main purpose of this article is to introduce concepts of criticality in economic dynamics based on the theory of statistical physics. As the modeling of economic behavior needs requirements and conditions that arise naturally between protagonists in real economic processes, we choose concepts that are relevant within the theory of many particle systems remaining in the framework of "statistical physics". Appropriate concepts that could detect Criticism in economic dynamics among others (e.g. Percolation theory) are the Criteria of "Entropy" defined by Shannon (1948) in the Article "Theory of Communication" and the "Leibler-Kullback Distance", which represent mathematical tools to extrapolate the mass of uncertainty in futural events. In this context, we use the S-Criterion of Klimontovich (1999) to estimate the relative degree of order from experimental data for unemployment in Germany, after the Second World War. Additionally, we apply fractional concepts bothering economic models, as economic dynamics are obviously of fractal structure. In this context, the development of Entropy in fractal structures is of significant importance. K.H. Hoffmann, J. Prehl, C. Essex, and others of the University of Chemnitz prove in their research papers that there is a significant relation between Entropy amount and fractality order (see Hoffmann et al., 2018; Prehl et al., 2016; Hoffmann et al., 2012). We focus especially on the LogNorm Distribution, as it dominates in the dynamics of personal income. Proceeding in the above-described sense, we handle the following central themes:

A) We analyze the yearly data rows of unemployment in Germany after the Second World War until 2018, using the S-criterion of Y. Klimontovich in order to detect Criticism with the help of the associated Entropy Concept. The dogmatic position of Klimontovich (1999, 2012) states that paths in dynamics, which are more entropic, are the more critical. The key study of heuristic character reveals elements of "Synergetics" and the degree of order in the free market economic system. First, we extrapolate the fractional exponents of the concrete unemployment dynamics in space and time with the help of an empirical relationship between themes.

B) We analyze the entropy development by systems, which create employment outside of the minimum systemic one, with the help of a specific model advanced for this question, using the Leibler-Kullback-Distance Criterion (see Kullback & Leibler, 1951).

C) We develop a specific Temperature-Term in the dynamics of unemployment that can consider social requirements in order to detect Criticism. Of major interest in this context is the question of whether classical concepts for criticality, like the Ising model, can define critical phases in specific dynamics. The research in this context points out the possibility of getting better results using the Renormalization-Group Ansatz, interpreting the last of the fractional point of view, by applying the theory of fractional oscillators.

D) The existence of Steady States in dynamics is an important argument and tool to detect metastability, which is evidence of the fact that specific dynamics constellations remain unchanged in the evolution of time. Of course, the existence of Steady States ensures the no emergence of critical phase transitions. We handle this theme using purely physical models. They are, in the concrete case, the fractional Langevin, the fractional Fokker-Planck-Ansatz and the fractional Master Equation by appropriate conditions. Looking at factors, which have a strong impact on employment dynamics, we extrapolate Steady States within the general model of Taxis, introducing here the concept of Econotaxis, as this behavior met in the biological evolution is of significant importance for stability in the free market economic concept. The applying differential equations are fractional. We refer within the extrapolation of Steady States to the considerable work (thesis) of Brockmann (2003) and Rehim (2006), which handle details of the topological diffusion and of the Continuous Time Random Walk implications (specifically time-fractional aspects), respectively.

E) The dynamics of personal income are characterized by a positive property for research. In this, dynamics dominate two characteristic statistical distributions. They are the LogNorm distribution for the major part of employers (95%) and the Power Law one for the corresponding small part (5%). Analyzing the behavior of this distribution by changing some characteristic parameters, we could locate values, which transform the LogNorm Distribution to the Power Law one. This is a critical phase transition, which enables the definition of Criticism in the personal income dynamics. Additively, we extrapolate the associated Entropy value of the LogNorm distribution changing characteristic parameters like expected value and variance. We can not extrapolate in this context an ideal and stable LogNorm distribution for society as it is a problem of political conception and dimension.

2. Theoretical Formalism

A.0: An overview of the solutions of the fractional diffusion equation within the CTRW formalism with the help of the Mittag-Leffler function.

A.0.1: Fractional Brownian Motion (FBM).

The dynamics, within the CTRW formalism, are well described by the appreciable work of Montroll and Weiss (1965), considering both space and time aspects of the statistical distributions. Kenneth Falconer (2003) refers in his Book, especially to the fractional Brownian Motion (FBM). Mainardi et al. (2007), in the Article "The Fundamental Solution of the Space-Time Fractional Diffusion", deliver the general Solutions to fractional Diffusion. The following categorization corresponds to dynamics with specific characteristics:

The distribution of the waiting time is generated by a power-law rule, while the distribution of the space jumps is characterized by a finite variance.

The Green Function corresponding to these dynamics in the Fourier domain reads as

$$\hat{G}(k,t) = E_{2,\beta}\left[-D_s k^2 t^\beta\right] \tag{1}$$

whereby β is the time fractional order. The inverse transform of (2) leads to the solution

$$W(x,t) = \frac{1}{2\pi}\int \mathrm{d}k \exp\left[-ikx\right] E_{2,\beta}\left[-D_s k^2 t^\beta\right] \tag{2}$$

In the Literature FBM is often called "time-fractional Diffusion (see Gorenflo

& Rehim, 2005; Thesis of Rehim, 2006).

A.0.2: Levy Flights.

Levy Flights are in the dynamic context of the reverse case of FBM. Jump probabilities are now generated by the power law rule, while the corresponding waiting time distribution possesses a variance of finite character. Mainardi et al. (2007) deliver in their Article "The Fundamental Solution of the Space-Time Fractional Diffusion Equation", in Fractional Calculus and Applied Analysis, 4(2), arXiv.cond.mat/0702418, the corresponding theory by detail. The solution in form of the Mittag-Leffler function degenerates on the Fourier domain to a relation of exponential character like the following:

$$\mathcal{F}\left(W\left(x,t\right);k\right)=\hat{f}\left(k,t\right)=\mathrm{e}^{-tD|k|^{\alpha}} \tag{1}$$

(see also details for Levy Flights and stable Distributions by Feller, 1991a, 1991b).

A.0.3: Ambivalent processes.

This case occurs in dynamics when both waiting-time distribution and jump distribution are of power law character. The corresponding formalism results in the following relations:

$$W\left(x,t\right)=\frac{1}{2\pi}\int \mathrm{d}k\exp\left[-ikx\right]E_{\alpha,\beta}\left[-D_s k^{\alpha}t^{\beta}\right] \tag{1}$$

with $\hat{G}\left(k,t\right)=E_{\alpha,\beta}\left[-D_s k^{\alpha}t^{\beta}\right]$ whereby α,β are the fractional order of the corresponding space and time dimension. D_s denotes here the diffusion coefficient. Especially in the case of ambivalent processes holds the following categorization:

For $0<\alpha\le 2$ and $0<\beta\le 1$;

If $\alpha<2\beta$ $\rightarrow$ Superdiffusive processes;

If $\alpha>2\beta$ $\rightarrow$ Subdiffusive processes;

If $\alpha=2\beta$ $\rightarrow$ Gauss processes.

Within the ambivalent dynamics probabilities with divergent moments and scale-free fluctuations in space and time occur (see for detail the Article "The Random Walk's Guide to Anomalous Diffusion" by Metzler & Klafter, 2000).

Remarkon Fundamental Work in Fractional Modeling. An Overview of economic dynamics with Memory from the Fractional Point of View delivers Articles by Tarasova and Tarasov (2017). Gorenflo and Mainardi (2015) introduce the essentials of fractional calculus. They contributed significantly to the Theory of Fractional Diffusion and the associated Dynamics through numerous Articles. Metzler and Klafter (2000) deliver the remarkable Article "The Random Walk's Guide", which contains fundamentals and an overview of the Theory of Anomalous Diffusion. The book of Podlubny (1998) refers in detail to the General Theory of Fractional Calculus and Fractional Differential Equations.

A.1: Extrapolating the fractality order from experimental data (time series).

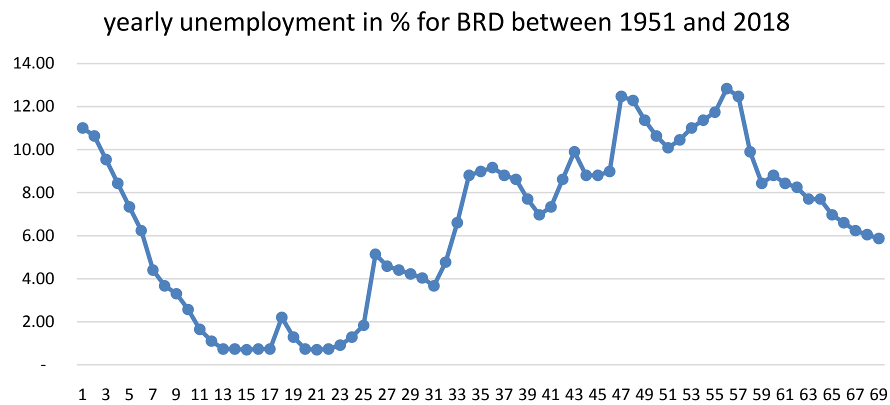
Comment on Graph 1: The development of unemployment in BRD between 1950 and 1960 during the first 10 years is characterized by a strong continuous decrease as a result of the Second World War. In the following years, a trend of many ups and downs occurs with an increasing unemployment average. These dynamics are not typical for industrial countries like USA or Japan at the same time.

A.1.1: Can we use averaged (means) data to extrapolate fractional exponents?

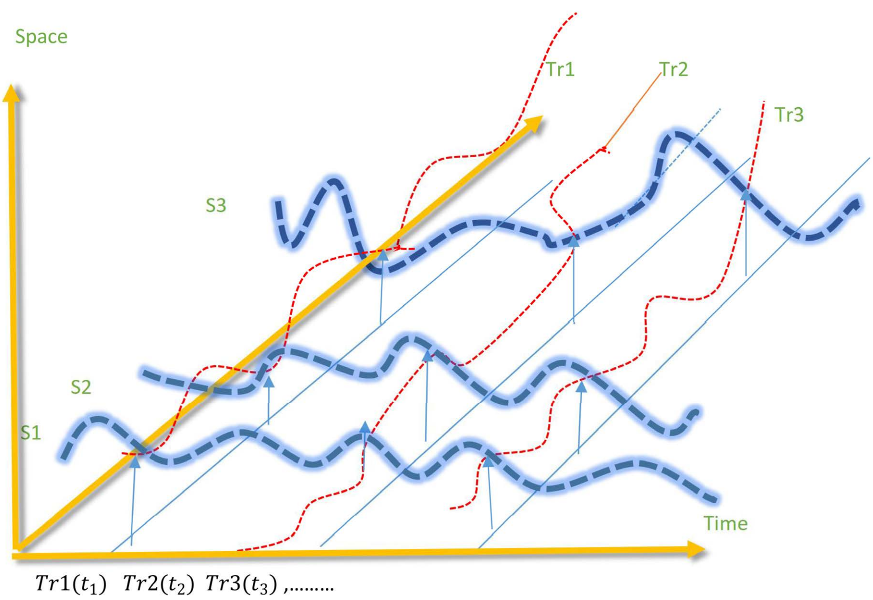
We answer this question with the help of the following Graph 2.

Extrapolating the fractality order with the help of Graph 2.

In order to illustrate the procedure that we use to extrapolate the space- and time-fractality of the employment dynamics from experimental data, we observe Graph 2. The thick dashed lines $S_1, S_2, S_3, \cdots, S_n$, constitute the trajectories of an ensemble of employment institutions (Enterprises and others), representing the employment level in space and time. We are thus interested in the trajectories



Graph 1. Yearly unemployment in % for Germany between 1951 and 2018 (Source: Statistisches Bundesamt).



Graph 2. Trajectories of employment ensembles dependent on space and time.

$t_{fix\,i}$ with $S_i = S_1, S_2, \cdots, S_n$, while we hold the time $t_{fix\,i}$ fixed (thin dashed lines). In this context, the observed process is a process discrete in space and time. We can certainly state that every trajectory $T_r\left(S_i, t_{fix\,i}\right)$ (dashed red line) belongs to the Levy-stable densities, which in limes (S_i, $i \to$ very large number) could tend to a Gauss-distribution. Now, regarding the α-stable laws we can assume that every sum of $T_r\left(S_i, t_{fix\,i}\right)$ constitutes a α-Levy-stable distribution too. Suppose now that in the time $t_{fix\,i} - t_{fix\,i-1}$ the α-Levy-stable distribution remains unchanged. In this case, we are allowed to observe the sum of $\sum S_i$ as a superordinated process, where the α-stable Levy parameter changes its value versus time. The process could be now observed as continuous in space and discrete in time. The statistical average in the form of the characteristic mean value for $\sum S_i$ within a specific but constant time is more or less only an approximation for the real trajectories. We argue that the process should be characterized additively by a real existing time fractality, which the statistical data does not give directly. Zolotarev and Uchaikin (1999) prove in their Book "Chance and Stability that if random variables X_i are stable distributed, then the Sum

$$\sum_{i=1}^{n} X_i\left(\alpha;\beta\right) = \frac{X_1 + X_2 + \cdots + X_n}{n},$$

with α = stable order of the Distributions, is a stable distribution too.

Result: Concerning a time series of experimental data that are consolidation and averages of more detailed available data, we could first, by an appropriate procedure, estimate the space-fractality ignoring the time-fractality. In the second stage, we could then extrapolate the time fractality exponent, identifying, which matches at best to the space one with the help of statistical treatment. The basis we use in this chapter to extrapolate the fractality order of the unemployment dynamics should be the CTRW formalism. According to this, a huge amount of time and space data are required in order to illustrate the associated tick- by tick-dynamics. As such data are usually not available, we must rescale space and time by compressed data series. While the time scales range from milliseconds to minutes, the space scaling seems to be empirically confined in an interval between 1% and 15% (rough assessment). The above assumptions could result in evidence, that unemployment dynamics should be categorized intuitively as fractional Brownian motion. This is not proven yet, despite the confined character of the space-fluctuations. From this point of view, the observed dynamics could be restricted in a potential well, excluding rare situations (crises). Unfortunately, the available experimental data are in form of annual or half-annual averages staking strongly the real-time flow. By the assumption of self-similarity, we could nonetheless gain a significant view of these dynamics. Due to this aim, we use the Method of Kenneth Falconer (2003). The procedure is based on an autocorrelation analysis of the time series. Falkoner calls the time series graph the image of a fractal function. At first, we can estimate the box dimension of the graph using the relation

$$D_{box}(\text{graph } f) = 2 - \lim_{h \to 0} \frac{\log(C(0) - C(h))}{2 \log h} \quad (1)$$

whereby the terms $C(h), C(0)$ are defined within the autocorrelation analysis procedure. By the assumption that the dynamics being in focus exhibits a time-fractional behavior, we get for the power-spectrum $S(\omega)$ the relations

$$S(\omega) \approx c/\omega^{-\alpha}, \quad C(0) - C(h) = bh^{a-1} \quad (2)$$

By the hypothesis $c = b = 1$ and $C(0) - C(h) = ch^{4-2D_{box}}$, we obtain the identities

$$4 - 2D_{box} = \alpha - 1 \text{ or } 4 - 2D_{box} + 1 = \alpha \quad (3)$$

The exponent α obtained above has the dimension of a space fractality order being assumed that the process itself is a space-time fractional one, as opposed to D_{box}, which possesses a pure geometrical property. Now, our goal is to extrapolate the time fractional exponent of a dynamics, the space fractional one being already estimated. To this task, there exists among others a procedure using an empirical relationship between the quadratic displacement of a variable in dynamics and the elapsed time in a fractional sense. The relationship to this reads as

$$\langle X^2 \rangle \equiv Kt^{\beta/\alpha} \quad (4)$$

with:

$\langle X^2 \rangle$ = Quadratic displacement of the variable X;

K = normalization factor;

β = time-fractional order (exponent);

α = space-fractional order (exponent);

t = elapsed physical time.

We obtain from (4) the relation

$$\ln(\langle X^2 \rangle) = \ln(K) + \frac{\beta}{\alpha} * \ln(t) \quad (5)$$

In the case of α= known, we can obtain the β value and inversely. The above way is of pure empirical character and often verified by investigating time series. A more sophisticated possibility consists in supposing the existence of the subordination relationship of the form

$$p(x,t) = t^{-\frac{\beta}{\alpha}} f_x\left(\frac{X}{t^{\beta/\alpha}}\right)$$

with $f_X = f(x)$ = function of the fractional image, $p(x,t)$ being a Probability Density Function (PDF) in space and time. Multiplying the above relation on both sides by $t^{\beta/\alpha}$, we obtain

$$p(x,t)t^{\beta/\alpha} = f_x\left(\frac{X}{t^{\beta/\alpha}}\right) \quad (6)$$

Brockmann (2008) handles this problem in the Article "Anomalous Diffusion and the Structure of Human Transportation Networks". Relation (6) allows extrapolating the order β, while we know the correspondingly "α", using a specific statistical treatment for maximal correlation. After extensive algebraic operations using "Mathematica" and, "Excell" the author extrapolated the following values (Exponents) for the unemployment statistics between 1951-2018.

D_{box} = Box-Dimension of the Graph = 1.93;

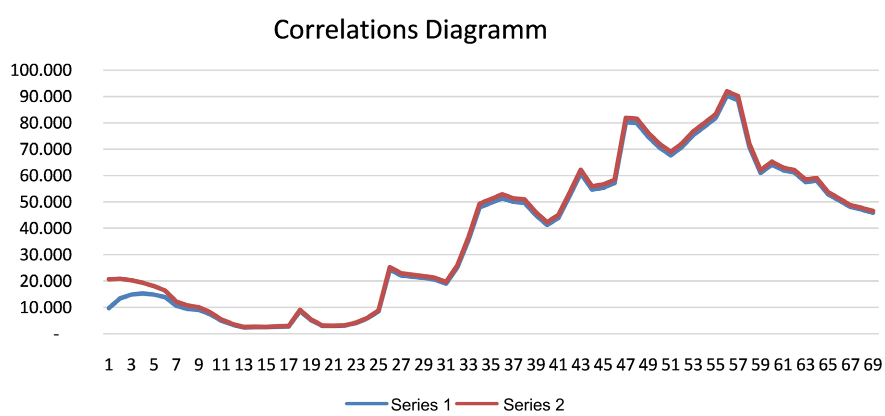
α = space-fractional Exponent = 1.3;

β = time-fractional Exponent = 0.08.

The following Graph 3, points out the maximal correlation (%) between α and β based on the above estimated values.

Comment to Graph 3: The correlation coefficient in this case turns out to be at ≈1. By the above-obtained values for α and β, we can assume that the unemployment process for the time 1951-2018 turns out to be an ambivalent multifractal process. As the employment dynamics of the B.R.D. have specific characteristics between 1951 and 2018, we found it convenient to compare two representative groups investigating the entropy development in relation to the associated fractality. The first group are represents the time between 1951 and 1975. During this time range the dynamics characterized by a low degree of unemployment with a sloping trend almost without ups and downs. This is certainly for BRD a consequence of a new economic impulse after the Second World War. The time range between 1970 and 2000 is the time of energy crises and intensive competitiveness between enterprises, which necessitated economic unities to reorganize their economic activities adapting to the new situation. This imprint is visible in the numerous ups and downs at the employment level. By direct comparison of the two time periods due to the concept of Klimontovich (1999), we extrapolated the second period as the period of physical chaos. That means that period II (1970-2000) should have a higher degree of entropy corresponding to a lower fractality exponent. In this context, we estimated for period II a fractality-exponent of **1.43** and for the first period a fractality-exponent of **1.61** respectively. These results validate the relation between entropy production and fractality order.

A.1.3: Investigating the fractal order of the unemployment dynamics in BRD between 1951-2019. Synergetics and the degree of order in employment.



Graph 3. Correlation between α and β for the values $\alpha = 1.3$ and $\beta = 0.08$.

Now, how could we interpret the increasing and volatile character of the entropy in BRD between 1950 and 2019, in the sense of Klimontovich's theory? At first, it must be said that the investigations in the above time horizon are not of theoretical interest as they do not represent the typical image of entropy development. Nevertheless, we can recognize certainly an important element within this investigation. The free market economic concept in BRD has achieved a degree of maturity within the range 1980-2018 being a result of its self-organized activities (see also Sornette, 2006) to ensure its own financial survival. As the free market economic concept needs some amount of entropy in the form of the associated unemployment degree, we can claim, that the transition from a lower to a higher unemployment degree is, in this case, a passage to a more ordered state. However, this position represents the entrepreneur side (owners and managers of enterprises) or the capital side within the free market concept. The question that arises here naturally is the following. Can unstable entropy dynamics within the free market economic concept reach an equilibrium state and what does this state look like? We answer this question partially in the chapter about steady states. Another interesting issue we will include further in this discussion clarifying the scope of this work. This is because, in the free market model, the capital side is not the single-acting protagonist. Workers and state are trying to smooth extremal employment dynamics by applying appropriate practices and policy strategies. This antagonistic environment does not help always to decrease entropy. Hence, it is important to assess what contributes to increasing or decreasing entropy from an economic point of view. We would fail the main scope of this work, we would not account for this by some certain results around the above question. Our central point in this context reads: How does entropy change financing unemployment by taxation? As the answer to this question is clear, we want to know if an increase in entropy, in this case, corresponds to a more ordered state in the sense of the S-criterion. This theme possesses two sides according to the point of view. While from the capital point of view, entropy growth could lead to a more ordered state, as opposed to the workers' point of view. However, it is of essential interest to ask whether workers and capital positions are always conflictive. This problem would however exceed the frame of this work. Now, at this stage, we want to clarify and emphasize some important critical notes. We extrapolated in this work the result that a low degree of unemployment represents the way corresponding to lower entropy. This can be realized by forcing extensively and governing the investment's potential to areas with intensive human inputs. This result is but of preventative character and represents a general strategy to reduce entropy. Thus social activities, which help to reduce entropy, need essential requirements. Social policies that support facilitating a superior educational and health system, for instance, contribute to an increase in employment and a decrease in social entropy.

B: How does entropy change by realizing an employment degree higher than the minimum systemic one?

To investigate this question it is first necessary to define what means the expression "the minimum systemic" whereby the entropy change refers to an economic performance or economic output. A certainly and exactly evaluated economic output can be realized by a minimum of human resources in dependence on the actual technological level. A higher input of employment (higher degree of employment) could lead to a change in the economic performance that is in focus. These relations can be formulated mathematically as follows:

Assume there is an economic constellation as 2-tuple (X_i, Y_i) with the following variables:

Y_i = Value of economic performance (output value);

X_i = Employment level in the specific performance =>;

=> $f(\lambda Q)$ with λ = general employment level as a macroeconomic index and Q = quantity of employable persons. Further the existence of a tuple (X_m, Y_m) with $X_m > X_i$ and $Y_m > Y_i$ is imaginable. It is realizable the existence of further tuples (X_j, Y_j) for which the following relation holds:

$$X_i < X_m < X_j \text{ and } Y_i < Y_j < Y_m \tag{1}$$

Relation (1) means that there exists a constellation by which a higher degree of employment does not cause a higher degree of economic output. All these combinations could be mathematically localized along a curve having of Log-Norm character. We must here denote that the dependence between employment degree and economic output is not linear and refers to a specific economic macro-model (socially oriented free market economy). An appropriate model to describe the above relations looks like

$$Y(\lambda, T_f) = \alpha(\lambda Q)^{\Psi} T_f^Z \tag{2}$$

with T_f = input of technical resources, $\Psi \in [0,1]$, $\alpha, Z \in [R+]$ and $\lambda \cdot Q$ like above.

An increase in employment about $\delta\lambda$ results in an increase of the economic output δY

The new median μ_N of the LogNorm distribution refers now to the term $\frac{Y + \delta Y}{\lambda + \delta\lambda}$.

(Notice 1: If for the Distributions Y, X with $X \to N(\mu; \sigma^2)$ the linear relation $Y = aX + b$ exists, then $Y \to N(a\mu + b; a^2\sigma^2)$ holds true, with $N(\mu; \sigma^2)$ = Gauss-Distribution.)

Our aim now is to extrapolate ΔY by use of the Legendre transformation starting from the above relation. Due to this goal, we obtain from Relation (2) the following formulas:

From $Y(\lambda, T_f) = \alpha(\lambda Q)^{\Psi} T_f^Z$, we get the formulas $g = Y - \partial_\lambda(Y) * \lambda$,

$$\alpha(\lambda Q)^{\Psi} T_f^Z - \alpha Q^{\Psi} T_f^Z * \Psi * \lambda^{\Psi-1} * \lambda = (\alpha - \alpha\Psi) * \lambda^{\Psi} Q^{\Psi} T_f^Z \tag{3}$$

From $\partial_\lambda(Y) = \alpha\Psi Q^{\Psi} * \lambda^{\Psi-1} T_f^Z$, we obtain the following formula

$$\lambda = \left[\frac{\partial_{\lambda}(Y)}{\alpha \Psi Q^{\Psi} T_f^Z} \right]^{\frac{1}{\Psi - 1}} \tag{4}$$

From $\begin{aligned} dg &= \alpha \Psi Q^{\Psi} \lambda^{\Psi-1} T_f^Z d\lambda + \alpha (\lambda Q)^{\Psi} * Z * T_f^{Z-1} + dZ \\ &- \lambda \alpha \Psi (\Psi - 1) Q^{\Psi} * \lambda^{\Psi-2} T_f^Z d\lambda - \alpha \Psi Q^{\Psi} \lambda^{\Psi-1} * T_f^Z d\lambda \end{aligned}$,

we gain the wanted formula as

$$dg = (\lambda Q)^{\Psi} * Z * T_f^Z dZ - \lambda \alpha \Psi (\Psi - 1) Q^{\Psi} * \lambda^{\Psi-2} T_f^Z d\lambda \tag{5}$$

Assuming constant technical inputs, we can state that the term dg could expect positive as well as negative values in dependence on the variable Ψ.

Using the relative entropy $d(p \| q)$ known as the Kullback-Leibler distance, we will focus on the formula

$$d(p \| q) = \int \left(\frac{1}{x\sigma_1\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) * \ln \left[\frac{\frac{1}{x\sigma_1\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right)}{\frac{1}{x\sigma_2\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_2^2} \right)} \right] \right) \mathrm{d}x \tag{6}$$

in order to extrapolate appropriate relations for dg and $\delta\lambda$. The following identities hold true:

$$\begin{aligned} d(p \| q) &= \int \left(\frac{1}{x\sigma_1\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) * \ln \left[\frac{\frac{1}{x\sigma_1\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right)}{\frac{1}{x\sigma_2\sqrt{2\pi}} * \exp\left(-\frac{(\ln x - \mu_2)^2}{2\sigma_2^2} \right)} \right] \right) \mathrm{d}x \\ &= \int \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) * \frac{1}{x\sigma_1} * \left(\ln \frac{\sigma_2}{\sigma_1} \left(\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) + \left(\frac{(\ln x - \mu_2)^2}{2\sigma_2^2} \right) \right) \right) \mathrm{d}x \end{aligned} \tag{7}$$

Substituting in (7) $\mu_2 = \mu_1 \frac{1 + \delta g}{\lambda + \delta\lambda}$, we get

$$d(p \| q) = \int \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) * \frac{1}{x\sigma_1} * \left(\ln \frac{\sigma_2}{\sigma_1} - \left(\frac{(\ln \ln x - \mu_1)^2}{2\sigma_1^2} \right) + \left(\frac{\left(\ln x - \mu_1 \frac{1+\delta g}{\lambda + \delta\lambda} \right)^2}{2\sigma_2^2} \right) \right) \right) \mathrm{d}x \tag{7.1}$$

A negative or positive value of the expression (7.1) depends only on the term

$$\ln \frac{\sigma_2}{\sigma_1} - \left(\frac{(\ln x - \mu_1)^2}{2\sigma_1^2} \right) + \left(\frac{\left(\ln x - \mu_1 \frac{1+\delta g}{\lambda + \delta\lambda} \right)^2}{2\sigma_2^2} \right) \tag{8}$$

as the exponential rest in (7.1) is always positive.

Assuming $\Theta = \mu_1 \frac{1 + \delta g}{\lambda + \delta\lambda}$ and setting $\sigma_2^2 = \sigma_1^2 * \Theta^2$, $\ln(x) = y$ in (8), we get the following quadratic equation of the variable y

$$y^2\left(1/2\Theta^2-\frac{1}{2}\right)+y\left(1-1/\Theta^2\right)+\ln\frac{\sigma_2}{\sigma_1}\geq 0 \tag{9}$$

The condition in (9) holds true if

$$\left(1-1/\Theta^2\right)^2-4*\left(1/2\Theta^2-\frac{1}{2}\right)*\ln\frac{\sigma_2}{\sigma_1}\geq 0 \tag{9.1}$$

The above Formula allows us Values of Θ to estimate in order to get Values of the Entropy changes.

Without l.o.g., the relation (9.1) leads also to the expression

$$\frac{1+\delta g}{\lambda+\delta\lambda}*\mu_1>1 \tag{10}$$

With $\frac{\sigma_2}{\sigma_1}=1$ the Condition (9.1) reads now $\Theta^2-1>0$.

Comment: The formula in (9.1) and (10) indicates the following relations:

The relative entropy regarding dg and $\delta\lambda$ could increase or decrease in dependence on the values of $\delta\lambda$ and ΔY. That means in words:

An increase in employment, without a simultaneous increase in the economic performance (output) generates a positive value of entropy (entropy increases). An increase in employment by an increase in economic performance generates negative entropy (entropy decreases). The above result is of immense importance for economic models outside of the free market economic concept.

C.0: The definition of an economic temperature-substitute related to the employment dynamics within the Langevin formalism.

First, we start with the differential equation

$$\frac{\partial u}{\partial t}=-\gamma*F(u)+\sigma*L(t) \tag{1}$$

with the notations $F(U)=-\frac{\partial}{\partial U}F(U)$ = drift force, generated by the potential $V(u)$, $L(t)$ = fluctuating stochastic force (perturbation, γ = friction coefficient, σ = magnitude of the perturbation (stochastic force). Interpreting the variable u in space and time dimension in an economic sense (e.g. as a social product), we could observe the formula $F(u)=\frac{\partial}{\partial t}V(u(t))$ as a force generating the growth of the social product. The stochastic perturbation $L(t)$ can be also interpreted as a force having an impact on the dynamics in a stochastic sense. This could be among others a social crisis, a technological change in production, changes in consumption and investment behavior, or many other exogenous factors. An inverse temperature can be defined within this formalism as

$$\beta=\frac{1}{KT}=\frac{\gamma}{\sigma^2} \tag{2}$$

In the above relation, σ is a fluctuating parameter interpreted in Einstein's fluctuation-dissipation theorem to have a Gauss character. Our aim is now to interpret the term $\beta = \frac{1}{KT} = \frac{\gamma}{\sigma^2}$ by terms of associated economic variables within a specific model.

Although the fluctuating parameter "σ" can be concerned in a general sense belonging to Levy-statistics. Undertaking it, we create the following model.

C.1: A model to define a temperature substitute in the dynamics of employment.

Assumptions: We start by the assumption that λ% of the employable people provides the social product, while the rest is unemployed (but are willing to work). We note here that the social product is defined on the level of factor costs. Further, we assume that a homogeneous output of X units is produced by K_f factor costs per unit. That means the identity

$$F = K_f * X \tag{1}$$

holds true, whereby F = social product on factor costs level. A taxation of q% on the social product of (1) with $q \in [0,1]$ results in the makroeconomic equilibrium equation

$$F(1+q) = X * P \tag{2}$$

where P has to be identified as the market price per unit of the produced good. In this context, the q% taxation on the social product of factor costs has been used to finance unemployment. By these simple definitions, we can estimate the spending power of people who are unemployed respective to people who are evolving in the production process. This can happen by the following algebraic calculations:

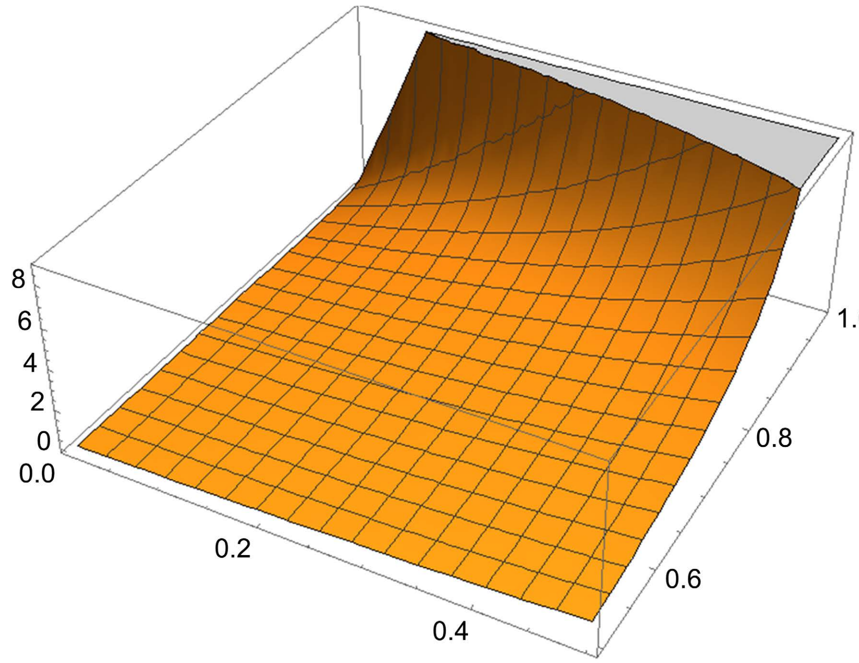
Let be $L_\lambda^N = \frac{(1-q)*F}{\lambda * N}$ the disposable personal income of employed people, where N is the amount of the employable people and $L_\mu^N = \frac{q*F}{(1-\lambda)*N}$ is the associated disposable personal income of unemployed people with $\mu = 1-\lambda$.

With the help of the above relations, we can estimate an indicator W_λ^μ of the relative "spending power" between these two classes. This can be formulated through the relations

$$W_\lambda^\mu = \frac{L_\mu^N}{L_\lambda^N} = \frac{q*\lambda}{(1-q)*(1-\lambda)} \tag{3}$$

$$\text{or alternatively } W_\mu^\lambda = \frac{(1-q)(1-\lambda)}{q\lambda} = \frac{1}{W_\lambda^\mu} \tag{4}$$

Example: By $q = 10\%$ and $\lambda = 96\%$, we obtain $W_\lambda^\mu = 2.666$. Relations (3) and (4) are not defined for $\lambda = 1$ and $q = 0$ respectively. The following Graph 4 indicates the spectrum of W_λ^μ values for λ and q restricted in the areas $\lambda \in [0.5, 0.99]$, and $q \in [0.01, 0.5]$ respectively. It is important to note that W_λ^μ increases almost with increasing λ, as opposed to the case of increasing q. However, we can estimate from Relation (3) by giving λ and q, at which W_λ^μ turns



Graph 4. Relative spending power for W_{λ}^{μ} versus, λ for $\{q,0.01,0.5\},\{\lambda,0.5,0.99\}$.

out to be equal to 1 (the same spending power of the two classes). **Graph 4** indicates these relations.

In a further step, we can estimate the weighted part of the two classes on the market volume. This can be carried out by the relations

$$G_{\lambda}^{\mu} = \frac{q * \lambda}{(1-q)*(1-\lambda)} * \frac{1-\lambda}{\lambda} = \frac{q}{1-q} \tag{5}$$

or alternatively

$$G_{\mu}^{\lambda} = \frac{(1-q)(1-\lambda)}{q\lambda} * \frac{\lambda}{1-\lambda} = \frac{1-q}{q} = \frac{1}{G_{\lambda}^{\mu}} \tag{6}$$

Notice: The taxation of the social product on factor-cost level is responsible for transferring income from the employing people to that without employment. However, this transfer has an impact on the market prices in the case of macroeconomic equilibrium. We define an appropriate economic temperature substitute by the multiplication of two terms as

$$T_{\mu}^{\lambda} = \frac{q}{1-q} * \frac{1-\lambda}{\lambda} \tag{7}$$

The first term (R.H.S.) represents the weighted part of the market volume for the two classes and has financial character. The second part is the absolute relation of the two classes being observed in the calculation. The temperature in this context has the form of social press. A low degree of taxation and low unemployment decreases the temperature in this way. A high degree of taxation and low employment increases the temperature ($1-q \to 0$, $\lambda \to 0$). This economic temperature definition coincides with a "Mainstream image" within the free market economy concept, is generated by economic logic, and represents a possibility to construct a bridge between thermodynamics and economic relations in the specific dynamics that are in focus. Another alternative mathematical form to define the temperature with respect to the above-mentioned conditions reads as

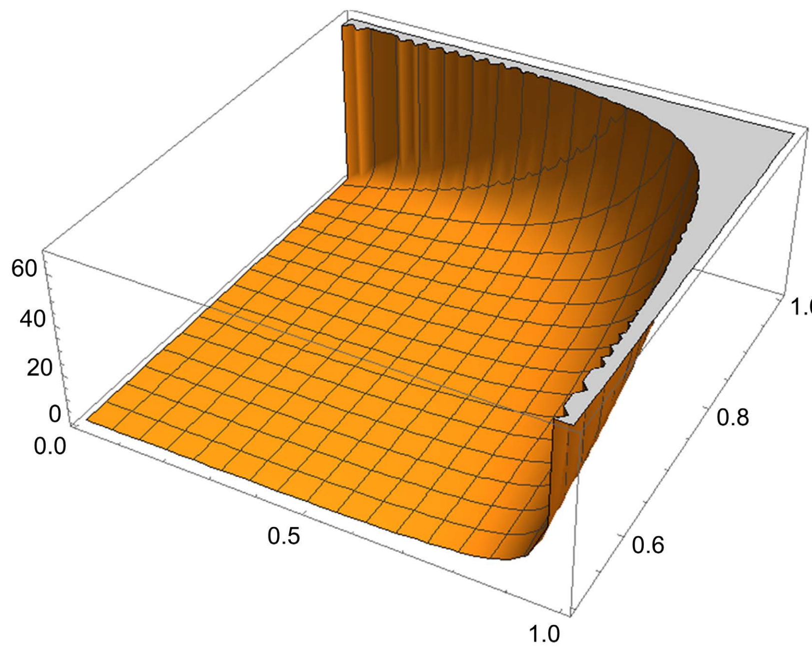
$$T_{\mu}^{\lambda} = e^{2q-1} e^{1-2\lambda} \tag{8}$$

q, λ being defined like above. The following Graph 5 gives an insight into the T_{μ}^{λ} values for $q \in [0.01, 1]$ and $\lambda \in [0.5, 1]$.

Within Formula (7), we can estimate pairs of q, λ values, they could have a higher probability to occur within the free market model. This fact leads meaningfully to the idea to define and detect the critical temperature with the help of the "Ising model". Another possibility consists in including such a temperature concept within a model of topologically induced diffusion in terms of quantum mechanics.

C.2: Applying the Ising Model to detect criticality in the employment dynamics.

Comment to the numerical results: The analysis of the classical Ising-model by a specifically defined temperature concept, leads to a paramagnetic state (low disorder) for the unemployment level of more than 60%. Concerning real situations within the free market economic model, we find this value as a rear event. The information provided by the classical Ising-model cannot capture quite widely the micro-behavior of the system, which could help to explain more informative relations on the macro level. The fractional point of view helps, in this



Graph 5. The economic temperature T_{λ}^{μ} versus q, λ.

case, to get simply a better insight due to specific economic dynamics. In this context, the concepts of critical point and phase transitions have to be redefined. The next chapters deal with this matter with the help of the concept of the fractional oscillator. The essential point for further investigations lies in the definition of a critical temperature in an economic sense. According to this, we recall the term $T_{\mu}^{\lambda} = \frac{q}{1-q} * \frac{1-\lambda}{\lambda}$ developed in Chapter C.1 that represents a temperature substitute concept in an economic sense. The first term (R.H.S.) represents the weighted part of the market volume of the two classes of people (employed or unemployed) and has a financial character. The second part is the absolute relation of the two classes, which is observed. A low degree of taxation and low unemployment decreases the temperature in this way. A high degree of taxation and low employment increases the temperature ($1-q \to 0$, $\lambda \to 0$). Now, we can imagine a constellation (values) of the first term (R.H.S.) represents the weighted part of the market volume of the two classes that have financial character. The second part is the absolute relation of the two classes, which is observed in the calculation, representing a kind of social press. A low degree of taxation and low unemployment decreases the temperature in this way. A high degree of taxation and low employment increases the temperature. Now, we could imagine a constellation of q and λ values at which both parties are satisfied.

C.3: Applying the economic temperature concept to detect criticality.

Now, we will try to obtain results considering economic aspects in the dynamics of employment (unemployment) from a fractional point of view. However, identifying the increasing unemployment degree by an increasing criticality without respecting any other criteria leads to nonmeaningful results. The reasons for this we analyzed in the previous chapter. The essential point for further investigations lies in the definition of a critical temperature in an economic sense. According to this, we recall the term $T_{\mu}^{\lambda} = \frac{q}{1-q} * \frac{1-\lambda}{\lambda}$ developed in Chapter C.2. Now, we could imagine a constellation of q and λ values at which both parties are convenient. Such a constellation is the result of two antagonistic forces. Within the free market model, the enterprises need some degree of unemployment, while the last is a systemic self-organized factor and the unemployed people quest to keep their receipts near the level of the employed one. The income for unemployed people, that is 80% of the income of employed people, could be for instance such a relation, while the associated q^* and λ^* can be easily estimated. This constellation is in accordance with the economic equilibrium state of the model. We could thus imagine the case of λ around 0.94 and q around 0.1, while these values satisfy both antagonistic factors. Indeed the last case is arranged outside of the economic equilibrium state. Both cases could then represent a critical temperature at around which fluctuations occur. We denote at this point that the economic temperature concept can be observed as dynamics in its own right. We are interested to follow the behavior of the term $\frac{T-T_c}{T_c}$, as this relation describes the system behavior near T_c. Observing these dynamics from a fractional point of view, we can use a specific approach based on the fractional oscillator. Due to this aim, we start from the initial value problem ${}_tD_*^{\beta}\langle x(t)\rangle = -b\langle x(t)\rangle$, $0<\beta<2$, which can be obtained from a fractional Fokker-Planck-model with drift. The term ${}_tD_*^{\beta}$ represents the "Caputo" fractional derivative operator of order β. The term $\langle x(t)\rangle$ can be interpreted as the mean (average) value of space fluctuations evolving in time and b is identical to the drift of the Fokker-Planck model that in our case could be the critical temperature or the equilibrium state of the term $T_{\mu}^{\lambda} = \frac{q}{1-q} * \frac{1-\lambda}{\lambda}$ by the associated q^* and λ^* values. Thus the above approach represents a time-fractional diffusion process with drift, whose solution leads to the formula

$$\langle x(t)\rangle = \langle x_0\rangle E_{\beta}\left[-bt^{\beta}\right] \tag{1}$$

where E_{β} denotes the Mittag-Leffler function given by the formula

$$E_{\beta}[z] = \sum\nolimits_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\beta)} \tag{2}$$

In this context, we observe a fractional relaxation process leading to an equilibrium state leaning on the classical Ising model. Although the similarities between the classical Ising model and the fractional approach are evident, it must be said that the fractional approach opens a new window that allows detecting more phase transitions, as the Mittag-Leffler function exhibits a specific mathematical behavior (finite number of zeros). We can further analyze the impact of the fractality order on the dynamics and therefore on its criticality. The classical Ising model primarily intends to analyze magnetization states in the dynamics of purely physical phenomena. Magnetization has been interpreted as a kind of order or disorder in the system. Now, how could we interpret the classical Ising model of a fractional point of view remaining within its mind? A possible way persists to apply the fractional Fokker-Planck ansatz with drift, which we mentioned above. A further possibility that respects the emergence of order and disorder in dynamics, consists in using the concept of the fractional oscillator as an initial value problem that is described in more detail in Chapter C.4.

C.4: Criticality in the dynamics of employment/the renormalization group in a fractional image. Fractional Oscillators.

In this chapter, we deal mostly with the entropy and entropy changes in fractal structures being the essential criterion of criticality in dynamics in this article. Additionally, we are concerned about the problem of steady states as a criticality factor (metastability), which is of major interest for economic policies. Finally, we define and analyze the criticality in fractional oscillators often met in dynamics of economic relevance, starting from the theory of the renormalization group. Although this theory represents a part of critical phenomena from a purely physical point of view, the fine focus on details, allows recognizing of similarities and bridges to fractional oscillators, which in turn enables to detect of critical behavior for dynamics outside of the pure physical frame.

C.4.1: The renormalization group from a fractional point of view.

The renormalization group is an important part of the theory of critical phenomena in a physical sense. In the following approach, we suppose that critical phenomena can be defined also for dynamics of economic relevance in the physical meaning. Although the central and essential criterion of criticality in this work has to be entropy, we will show that fractional modeling is able to represent critical phenomena and phase transitions on dynamics in accordance with the classical theory of the renormalization group in a fascinating and elegant way. In this context, it is helpful and meaningful to define temperature in the economic sense. The scope of the renormalization group is to derive analytical tools and solutions for the behavior of a system on the macro level, considering the behavior on the micro level. In this context, the free energy of a spin system turns out to be of major importance and can be used to extrapolate micro-macro behavioral relations.

General Frame: The evolution of the free energy F in a system in which the temperature exhibits continuous changes and tends to approach a critical value, could be modeled by the fundamental equation:

$$F(x) = g(x) + \frac{1}{\mu} F\left[\varphi(x)\right] \tag{1}$$

Sornette (2006) and Kadanoff (2000) handle with detail the above theme. The variable x represents the distance of the temperature T between the actual value and the critical value T_c. Thus, the function $F(x)$ is a mesh of the free energy in dependence on the temperature distance to its critical value. The function $g(x)$ represents the non-singular part of the function $F(x)$. The constant μ is the rescaling factor due to the differential distance $\mathrm{d}\left(F\left[\varphi(x)\right]\right)/\mathrm{d}x$. We can rewrite Relation (1) using the following formula:

$$F\left(\left|T - T_c\right|\right) = g\left(\left|T - T_c\right|\right) + \frac{1}{\mu} \partial F\left(\left|T - T_c\right|\right) / \partial\left(\left|T - T_c\right|\right) \tag{2}$$

Relation (2) means in words that the free energy of a system turns out to be a function of a non-singular term of itself and also of the differential changes of the independent variable in a linear manner. In a broader sense, this relation is but a linear form of a differential equation of first order. At this point, we will mention an important assumption. We proceed further under the assumption that the differential changes of the free energy due to the changes in the temperature distance to critical value are of fractional nature. This assumption may be arbitrary and not proven, but many observations tend to confirm it. Before interpreting Equation (2) in a fractional sense it is important to make the following notices. Approaching the temperature distance value to the critical point the identity $F(x) = F(T_c - T) = F(0)$ holds true. This free energy position can be observed as a strange attractor on a dynamics or a permanent changing initial value problem. Therefore, this position can signalize the existence of a phase transition in the dynamics. In a mathematical sense the position $F(0)$ =“S” indicates a critical situation of the behavior of the free energy at which the derivative $\partial F\left(|T-T_c|\right)/\partial\left(|T-T_c|\right)$ turns out to be infinite. Position “S” can be observed as a singular point in the dynamics. Further, we will notice that the solution of (2) can be represented by the use of iterative procedures. In this context, the solution can be expressed by the formula

$$f_n(x)=\sum_{i=1}^{n}\frac{1}{\mu^i}g\left[\varphi^i(x)\right] \quad \text{with} \quad f_{n+1}=g(x)+\frac{1}{\mu}f_n\left[\varphi(x)\right],\quad n=0,1,2,\cdots$$

The subscripts “n” should be interpreted as $\varphi^n(x)=\varphi\left[\varphi^{n-1}(x)\right]$.

This procedure is a dynamical mapping of the free energy while changing the temperature distance.

We can rewrite Equation (1) as: $\frac{1}{\mu}F\left[\varphi(x)\right]=g(x)-F(x)$ whereby

$$F\left[\varphi(x)\right]=\frac{\partial F\left[\varphi(x)\right]}{\partial x}=\frac{\partial F\left[|T_c-T|\right]}{\partial|T_c-T|}$$

The above relation establishes a well-defined fractional “Eigen Value” problem of the form $\frac{\partial F^{\alpha}}{\partial^{\alpha}T}=-\lambda F+q(T)$ with the help of an adequate transformation $\lambda\to\mu^{1/\alpha}$ of fractional order α. The above fractional “Eigen Value” problem is first defined as a space fractional differential equation. In the following steps, we can observe the evolution of the above differential equation as a time-fractional problem, which can be solved within the Caputo formalism. However, the solutions to this problem allow statements about the criticality of the dynamics standing in observation.

C.4.2: About the solution of the fractional differential equation $\frac{\partial F^{\alpha}}{\partial^{\alpha}T}=-\lambda F+q(T)$.

The space fractional aspect of the fractional oscillator.

The Book “Theory and Applications of Fractional Differential Equations” (see Kilbas et al., 2006) deliver a solution to the above equation. To solve the above fractional differential equation it is meaningful to apply the multi-dimensional FourierTransform method for nonhomogeneous differential equations by the Riesz-fractional derivatives. The solution reads

$$F(T)=\int G_{\alpha}^{F}(T-t)q(t)\,\mathrm{d}t \tag{1}$$

that represents a convolution integral, where

$$G_{\alpha}^{F}(T)=\frac{|T|^{\frac{2n-2}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{\infty}\frac{1}{|\rho|^{\alpha}+\lambda}\rho^{\frac{n}{2}}J_{\left(\frac{n}{2}-1\right)}\left(\rho|T|\right)\mathrm{d}\rho \tag{2}$$

with: n = space dimension and $J_{\nu}(z)$ = Integral involving the Bessel function of the first kind (see details by Kilbas et al., 2006).

C.5: Fractional oscillators and their affinity to the renormalization group ansatz.

One of the main reasons to interpret the renormalization group in a time sense is the principle of subordination. Within this scenario the process $X(t)=\breve{X}(\tau(t))$ evolves with an operational time $\tau(t)$ which is a fluctuating function of the physical time *t*. This irregular temporal behavior is often met near phase transitions. This results in statements, that a process that approaches a critical time t_c can be formulated in terms of the renormalization ansatz with respect to time.

Mathematically, this idea leads to the ansatz of the form

$$F(|t-t_c|)=g(|t-t_c|)+\frac{1}{\lambda}\partial F(|t-t_c|)/\partial(|t-t_c|) \tag{1}$$

Under the assumption that the derivative $\partial F(|t-t_c|)/\partial(|t-t_c|)$ behaves fractal, we can rewrite Formula (1) as

$$\frac{\partial^\beta F}{\partial t^\beta}=\lambda F(|t-t_c|)+\lambda g(|t-t_c|) \tag{2}$$

At this stage, we recall the time fractional integral equation of the CTRW process with memory whereby the memory function has a power law time decay.

This has been formulated as

$$\frac{\partial^\beta p(x,t)}{\partial t^\beta}=-p(x,t)+\int_{-\infty}^{+\infty} p(x,t)*w(x-z)\,\mathrm{d}z \tag{3}$$

Between the renormalization group ansatz, the fractional oscillator of the form $D_*^\beta u(t)=D^\beta\{u(t)-u(t=0)\}=-u(t)+q(t)$ and the fractional integral equation of the CTRW process with memory (3) there exists a significant relation of essential character. At first, the relation in (3) can be interpreted in terms of the distance $(|t-t_c|)$ whereby this distance could be identified by the location *X*. In this way. the equation in (3) looks like

$$\frac{\partial^\beta p(|t-t_c|,t)}{\partial t^\beta}=-p(|t-t_c|,t)+\int_{-\infty}^{+\infty} p(|t-t_c|,t)*w(|t-t_c|-\tau)\,\mathrm{d}\tau \tag{4}$$

In a further step, we recognize that the second term (R.H.S.) of the above relation is a convolution integral which corresponds to the $q(t)$ term of the fractional oscillator being the impulse response solution of the oscillator or the system's answer. All the above relations could be expressed by the following words: A time fractional oscillator approaches a critical point of the dynamics in time steps of fractional character (subordination of the physical time). This behavior corresponds to the Renormalization Group ansatz if the location distance $(X-X_C)$ could be substituted by a time distance $|t-t_c|$, t_c, which is now the critical time point of the ansatz (isofractality). The convolution term of the fractional integral CTRW equation, which constitutes the memory of the dynamics, corresponds to the impulse response solution of the fractional oscillator. These relations correspond meaningfully to the classical ansatz of the renormalization group.

Result: The solution of the time-fractional oscillator equation represented by the Mittag-Leffler equation can be used as an examination tool to detect criticality in associated dynamics. Due to this aim, we point to the article "Fractional Oscillations and Mittag-Leffler Functions", by Gorenflo and Mainardi (1996), which extrapolates analytically the solutions of the fractional oscillator Ansatz.

D: Steady states.

D.0: Steady states in the dynamics of employment.

In this chapter, we extrapolate steady states on the dynamics of employment using various physical and statistical modeling. We begin first with the fractional Fokker-Planck models assuming that an economic force (impulse) could induce an increase or decrease in employment.

D.1: Steady states on the employment dynamics using a fractional Fokker-Planck ansatz for Levy Flights (space-fractionality).

D.1.1: The case of symmetric Levy perturbation.

In this chapter, we interpret the Article" Stationary States in Single-Well Potentials under Symmetric Levy Noises", expanding it with specific algebraic manipulations, and the Article "Relaxation in Stationary States for Anomalous Diffusion" (see Dybiec et al., 2010, 2011).

We start assuming that a deterministic force has an impact on the employment level whereby a stochastic perturbation can further influence the dynamics. The associated Langevin-Ansatz reads as

$$\frac{\partial X}{\partial t} = f(x) + \xi(t) \quad \text{with} \quad f(x) = \text{deterministic part}, \quad \xi(t) = \text{stochastic part} \tag{1}$$

The term $f(x)$ could be further interpreted as a force resulting from an existing potential $V(x)$. In this case, holds true the equation $\frac{\mathrm{d}V(X)}{\mathrm{d}X} = -f(x)$. That means that Equation (1) can be reformulated as

$$\frac{\partial X}{\partial t} = -V(x) + \xi(t) \tag{2}$$

Assuming a subharmonic potential of the form $V(x) = |x|^c$ with $c \in (0,2)$, we obtain by integration of Formula (2) the following result

$$x(t) = x(0) - \int_0^t V(x(\tau))' \mathrm{d}\tau + \int_0^t \xi(\tau) \mathrm{d}\tau \tag{3}$$

or the equivalent

$$x(t) = x(0) - \int_0^t V(x(\tau))' \mathrm{d}\tau + L_\alpha(t) \tag{4}$$

The integral $\int_0^t \xi(\tau)\mathrm{d}\tau = L_\alpha(t)$ might define a Levy-stable process evolving in time.

The relations in (3) and (4) constitute the following fractional Fokker-Planck ansatz

$$\frac{\partial P(x,t)}{\partial t}=\left[\frac{\mathrm{d}}{\mathrm{d}x}V'(x,t)+\sigma^{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\right]P(x,t) \tag{5}$$

where α is the order of the associated Levy-stable process.

The term $\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}$ denotes the fractional Riesz-Feller derivative operator of order α.

Transferring the above operator onto the Fourier domain, we obtain the formula

$$\mathcal{F}\left[\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}f(x)\right]\overset{F}{\Rightarrow}-|k|^{\alpha}\hat{f}(k) \tag{6}$$

It is known that the solutions of the fractional Fokker-Planck ansatz can be extrapolated only in a few rare cases. Thus, it is needful to apply approaches to get solutions for the FFP-ansatz.

Assuming the existence of steady states, we start with the conditions that should hold true

$$\frac{\partial P(x,t)}{\partial t}=\left[\frac{\mathrm{d}}{\mathrm{d}x}V'(x,t)+\sigma^{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\right]P(x,t)=0 \tag{7}$$

$$\text{or}\quad\left[\frac{\mathrm{d}}{\mathrm{d}x}V'(x)+\sigma^{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\right]P(x)=0 \tag{8}$$

Assuming further that $P(x)$ should be of a power law form, we set

$$P(x)\approx|x|^{-\omega}\quad\text{for}\quad x\to\infty\quad\text{with}\quad\omega>0 \tag{9}$$

Substituting (9) and $V'(x)=C|x|^{c-1}$ in (8), we obtain the relation

$$\frac{\partial}{\partial x}\left[c|x|^{c-1}*P(x)\right]=c(c-1)*x^{c-2-\omega}=-\sigma^{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}*|x|^{-\omega} \tag{10}$$

We suppose further that the acting of the Riesz-Feller derivative operator on the power law function leads again to a power-law function. Due to that, we can state

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}|x|^{\alpha}}P(x)=x^{-(1+\alpha)} \tag{11}$$

Setting $c(c-1)\approx 1$, we extrapolate from (10) the relation

$$x^{c-2-\omega}=x^{-(1+\alpha)}\quad\text{or}\quad c-2+(1+\alpha)=\omega. \tag{12}$$

With the help of the above relation, it is not difficult to estimate the steady-state distribution

$$P_{st}(x)=|x|^{-\omega}=|x|^{-(c+\alpha-1)} \tag{13}$$

Relation (13) means: If empirical data are available that allows us to estimate α and assuming that the exponent of the harmonic potential is known, then we can define a steady-state probability distribution.

Normalizing $P_{st}(x)$ by the condition $\int_{-\infty}^{+\infty} P_{st}(x)\mathrm{d}x = 1$, we obtain at first that the relation $\omega > 1$ must hold true. That means that $c > 2-\alpha$ must hold true.

D.1.2: The case of an asymmetric Levy perturbation.

The Article "Stationary States in Langevin Dynamic under Asymmetric Levy noises" (see Dybiec et al., 2007), deliver a Fokker-Planck ansatz for Levy-Flights using an asymmetric perturbation Term. In the following text, we interpret this article expanding it with algebraic manipulations by the author.

The extrapolation of steady states in the case of an asymmetric Levy perturbation has some more difficulties than the symmetric one. The characteristic function $\hat{\phi}(k)$ of a stable Levy distribution of order α can be formulated on the Fourier domain as

$$\hat{\phi}(k) = \exp\left[ik\mu - \sigma^{\alpha}|k|^{\alpha}\left(1 - i\beta\,\mathrm{sign}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right] \quad (1)$$

with: α = Stability-Exponent, $\alpha \in (0,1)\cup(1,2)$;

σ = Scale-Parameter;

μ = Location-Parameter;

β = Skewness-Parameter, $\beta \in [-1,1]$.

Relation (1) allows us to define within the Fokker-Planck formalism the following ansatz for α # 1

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left[\mu - V(x,t)'\right]P(x,t) \\ &\quad + \sigma^{\alpha}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}P(x,t) + \beta\tan\left(\frac{\pi\alpha}{2}\right)\frac{\partial}{\partial x}\frac{\partial^{\alpha-1}}{\partial|x|^{\alpha-1}}P(x,t)\right]\end{aligned} \quad (2)$$

We can presume without problems $\mu = 0$ and $\sigma = 1$.

Similar to the case 14.1.1 (symmetric perturbation), we suppose again the existence of a subharmonic potential $V(x) = |x|^{c}$ with $c \in (0,2)$ und the existence of the steady state probability distribution of the form $P_{st}(x) = |x|^{-\omega}$, $\omega > 0$. The condition for steady states leads to the equation

$$0 = -\frac{\partial}{\partial x}\left[-V(x,t)'\right]P(x,t) + \sigma^{\alpha}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}P(x,t) + \tan\left(\frac{\pi\alpha}{2}\right)\frac{\partial}{\partial x}\frac{\partial^{\alpha-1}}{\partial|x|^{\alpha-1}}P(x,t)\right] \quad (3)$$

Assuming that $\frac{\partial^{\alpha}}{\partial x^{\alpha}}P_{st}(x) \cong x^{-(\alpha+1)}$, we obtain $\frac{\partial^{\alpha-1}}{\partial|x|^{\alpha-1}} = x^{-\alpha}$. Substituting these relations into (3), we obtain

$$-\frac{\partial}{\partial x}\left(-cx^{c-1}\right)|x|^{\omega} + x^{-(\alpha+1)} - \beta\tan\left(\frac{\pi\alpha}{2}\right)\alpha x^{-(\alpha+1)} = 0 \quad (4)$$

$$\text{or}\quad c*(c-1)x^{c-2-\omega} + x^{-(\alpha+1)}\left[1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right] = 0 \quad (5)$$

Substituting $c*(c-1) = Y$ and $1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right) = Z$ in (5), we obtain

$$Y * x^{c-2-\omega} + Z * x^{-(\alpha+1)} = 0 \quad (6)$$

Assuming $Z < 0$, we can formulate Relation (6) as

$$\log\left(Y * x^{c-2-\omega}\right) = \log\left(-Z * x^{-(\alpha+1)}\right) \quad (7)$$

$$\text{or} \quad \log(Y) + (c-2-\omega)\log(x) = \log(-Z) - (\alpha+1)\log(X) \quad (8)$$

Assuming that $c*(c-1) \approx 1$ and $-\alpha\beta\tan\left(\frac{\pi\alpha}{2}\right) = 1$, we obtain from (5) the relation

$$c - 2 - \omega = -(\alpha+1) \quad (9)$$

(end of the interpretation of the Article "Stationary States in Langevin Dynamic under Asymmetric Levy Noises" by Dybiec et al., 2007).

Extension: The identities $-\alpha\beta\tan\left(\frac{\pi\alpha}{2}\right) = 1$ and $1-2-\omega = -(\alpha+1)$ constitute an equation system for the variables ω and β if α is known. The solution allows to estimate ω for some specific β. This problem could be formulated in a more general form. By $c*(c-1)\#1$, we have to solve the equation system

$$c*(c-1) = \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right) - 1 \quad \text{and} \quad c - 2 - \omega = -(\alpha+1) \quad (10)$$

The equation $c*(c-1) = \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right) - 1$ is a quadratic equation of the variable c with the solution

$$c = -\frac{1}{2} \pm \sqrt{\frac{1 - 4\left(1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{2}} \quad (11a)$$

Real solutions exist if

$$1 - 4\left(1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right) \geq 0 \quad (11b)$$

Substituting (11a) into $c - 2 - \omega = -(\alpha+1)$, we extrapolate the relation

$$\omega = \alpha - 1.5 \pm \sqrt{\frac{1 - 4\left(1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{2}} \quad (12)$$

This above relation results in

$$P_{st}(x) = |x|^{-\left[\alpha - 1.5 \pm \sqrt{\frac{1 - 4\left(1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{2}}\right]} \quad (13)$$

whereby β must fulfill the inequation $1 - 4\left(1 - \alpha\beta\tan\left(\frac{\pi\alpha}{2}\right)\right) \geq 0$.

D.2: Steady states on the employment dynamics using the Fokker-Planck ansatz in the case of a space-time fractional subdiffusive model.

Risken (1991) handles in the Article "The Fokker-Planck Equation" with details the Theory of the Fokker-Planck Ansatz. Considering both the space and time fractionality the above Ansatz is formulated as

$$\frac{\partial}{\partial t}P(x,t)=D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}\frac{V(x)'}{n_\alpha}+k_\alpha\frac{\partial^\mu}{\partial x^\mu}\right]P(x,t)$$

with

α = time fractal exponent;

μ = space fractal exponent;

$D_t^{-\alpha}x(t)$ = Riemann-Liouville fractional integral of order α.

We start from the definition for the Riemann-Liouville fractional derivative

$$D_t^\alpha\left(u(x,t)-u(x,0)\right)=L_{FP}u(x,t) \tag{1}$$

with

$$L_{FP}u(x,t)=-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha}+k_\alpha\frac{\partial^\mu}{\partial x^\mu}$$

Integrating (1) by the Rieman-Liouville fractional integral $D_t^{-\alpha}$, we obtain

$$D_t^{-\alpha}D_{t*}^\alpha\left(u(x,t)-u(x,0)\right)=D_t^{-\alpha}L_{FP}u(x,t) \tag{2}$$

$$\text{or}\quad u(x,t)-u(x,0)=D_t^{-\alpha}L_{FP}u(x,t) \tag{3}$$

Applying the conventional (integer) derivative on (3), we obtain

$$\frac{\partial}{\partial t}\left(u(x,t)-u(x,0)\right)=D_t^{1-\alpha}L_{FP}u(x,t) \tag{4}$$

with $D_t^{1-\alpha}=\frac{\partial}{\partial t}D_t^{-\alpha}$ and $D_t^{-\alpha}=J^\alpha$ = Riemann-Liouville integral operator.

In order to extrapolate steady states, the following condition must hold true

$$D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha}+k_\alpha\frac{\partial^\mu}{\partial x^\mu}\right]P(x,t)=0 \tag{5}$$

The above relation means

$$D_t^{-\alpha}\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha}+k_\alpha\frac{\partial^\mu}{\partial x^\mu}\right]P(x,t)=K\quad(\text{Constant \# 0}) \tag{6}$$

as

$$D_t^{1-\alpha}\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha}+k_\alpha\frac{\partial^\mu}{\partial x^\mu}\right]P(x,t)=0 \tag{7}$$

The above form (7) can be observed as a space fractional diffusion equation with a drift evolving in time. We can formulate Relation (7) with the help of the Caputo fractional derivative as

$$D_{t*}^\alpha\left(u(x,t)-u(x,0)\right)=L_{FP}u(x,t)=0 \tag{8}$$

According to the above form the solutions for stationarity can be extracted with the help of the Mittag-Leffler function. It is known that the solution of the space-time fractional diffusion equation $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^\mu u(x,t)}{\partial x^\mu}$ is given on the

Fourier domain by the formula $\hat{U}(k,t) = E_{\mu,\alpha}\left[k^{\mu}t^{\alpha}\right]$. Including the drift term, we can represent the solution in an explicit form.

The case of $D_t^{-\alpha}\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha} + k_\alpha\frac{\partial^{\mu}}{\partial x^{\mu}}\right]P(x,t) = K = \text{constant} \# 0$, is more complicated.

We will deal with the simple case of symmetric perturbation assuming the existence of a harmonic potential of the form $|X|^C$, varying C in the range $C \in (0,2)$.

We start assuming the existence of a steady states probability distribution of the form $P_{st}(t) = |t|^{-\omega}$ like in the above case (14.1.1) obtaining again the relation

$$P_{st}(t) = |t|^{-\omega} = |t|^{-(c+\alpha-1)} \tag{9}$$

Thus, substituting (8) in the kernel $\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha} + k_\alpha\frac{\partial^{\mu}}{\partial x^{\mu}}\right]P(x,t)$, the following relation must hold true

$$\left[-\frac{\partial}{\partial x}\frac{V(x)}{n_\alpha} + k_\alpha\frac{\partial^{\mu}}{\partial x^{\mu}}\right]P(x,t) \to C*(C-1)*|X|^{C-1}*|x|^{-(c+\alpha-1)} + |x|^{-(c+\alpha-1)} \tag{10}$$

The above relation results in

$$D_t^{-\alpha}\left\{C*(C-1)|t|^{C-1}*|t|^{-(c+\alpha-1)} + |t|^{-(c+\alpha-1)}\right\} = K \tag{11}$$

The Riemann-Liouville fractional integral $J^{\alpha} = D_t^{-\alpha}$ applied on a function $f(t)$ is defined as

$$J^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} * \int_0^x f(t)(t-\xi)^{\alpha-1}\,\mathrm{d}\xi .$$

Hence, respecting further (11), we extrapolate

$$\frac{1}{\Gamma(\alpha)} * \int_0^t C*(C-1)t^{-\alpha} * (t-\xi)^{\alpha-1}\,\mathrm{d}\xi + \frac{1}{\Gamma(\alpha)}\int_0^t t^{-(c+\alpha-1)}(t-\xi)^{\alpha-1}\,\mathrm{d}\xi = K \tag{12}$$

getting a condition between α and C which allows to define the steady state probability distribution for the fractional subdiffusive Fokker-Planck ansatz. We gain from the integrals of (12) the solution

$$S = C*(C-1)*\frac{1}{\alpha} + \frac{t^{1-c}+1}{\alpha} \tag{13}$$

Thus the relation in (12) holds true only for $C=1$ leading to $S = 1/\alpha$.

Substituting $C=1$ into $P_{st}(t) = |t|^{-\omega} = |t|^{-(c+\alpha-1)}$, we extrapolate for the stationary distribution the formula $P_{st}(t) = t^{-\alpha}$ with $t>0$ (see Thesis of Dybiec et al., 2011).

D.3: Steady states (stationarity) on the fractional master equation (see thesis of Brockmann, 2003).

Pure stochastic performance for various dynamics can be derived within the theory of CTRW. In this context, using a memory function associated to the survival and waiting time distribution, we can formulate the integral equation of the CTRW in form of a general master equation reading as

$$\int_0^t \nu(t-\tau)\frac{\partial}{\partial \tau} p(x,\tau)\mathrm{d}\tau = -p(x,t) + \int_{-\infty}^{\infty} w(x-y)\,p(y,t)\mathrm{d}y \tag{1}$$

If $\nu(t)=\delta(t)$ with $\nu(t)$ = memory function, then the associated waiting time distribution $y(t)$ has exponential character (Markov-process).

If $\nu(t)=\delta(t)/\lambda$ with λ = constant, we obtain $y(t)=\lambda e^{-\lambda t}$ representing the general compound Poisson process. If we assume $\tilde{\nu}(t)$ # constant (on the Laplace domain), we deal with general non-markovian processes. A special case of non-markovian processes is that at which the waiting time distribution takes the specific form $\nu(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}$ having in Laplace domain the shape $\tilde{\nu}(s)=\frac{1}{s^{1-\beta}}$. In this case, Equation (1) turns out to be

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = -p(x,t) + \int_{-\infty}^{\infty} w(x-y)\,p(y,t)\mathrm{d}y \tag{2}$$

Starting from Relation (1), steady states could be achieved by the condition

$$p(x,t) = \int_{-\infty}^{\infty} w(x-y)\,p(y,t)\mathrm{d}y \tag{3}$$

In case of $\frac{\partial^\beta}{\partial t^\beta} = D_*^\beta$, D_*^β being the Caputo fractional derivative operator, we obtain the relation for steady states as following

$$D_*^\beta p(x,t) = 0 = D^\beta p(x,t) - D^\beta f(x_0,t_0) \tag{4}$$

with D^β = Riemann-Liouville derivative operator of order β and $f(x_0,t_0)$ = initial condition.

D.3.1: A short referring to the dynamics of earnings/econotaxis and associated steady states.

Researching the activities of an enterprise one would recognize that the results of the activities of various parts of the enterprise especially on the operational level are fractal. This is however the result due to the main goal of every enterprise within the free market economic model, namely to maximize the earnings. This goal forces the enterprise to a flexible, fluctuating, and often unpredictable behavior on the operative level. Further, it should be said, that investments often are realized in order to reduce the costs of the production processes or of other economic activity fields. These strong goal-oriented action operations have a strong impact on the employment level. Economic acting subjected to a goal like "earning maximization" can be observed in biological processes, especially in the "chemotaxis-behavior". The above-mentioned behavior can be met by the motion of microorganisms on the meso- and macro-level to an attractant and removal from it. Biological organisms move in direction of a food source as long as it remains attractive. On the way to the attractant and while their sojourn at it (food source) they leave signals (information) about the concentration of food called CAS. These signals are received from other organisms moving to or off the source. This information-process results in a higher or lower concentration of microorganisms at the food source. The most known model describing chemotactic processes is that of Segel and Keller (see Roberts et al., 1970).

The model consists of two coupled parabolic differential equations one of the CAS concentration and the other of the concentration of organisms at the source. This biological model is of high adequacy for the free-market economic model with no strong restrictive frame conditions. The concentration of biological species can be identified by the concentration of enterprises in a specific economic area (for example, vehicle production, chemicals, or various services). The CAS concentration could be identified by the profitability of the capital that has been invested (capital rentability as % of the investment). In this context, we call such an economic process as "econotaxis-dynamics". An econotaxis model could be formulated in terms of an appropriate CTRW process or as a stochastic master equation. Due to this aim, we will quantify such a model using mathematical relations. According to this goal, we formulate the following ansatz (model)

$$\frac{\partial I(x,t)}{\partial t}=\frac{\partial^2 I}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[I(x,t)*\frac{\partial C(x,t)}{\partial x}\right] \tag{1}$$

with the variables:

$I(x,t)$ = Amount (finance-value) x of investment that are realized at instant time t;

$C(x,t)$ = capital rentability concentration;

Φ = econotaxical coefficient (an analogon to diffusion coefficient);

Equation (1) can be interpreted in the fractional image.

<u>Version 1: Time-fractional interpretation</u>

$$D_{t*}^{\beta}I(x,t)=D\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right] \tag{2}$$

with D = diffusion coefficient and D_{t*}^{β} = time fractional derivative operator in the sense of M. Caputo. The Differential $\frac{\partial}{\partial x}$ represents here the common (integer) differential gradient.

<u>Version 2: Time-space fractional interpretation</u>

In this case, one obtains the diff. equation

$$D_{t*}^{\beta}I(x,t)=D\frac{\partial^{\alpha} I(x,t)}{\partial x^{\alpha}}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right] \tag{3}$$

with α,β the corresponding space-and time-fractional order respectively.

The differential Equation (3) can be interpreted as a space-time diffusion ansatz with drift.

<u>Steady states on the econotaxis model</u>

We start from the following formula for version 1

$$D_{t*}^{\beta}I(x,t)=D\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right] \tag{4}$$

For the Caputo fractional differential operator holds true the relation

$$D_t^{\beta}\left[I(x,t)-I(x_0,t_0)\right]=D_{t*}^{\beta}I(x,t)=L_{FP}I(x,t) \tag{5}$$

whereby D_t^{β} = Riemann-Liouville fractional derivative operator and
D_{t*}^{β} = Caputo fractional derivative operator

$$L_{FP}I(x,t)=D\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right] \tag{6}$$

Integratig both sides of (6) by $J^{\beta}=D_t^{-\beta}$, we obtain

$$I(x,t)-I(x_0)=D_t^{-\beta}\left[K\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right]\right] \tag{7}$$

as $D_t^{-\beta}*D_t^{\beta}=I$ = identity operator.

From (7), we obtain

$$\frac{\partial}{\partial t}\left(I(x,t)-I(x_0)\right)=D_t^{1-\beta}\left[K\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial}{\partial x}\left[\frac{\partial C(x,t)}{\partial x}*I(x,t)\right]\right] \tag{8}$$

Relation (8) could be reformulated as

$$\frac{\partial I}{\partial t}=\frac{\partial}{\partial t}J^{\beta}\left[K\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial^2 C(x,t)}{\partial x^2}*I(x,t)\right] \tag{9}$$

and interpreted as a Fokker-Planck Ansatz with $\frac{\partial C(x,t)}{\partial x}$ as an economic impulse.

(Capital-earnings)

In the case that the function $C(x,t)$ can be formulated by a smooth algebraic form, then one could define steady states using the condition

$$\frac{\partial}{\partial t}J^{\beta}\left[K\frac{\partial^2 I(x,t)}{\partial x^2}-\Phi\frac{\partial^2 C(x,t)}{\partial x^2}*I(x,t)\right]=0 \tag{10}$$

E.1: The "Kullback-Leibler" distance and its meaning for the Entropy-change by the LogNorm distribution (a mathematical Insertion).

(See the Article "On Information and Suffiency" by Kullback & Leibler, 1951)

Notice: We handle the Criticism of the personal income dynamics with the help of changes in the LogNorm Distribution. This is presupposed by the fact that more than 95% personal income balg is characterized through the LogNorm Distribution.

The Kullback-Leibler distance $D(p\|q)$ between two distributions p,q is a measure for the inefficiency of the assumption that a distribution could degenerate in a *q*-form although its real form is of *p*-form.

This can be represented by the formula

$$D(p\|q)=\sum_X p(x)*\ln\left(\frac{p(x)}{q(x)}\right) \tag{1}$$

The conditional entropy *S* for two given Probability Density Functions (mutual-Information) can be formulated by the following relations:

$$S[X|Y] = -\int_0^\infty P(X,Y) * \ln[P(X|Y)]\,\mathrm{d}x\mathrm{d}y \tag{2}$$

with $P(X,Y)$ = marginal (conditional) probability between $P(X), P(Y)$.

In the concrete case of the transition of a LogNorm distribution with parameter μ_0 in a LogNorm distribution by a new parameter μ_{new}, the above formula can be simplified as follows

$$S[X|Y] = S[X(\mu_0)] + \int_0^\infty P(X(\mu_{new})) * \ln[P(X(\mu_{new}))]\mathrm{d}x$$
$$\rightarrow S[X] - S[X|\mu_{neu}] \equiv S[\mu_0] - S[X|\mu_{neu}] \tag{3}$$

The variable $S[X|Y]$ can be interpreted as the information $I[X|\mu_{new}]$ and the term $\int_0^\infty P(X(\mu_{new})) * \ln[P(X(\mu_{new}))]\mathrm{d}x$ can be characterized as the factor, which contributes to increasing or decreasing of the information gain.

The variation of specific parameters at the above relation allows to identify the entropy-changes in the dynamics.

The entropy may increase if the term $\int_0^\infty P(X(\mu_{neu})) * \ln[P(X(\mu_{neu}))]\mathrm{d}x$ turns out to be positive. Primarily the relation holds true

$$\begin{aligned} &\int_0^\infty P(X(\mu_{neu})) * \ln[P(X(\mu_{neu}))]\mathrm{d}x \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}} * \frac{1}{\sigma_{new} * x} * \exp^{(\ln x - \mu_{new})^2 / 2\sigma_{new}^2} \\ &\quad * \ln\left\{\frac{1}{\sqrt{2\pi}} * \frac{1}{\sigma_{new} * x} * \exp^{(\ln x - \mu_{new})^2 / 2\sigma_{new}^2}\right\}\mathrm{d}x \\ &= A \end{aligned} \tag{4}$$

After algebraic operations, we obtain

$$\begin{aligned} A &= \int_0^\infty \frac{1}{\sqrt{2\pi}} * \frac{1}{\sigma_{new} * x} * \exp^{(\ln x - \mu_{new})^2 / 2\sigma_{new}^2} \\ &\quad * \left(-\ln(x\sigma_{new}) - \ln(2\pi) + (\ln x - \mu_{new})^2 / 2\sigma_{new}^2\right)\mathrm{d}x \end{aligned} \tag{5}$$

Relation (5) turns out to be positive by the condition

$$-\ln(x\sigma_{neu}) - \ln(2\pi) + (\ln x - \mu_{neu})^2 / 2\sigma_{neu}^2 > 0. \tag{6}$$

with $\mu_{neu} = \mu_0(\lambda(1-q) + q(1-\lambda))$ and $\sigma_{neu} = (\lambda(1-q) + q(1-\lambda)) * \sigma_0$. Therefore, we obtain

$$-\ln x - \ln(\sigma_{neu}) - K_1 + (\ln x^2 + \mu_{neu}^2 - 2\ln x * \mu_{neu}) / K_2 = 0 \tag{7}$$

with $K_1 = \ln(2\pi)$, $K_2 = 2\sigma_{neu}^2 = \text{Constant}$.

Relation (7) is a quadratic equation in the factor $\ln x$ of the form

$$\alpha(\ln X)^2 + \beta \ln x + \gamma$$

with $\alpha = 1/K_2$, $\beta = -(1 + 2\mu_{neu})/K_2$ and $\gamma = -\ln(\sigma_{neu}) + \frac{\mu_{neu}^2}{K_2} - K_1$.

Analyzing the solutions of the above quadratic equation, we can extrapolate relations between the parameters α, β and γ that are needed to interpret the condition in (6).

E.2: The entropy production on the LogNorm distribution applying the Kullback-Leibler distance by the help of an specific macroeconomic Ansatz

In this chapter, we focus on the statistical transformation, which appears, in the case that a part of the national product of factor costs will be used to finance (support) unemployment in form of a social income. Assuming that the distribution of the personal income at equilibrium (before starting the social income action) is of LogNorm form, we will investigate the entropy difference, which is caused by the social income intervention approaching that by the Kullback-Leibler distance. To do it, we recall the definitions and results of Chapter D.1 where we define specific macroeconomic Indices and an economic temperature.

If we utilize a part q (%) of the social product to factor costs through taxation in order to finance people being unemployed the following relations hold true:

A λ % part of the employable population $\lambda \in [0,1]$, yields a social product to factor costs of X unities the last having a finance value F. The q (%) part of it is utilized to finance the $(1-\lambda)$ % population being unemployed. In this case, the following new equilibrium situation holds true

$$F(1+q) = X * P \tag{1}$$

while P is the final market price. Employed people receive a disposable income (after taxation)

$$L_\lambda^N = \frac{(1-q) * F}{\lambda * N} \tag{2}$$

and unemployed people receive respectively an income amount of

$$L_\mu^N = \frac{q * F}{(1-\lambda) * N} \tag{3}$$

with $\mu = 1-\lambda$ and N = whole employable population.

By these assumptions, we gain the following indexes

1) $W_\lambda^\mu = \frac{L_\mu^N}{L_\lambda^N} = \frac{q * \lambda}{(1-q)(1-\lambda)}$ = absolute relationship between social- and regular-income (spending power) (4)

2) $G_\lambda^\mu = W_\lambda^\mu * \frac{1-\lambda}{\lambda} = \frac{q}{1-q}$ = weighted potency of regular and social income on the market level (market potency) (5)

or alternatively

3) $$G_\mu^\lambda = \frac{(1-q)(1-\lambda)}{q\lambda} * \frac{\lambda}{1-\lambda} = \frac{1-q}{q} = \frac{1}{G_\lambda^\mu} \tag{6}$$

Notice: The taxation of the social product on the factor-costs level is responsible for transferring income from the employing people to that without employment. However, this transfer has an impact on the market prices. Now, we assume that the distribution, before the social income intervention acts, remains in LogNorm form. After the intervention, we have two classes of income. The employed class remains in LogNorm form while a new class arises, which has a uniform distribution. At this point, two scenarios for social income are imaginable:

a) The income is oriented to the earlier corresponded employment income;

b) The social income has a relatively fixed character and is oriented only to the amount of the taxation of the social product, which is splatted uniformly by the $(1-\lambda)N$ unemployed population.

We assume further that the new distribution that indicates the income of the employed class after taxation, remains by the LogNorm structure, while this of the unemployed class is approximately uniformly distributed. It must be said, at this point, that the statistical parameters of the new LogNorm distribution receive new values. The realization of the unconditional basic income (being actually in discussion) does not probably lead to new investment and thus can be characterized as a hidden social income forcing only consumption. The mathematical treatment to capture these relations could be gained by following thoughts:

If $X \to N(\mu,\sigma^2)$ represents a normal distribution then the stochastic variable Y with $Y = aX + b$ results in the distribution $Y \to N(a\mu + b, a^2\sigma^2)$.

Using the relations $\mu_{new} = \lambda(1-q)\mu_0 + q * \dfrac{Z}{(1-\lambda)*N}$ and

$\sigma_{new}^2 = (\lambda(1-q))^2 \sigma_0^2$, we gain a new LogNorm distribution of the form

$$f(y,\sigma_{new},\mu_{new}) = \frac{1}{\sqrt{2\pi}} * \frac{1}{\sigma_{new} * y} * \exp^{(\ln y - \mu_{new})^2/2\sigma_{new}^2} \to$$

$$\frac{1}{\sqrt{2\pi}} * \frac{1}{\lambda(1-q)\sigma_0 Y} * \exp\left(\left(\ln y - \left(\lambda(1-q)\mu_0 + q * \frac{Z}{(1-\lambda)*N}\right)\right)\Big/\left(2\sigma_0^2\left(\lambda(1-q)\right)^2\right)\right) \quad (7)$$

with Z = social product to factor costs, λ, q macroeconomic variables defined above and N = whole population of the employable people.

E.2.1: Extrapolating the associated Entropy by the Kullback-Leibler Distance.

The Kullback-Leibler Distance is formulated by

$$D(p \parallel q) = \sum_X p(x) * \ln\left(\frac{p(x)}{q(x)}\right) \quad (1)$$

The conditional entropy $S[X \mid Y]$ (mutual information) of two given probabilities $P(X)$ and $P(Y)$ can be expressed by the following form

$$S[X \mid Y] = -\int_0^\infty P(X,Y) * \ln[P(X \mid Y)] \mathrm{d}x\mathrm{d}y \quad (2)$$

with $P(X \mid Y)$ = marginal (conditional) probability between $P(X)$ and $P(Y)$.

As Formula (2) corresponds to the definition in (1), we can estimate the entropy change (Kullback-Leibler distance) for a LogNorm distribution transferred to a new LogNorm distribution with the help of Relation (2).

Relation (2) can be reformulated as

$$S[X|Y]=S\left[X\left(\mu_0,\sigma_0\right)\right]+\int_0^{\infty}P\left(X\left(\mu_{new},\sigma_{new}\right)\right)*\ln\left[P\left(X\left(\mu_{new},\sigma_{new}\right)\right)\right]\mathrm{d}x \quad (3)$$

as

$$S[X]-S[X\mid\mu_{neu},\sigma_{new}]\equiv S[\mu_0,\sigma_0]-S[X\mid\mu_{new},\sigma_{new}] \quad (4)$$

From Relation (3), we recognize that entropy could increase if the term $\int_0^{\infty}P\left(X\left(\mu_{new},\sigma_{new}\right)\right)*\ln\left[P\left(X\left(\mu_{new},\sigma_{new}\right)\right)\right]\mathrm{d}x$ increases. We obtain

$$\begin{aligned}&\int_0^{\infty}P\left(X\left(\mu_{new},\sigma_{new}\right)\right)*\ln\left[P\left(X\left(\mu_{new},\sigma_{new}\right)\right)\right]\mathrm{d}x\\&=\int_0^{\infty}\frac{1}{\sqrt{2\pi}}*\frac{1}{\sigma_{new}*x}*\exp^{\left(\ln x-\mu_{new}\right)^2/2\sigma_{new}^2}\\&\quad*\ln\left\{\frac{1}{\sqrt{2\pi}}*\frac{1}{\sigma_{new}*x}*\exp^{\left(\ln x-\mu_{new}\right)^2/2\sigma_{new}^2}\right\}\mathrm{d}x\\&=A\end{aligned} \quad (5)$$

We can (5) rewrite as

$$\begin{aligned}A=&\int_0^{\infty}\frac{1}{\sqrt{2\pi}}*\frac{1}{\sigma_{new}*x}*\exp^{\left(\ln x-\mu_{new}\right)^2/2\sigma_{new}^2}\\&*\left(-\ln\left(x\sigma_{new}\right)-\ln(2\pi)+\left(\ln x-\mu_{new}\right)^2/2\sigma_{new}^2\right)\mathrm{d}x\end{aligned} \quad (6)$$

The above relation turns out to be positive if

$$-\ln\left(x\sigma_{new}\right)-\ln(2\pi)+\left(\ln x-\mu_{new}\right)^2/2\sigma_{new}^2>0 \quad (7)$$

The expression in (7) can be reformulated as

$$-\ln x-\ln\left(\sigma_{new}\right)-K_1+\left(\ln x^2+\mu_{new}^2-2\ln x*\mu_{new}\right)/K_2 \quad (8)$$

Substituting the terms $\mu_{new}=\lambda(1-q)\mu_0+q*\frac{Z}{(1-\lambda)*N}$ and $\sigma_{new}^2=\left(\lambda(1-q)\right)^2\sigma_0^2$ in (8), we obtain a quadratic equation in $\ln x$ with $K_1=\ln(2\pi)$, $K_2=2\sigma_{new}^2$ = known of the form $a(\ln X)^2+b\ln x+c$ with $a=1/K_2$, $b=-\left(1+2\mu_{new}\right)/K_2$ and $c=-\ln\left(\sigma_{new}\right)+\frac{\mu_{new}^2}{K_2}-K_1$.

The solution of the quadratic form leads to conditions between a,b,c should hold true in order to obtain an increased value of entropy. Entropy changes of the LogNorm distribution can be generally extrapolated using the S-theorem and varying the averaged value μ_0 and the standard deviation σ. For instance, smaller value of σ corresponds to a higher concentration of probability mass at the averaged value. This could be in a social sense of essential importance. This can be realized using the formula $\int_0^{\infty}P\left(X\left(\mu_{new},\sigma_{new}\right)\right)*\ln\left[P\left(X\left(\mu_{new},\sigma_{new}\right)\right)\right]\mathrm{d}x$ and substituting μ_{new},σ_{new} by $\mu_{new}=\mu_0\mp\delta\mu$ and $\sigma_{new}=\sigma_0\mp\delta\sigma$.

Within our investigations referring to personal income, we have got the following results:

R1) A transition of the LogNorm distribution to a power-law one can be characterized as critical;

R2) Investigating the LogNorm distribution, we can detect regions with power law characteristics signalizing critical behavior (see Newmann, 2005);

R3) If we could imagine a future-personal income constellation at which a significantly high degree of unemployment is supported by social wages, this constellation can be characterized as entropic compared with situations with a relatively low unemployment degree (<3%);

R4) We have estimated the corresponding entropy development in the case of financing unemployment by taxation with the help of an economic temperature concept leading to very specific solutions;

R5) Unemployment is often connected to inequality within the frame conditions of health services, pension funds and other social institutions, generating additively a big amount of volatile social entropy.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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