Theoretical Economics Letters, 2023, 13, 650-665
https://www.scirp.org/journal/tel
ISSN Online: 2162-2086
ISSN Print: 2162-2078

Fourier-Cosine Method for Pricing Equity-Indexed Annuity under Heston Model

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How to cite this paper: Gu, X. C., & Wei, X. (2023). Fourier-Cosine Method for Pricing Equity-Indexed Annuity under Heston Model. *Theoretical Economics Letters, 13,* 650-665.
https://doi.org/10.4236/tel.2023.133039

Received: April 18, 2023
Accepted: June 27, 2023
Published: June 30, 2023

Abstract

In this paper, a pricing method based on the Fourier-Cosine series expansion is introduced for pricing equity-indexed annuities (EIAs) under the Heston model. By means of the Fourier-Cosine series expansion, the density function of the underlying indexed is recovered from its characteristic function, and then yields an efficient way for EIAs pricing. To show the accuracy of the Fourier-Cosine expansion method, numerical experiments, we provide the numerical results of EIAs price for the classical Black-Scholes model. It is shown that the computation results obtained by the Foueier-Cosine series are as accurate as those obtained by using the Monte Carlo simulation method. The Fourier-Cosine expansion method can be used to obtain the break-even participation rate under the Heston model with or without a cap in simple ratchet EIAs.

Keywords

Fourier-Cosine Expansion, Equity-Indexed Annuity, Heston Model, Pricing

1. Introduction

Equity-indexed annuities (EIAs) were introduced in 1995 by Keyport Life Insurance Corporation. They are designed to provide not only certain annuity benefits, but also returns that are linked to the underlying equity index. EIAs attract investors because policyholders can share in financial market growth and their properties can be protected by the minimum guaranteed benefits provided by EIAs.

The EIA pricing problem can be traced back to 1976, which introduced modern option pricing theory and used its techniques to equity-indexed insurance pricing and minimum risk investment strategy (Brennan & Schwartz, 1976). This theory assumed the underlying equity dynamic was the classical Black-Scholes model, a geometric Brownian motion with a constant interest rate and volatility.

Under this classical Black-Scholes model, Gerber and Shiu (1994) consider derivative pricing in an incomplete market by Esscher transform. Then, three common EIA pricing, which were point-to-point, the cliquet and the lookback, were introduced by Tiong (2000). Four path-dependence option pricing was introduced by Lee (2003) to improve the participation rate. As the classical Black-Scholes model is a simplified description of a real financial market, subsequent research began to improve the model assumption to better match real life. For example, the constant interest rate assumption is irrational in the EIAs, given the long maturity of RIA contracts, which may cause EIA pricing bias. Thus, a stochastic interest rate model was proposed by Lin and Tan (2003), and it was shown by Monte Carlo simulation that the stochastic interest rate affects EIA premiums and break-even participation rates. Subsequently, a ratchet EIA pricing was derived under the Vasicek interest rate model (Kijima & Wong, 2008). Ballotta (2010) studied the EIA pricing under the variance gamma model. A regime-switching model was introduced for pricing and hedging long-term equity-linked insurance products by Hardy (2001). And Lin et al. (2009) studied the equity-indexed annuity pricing by Esscher transform under this model. In Chang (2014), a tree method was used to price EIAs with surrender risk under the Vasicek interest rate model and CIR++ model (Wei et al., 2013). A unified pricing framework for EIAs under a regime-switching Lévy model was presented by Kolkiewicz and Lin (2017). In Chiu et al. (2019), an exchange model was added to the Black-Scholes model to explain the contract and market parameter's influence on pricing EIAs. And Hawkes jump diffusions model was studied for EIAs pricing and a numerical result was given (Sharma et al., 2021).

The stochastic model is proposed to better match reality. In Merton (1976), a jump-diffusion model was proposed and subsectionly, a widely jump-diffusion model, named variance gamma model, was proposed by Madan et al. (1998). In the research on stochastic volatility, Hull and White (1987) assumed that volatility was a stochastic process and built the Hull-White model. In Stein and Stein (1991), stochastic volatility was added to the underlying stock price. A square root model was introduced and named the Heston model (Heston, 1993).

The Foueier-Cosine expansion method is used to price ratchet EIAs in this paper. The Foueier-Cosine expansion method was introduced by Carr and Madan (1999) and became a popular and efficient option pricing tool. Fourier-Cosine expansion method can be applied to any model with a known characteristic function for an underlying asset regardless of whether its density function is known. Previously, it was used for European options, Bermudan options (Fang & Oosterlee, 2008) and Asian options (Zhang & Oosterlee, 2013). Recently, this method has been introduced for pricing insurance contracts in Goudenége et al. (2018).

The contribution of this paper is two-fold. On the theoretical side, we show that the Fourier-Cosine expansion method can be applied to price EIAs under the stochastic volatility model. On the application side, we provide the formula for pricing EIAs under the Heston model and analyze the sensitivity of the price to the model and product parameters. The innovation of this paper is that we propose the stochastic volatility model in the pricing of EIAs, which allows us to consider the evolution of the volatility effect on the price of EIAs.

The rest of the paper is organized as follows. We describe the ratchet EIA and introduce the model setup in Section 2. We apply the Fourier-Cosine expansion method to the pricing EIAs in Section 3. Finally, numerical tests are presented in Section 4 to demonstrate the accuracy of the proposed method.

2. Model Description and Ratchet EIAs Contract

2.1. Models Setup and Related Characteristic Functions

Assume that the payment EIA contract with maturity at *T*-year is based on the return of the equity index. Let $S(t), t \geq 0$ denote the equity index value at time *t*. Consider a frictionless financial market with continuous trading. Assume that taxes, transaction costs and restrictions on borrowing or short sales do not exist and that all securities are perfectly divisible.

We assume that the equity index $S(t)$ underlying the EIAs contract follows the Heston model, which satisfies the following system:

$$\begin{cases} \mathrm{d}S(t) = (r-q)S(t)\mathrm{d}t + \sqrt{v(t)}S(t)\mathrm{d}W_S(t) \\ \mathrm{d}v(t) = k(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t) \end{cases} \tag{2.1}$$

where $r > 0, q \geq 0$ are constants representing the riskyless interest rate and the dividend rate respectively, $\{v(t), t \geq 0\}$ is variance process of the equity value, $\bar{v} > 0, \gamma > 0$ is the mean and the variance of volatility of the equity index, $k > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $W_S(t), W_v(t)$ are two different Brownian motions with correlation $\mathrm{d}W_S(t)\mathrm{d}W_v(t) = \rho\mathrm{d}t$ under a risk-neutral measure Q.

Assume $X(t) := \ln(S(t))$ denote the log-equity index and have an initial value: $X(0) = \ln(S(0)) = x$. By (2.1), $X(t)$ satisfies the following system:

$$\begin{cases} \mathrm{d}X(t) = \left(r - q - \frac{1}{2}v(t)\right)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_S(t) \\ \mathrm{d}v(t) = k(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t) \end{cases} \tag{2.2}$$

Under Heston model (2.2), the conditional characteristic function of the log-asset price $X(T)$ given that $X(t) = x$ is given by Feyman-Kac formula:

$$\begin{aligned} \psi(u,T) &:= \mathbb{E}^{Q}\left[\mathrm{e}^{iuX(T)}\right] \\ &= \exp\left(iu(r-q)\tau + \frac{v_0}{\gamma^2}\left(\frac{1-\mathrm{e}^{-D\tau}}{1-G\mathrm{e}^{-D\tau}}\right)(k - i\rho\gamma u - D)\right) \\ &\quad \cdot \exp\left(\frac{k\bar{v}}{\gamma^2}\left(\tau(k - i\rho\gamma u - D) - 2\ln\left(\frac{1-G\mathrm{e}^{-D\tau}}{1-G}\right)\right)\right) \cdot \exp(iux), \end{aligned} \tag{2.3}$$

where $\tau = T - t, D = \sqrt{(k - i\rho\gamma u)^2 + (u^2 + iu)\gamma^2}, G = \frac{k - i\rho\gamma u - D}{k - i\rho\gamma u + D}$.

When the variance $\{v(t), t \geq 0\}$ degenerates to a constant, i.e. $\kappa = \gamma = 0$, the Heston model reduces to the classical Black-Scholes model, which is given as follows:

$$\mathrm{d}S(t) = (r-q)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_S(t), \tag{2.4}$$

Thus, the conditional characteristic function of log-asset price $X(T)$ given $X(0) = x$ can be given by:

$$\psi(u,T) := \mathbb{E}^{Q}\left[e^{iuX(T)}\right] = \exp\left[iu\left(x + \left(r - q - \frac{1}{2}\sigma^2\right)T\right) - \frac{1}{2}\sigma^2 u^2 T\right]. \tag{2.5}$$

In this paper, we consider the Heston model as a risky equity index model for pricing EIAs, we derive the pricing formula for evaluating EIA contracts under Heston model. To show the accuracy of proposed method, we present the price of EIAs under Black-Scholes model for comparison with the existing result, since there are only results under Black-Scholes model in former research. We also provide some numerical test for pricing EIAs under Heston models.

2.2. EIAs Contracts and Their Payoffs

Consider an EIA contract with maturity at T years. A ratchet EIA is one whose equity index is reset at some frequency. We consider EIAs with an annual reset, though the model and pricing methodology can easily be extended to an arbitrary reset frequency. Denote the index return over the jth year as follows:

$$R_j = \frac{S(j)}{S(j-1)}, \quad \text{for } j = 1, 2, \cdots, T \tag{2.6}$$

and denote the index return series from year 1 to year T by the following:

$$R_0^T := \{R_j, j = 1, 2, \cdots, T\}.$$

There are two ways to compute the payment of the ratchet EIAs: the simple ratchet and the compound ratchet. The payoff of the EIAs is the greater of either the index return R_j times a participation rate α or a minimum guaranteed return g. For the simple Ratchet, the payment at maturity T-year, denoted by $V(T, R_0^T)$, is defined as

$$V(T, R_0^T) = 1 + \sum_{j=1}^{T} \max\left[g, \alpha(R_j - 1)\right], \tag{2.7}$$

For the compound Ratchet, its payment at maturity T-year is as

$$V(T, R_0^T) = \prod_{j=1}^{T} \max\left[1 + g, 1 + \alpha(R_j - 1)\right]. \tag{2.8}$$

To reduce the cost of the EIAs, a cap design is imposing to the contract, thus the payoff of the simple Ratchet EIAs and the compound Ratchet EIAs with cap $c > g$ take the following forms respectively:

$$V(T, R_0^T) = 1 + \sum_{j=1}^{T} \min\left[\max\left[g, \alpha(R_j - 1)\right], c\right], \tag{2.9}$$

$$V\left(T,R_0^T\right)=\prod_{j=1}^{T}\min\left[\max\left[1+g,1+\alpha\left(R_j-1\right)\right],1+c\right]. \tag{2.10}$$

Note that the simple or compound Ratchet EIAs without cap can be taken as simple or compound Ratchet EIAs with cap $c=+\infty$.

3. Pricing EIAs by Fourier-Cosine Method

Under the selected risk-neutral measure Q, the time 0 price of the EIAs with payment $V\left(T,R_0^T\right)$ at maturity T-year is

$$V(0;\alpha):=\mathrm{e}^{-rT}\mathbb{E}^{\mathrm{Q}}\left[V\left(T,R_0^T\right)\right]. \tag{3.1}$$

Assume that an initial premium of 1 unit is invested in the EIAs contract. The task of pricing the EIAs contract is to find the break-even participation rate α^* such that the value of the EIAs at time 0 breaks even, that is,

$$V\left(0;\alpha^*\right)=1. \tag{3.2}$$

For the simple Ratchet EIAs with and without cap whose payment as given in (2.7) and (2.9) respectively, we have

$$\begin{aligned}V(0;\alpha)&=\mathrm{e}^{-rT}\mathbb{E}^{\mathrm{Q}}\left[1+\sum_{j=1}^{T}\max\left[g,\alpha\left(R_j-1\right)\right]\right]\\&=\mathrm{e}^{-rT}\left(1+\sum_{j=1}^{T}\mathbb{E}^{\mathrm{Q}}\left[\max\left[g,\alpha\left(R_j-1\right)\right]\right]\right)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}V(0;\alpha)&=\mathrm{e}^{-rT}\mathbb{E}^{\mathrm{Q}}\left[1+\sum_{j=1}^{T}\min\left[\max\left[g,\alpha\left(R_j-1\right)\right],c\right]\right]\\&=\mathrm{e}^{-rT}\left(1+\sum_{j=1}^{T}\mathbb{E}^{\mathrm{Q}}\left[\min\left[\max\left[g,\alpha\left(R_j-1\right)\right],c\right]\right]\right).\end{aligned} \tag{3.4}$$

For the compound Ratchet EIA with and without cap whose payment as given in (2.8) and (2.10) respectively, we have

$$V(0;\alpha)=\mathrm{e}^{-rT}\mathbb{E}^{\mathrm{Q}}\left[\prod_{j=1}^{T}\max\left[1+g,1+\alpha\left(R_j-1\right)\right]\right] \tag{3.5}$$

and

$$V(0;\alpha)=\mathrm{e}^{-rT}\mathbb{E}^{\mathrm{Q}}\left[\prod_{j=1}^{T}\min\left[\max\left[1+g,1+\alpha\left(R_j-1\right)\right],1+c\right]\right]. \tag{3.6}$$

Notice that $\Delta_j, j=1,2,\cdots,T$ is not independent under Heston model. By (3.4), R_j is not independent, the computation of expectation in (3.5) and (3.6) requires the joint distribution of R_j, which make it hard to use Fourier-Cosine expansion directly. Thus, we consider only the simple Ratchet EIAs with and without cap whose the time 0 price as given in (3.3) and (3.4), the case of compound Ratchet EIAs will be treated in a subsequent research.

3.1. Fourier-Cosine Expansion Method

To compute the time 0 price of the EIAs, it is essential to compute the expectation about R_j in (2.7) and (2.9). In fact for any function $h(\cdot)$ about R_j, its expectation can be computed as

$$\mathbb{E}^{Q}\left[h\left(R_j\right)\right]=\mathbb{E}^{Q}\left[h\left(\exp\left(\Delta_j\right)\right)\right]=\int_0^{\infty}h\left(\exp\left(y\right)\right)f_{\Delta_j}\left(y\right)\mathrm{d}y, \tag{3.7}$$

where $f_{\Delta_j}(y)$ is the probability density function of Δ_j. Since $f_{\Delta_j}(y)$ is usually not know for Black-Scholes model and Heston model, we will replace it by its Fourier-Cosine expansion

$$f_{\Delta_j}\left(y\right)=\sum_{k=0}^{+\infty}{}' A_k\cos\left(k\pi\frac{y-a}{b-a}\right) \tag{3.8}$$

with

$$A_k:=\frac{2}{b-a}\int_a^b f_{\Delta_j}\left(y\right)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y\approx\frac{2}{b-a}\int_0^{\infty}f_{\Delta_j}\left(y\right)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \tag{3.9}$$

where a and b are the truncated bounds of the integration interval such that the truncated error is under control and the prime in the summation indicates that the first term in the summation is weighted by one-half. Comparing the approximation of A_k in (3.9) with the characteristic function of Δ_j, we have

$$A_k\approx\frac{2}{b-a}\mathrm{Re}\left[\phi_{\Delta_j}\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ak\pi}{b-a}\right)\right], \tag{3.10}$$

where $\phi_{\Delta_j}(u)$ is the characteristic function of Δ_j and $\mathrm{Re}\{\cdot\}$ denotes the real part of the argument.

Subsituting $f_{\Delta_j}(y)$ in (3.7) by its cosine expansion, we have

$$\mathbb{E}^{Q}\left[h\left(R_j\right)\right]=\int_0^{\infty}h\left(\exp\left(y\right)\right)\sum_{k=0}^{+\infty}{}' A_k\cdot\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \tag{3.11}$$

then applying the expression (3.10) of A_k and interchanging the summation and integration yields

$$\begin{aligned}&\mathbb{E}^{Q}\left[h\left(R_j\right)\right]\\&=\int_0^{\infty}h\left(\exp\left(y\right)\right)\sum_{k=0}^{+\infty}{}'\frac{2}{b-a}\mathrm{Re}\left[\psi_{\Delta_j}\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ak\pi}{b-a}\right)\right]\cdot\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \quad (3.12)\\&\approx\sum_{k=0}^{N-1}{}'\mathrm{Re}\left[\phi_{\Delta_j}\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ak\pi}{b-a}\right)\right]V_k,\end{aligned}$$

where N is the truncated number for the Fourier-Cosine expansion and V_k is the so-called the COS coefficient defined as

$$V_k:=\frac{2}{b-a}\int_a^b h\left(\exp\left(y\right)\right)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y.$$

$\{V_k;k=1,2,\cdots\}$ can be analytically derived as a close formula when the function $h(\exp(y))$ is specified for simple Ratchet and with or without cap, we will derive the formulas in next section.

3.2. Coefficients V_k for Ratchet EIAs

For the simple Ratchet EIAs with or without cap with payment as given in (2.7) and (2.9), the function $h(R_j)$ is specified as

$$h(R_j) = \min\left[\max\left[g, \alpha(R_j - 1)\right], c\right],$$

thus the coefficient V_k can be derived as

$$\begin{aligned}
V_k &= \frac{2}{b-a}\int_a^b h(\exp(y))\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
&= \frac{2}{b-a}\int_a^b \min\left[\max\left[g, \alpha(\exp(y)-1)\right], c\right]\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
&= \frac{2}{b-a}\left[\int_a^{\ln\left(\frac{g}{\alpha}+1\right)} g\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y + \int_{\ln\left(\frac{g}{\alpha}+1\right)}^{\ln\left(\frac{c}{\alpha}+1\right)} \alpha\left(\mathrm{e}^y - 1\right)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \right.\\
&\quad \left. + \int_{\ln\left(\frac{c}{\alpha}+1\right)}^{b} c\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y\right] \\
&= \frac{2}{b-a}\left[g\phi_k\left(a, \ln\left(\frac{g}{\alpha}+1\right)\right) + \alpha\chi_k\left(\ln\left(\frac{g}{\alpha}+1\right), \ln\left(\frac{c}{\alpha}+1\right)\right)\right.\\
&\quad \left. - \alpha\phi_k\left(\ln\left(\frac{g}{\alpha}+1\right), \ln\left(\frac{c}{\alpha}+1\right)\right) + c\phi_k\left(\ln\left(\frac{c}{\alpha}+1\right), b\right)\right],
\end{aligned} \tag{3.13}$$

where Equation (3.13) is obtained because the integration bound $[a,b]$ is sufficiently large that $a < \ln\left(\frac{c}{\alpha}+1\right)$ and $b > \ln\left(\frac{g}{\alpha}+1\right)$, and $\forall(p,q) \subset (a,b)$,

$$\begin{aligned}
\phi_k(p,q) &:= \int_p^q \cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
&= \begin{cases} \left[\sin\left(k\pi\frac{q-a}{b-a}\right) - \sin\left(k\pi\frac{p-a}{b-a}\right)\right]\cdot\frac{b-a}{k\pi}, & k \neq 0 \\ q - p, & k = 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\chi_k(p,q) &:= \int_p^q \mathrm{e}^y\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
&= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{q-a}{b-a}\right)\mathrm{e}^q - \cos\left(k\pi\frac{p-a}{b-a}\right)\mathrm{e}^p\right.\\
&\quad \left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{q-a}{b-a}\right)\mathrm{e}^q - \frac{k\pi}{b-a}\sin\left(k\pi\frac{p-a}{b-a}\right)\mathrm{e}^p\right].
\end{aligned} \tag{3.14}$$

Based on the computation formulae for the time 0 value for the EIAs contracts given above, we can find the break-even participation rate α^* by solving the equation of (3.2). The participation rate is used as a benchmark for the insurer to set the participation rate α for the EIAs contracts.

4. Numerical Test

4.1. Numerical Calculation under Black-Scholes Model and Heston Model

To test the accuracy and efficiency of the Fourier-Cosine expansion method, we use the formula by Fourier-Cosine expansion method in Section 3 to obtain both the time 0 price $V(0,\alpha)$ of ratchet EIAs for some given participation rate α to compare with the existing results by Monte Carlo simulation under the Black-Scholes model given in Hardy (2004) and the break-even participation rate α^* to compare them with the existing results under the Black-Scholes model given in Tiong (2000). We also provide the break-even participation rate computed by the Fourier-Cosine expansion method Heston model for some parameter cases.

To compare the results between the Fourier-Cosine expansion method in this paper and Monte Carlo simulation in Hardy (2004), we set the parameters of both the model and EIAs contract the same as in Hardy (2004), which the interest rate $r = 6\%$, volatility $\sigma = 0.25$, the minimum guaranteed return rate $g = 0\%$, the dividend rate $q = 2\%$, maturity time $T = 7$ and participation rate α and cap rate c are presented in **Table 1**. In **Table 1**, we present the price of EIAs under the Black-Scholes model with the Fourier-Cosine expansion method and the benchmark result by Monte Carlo simulation given in Hardy (2004). And we also present the absolute error in **Table 1**. The definition of absolute error is in the following equation:

$$\text{Absolute error} := \text{AE} = |\text{FC} - \text{MC}|,$$

where FC is the price of EIAs in the Fourier-Cosine expansion method and MC is the price of EIAs in the Monte Carlo simulation. As it is shown in **Table 1**, the approximation of Fourier-Cosine expansion results is as accurate as those obtained using the Monte Carlo simulation in Hardy (2004) up to two decimal places.

Table 1. The price and computation error of the simple ratchet EIA of Black-Scholes models. FC method stands for the price by Fourier-Cosine expansion method, and MC stands for the price by Monte Carlo simulation provided in Hardy (2004), % of price, AE stands for the absolute errors of the Fourier-Cosine expansion method to the Monte Carlo simulation.

Price of EIA Partition Rate α	Method	Cap Rate c			
		10%	15%	20%	30%
0.6	FC	83.6851	89.1147	92.8456	96.9644
	MC	83.6853	89.1154	92.8469	96.9651
	AE	0.0002	0.0007	0.0013	0.0007
0.8	FC	84.9961	91.7378	96.9181	103.7266
	MC	84.9959	91.7387	96.9184	103.7272
	AE	0.0002	0.0009	0.0003	0.0006
1.0	FC	85.8197	93.4476	99.6855	108.7400
	MC	85.8198	93.4488	99.6871	108.7416
	AE	0.0001	0.0012	0.0016	0.0016
1.2	FC	86.3831	94.6419	101.6656	112.5247
	MC	86.3830	94.6426	101.6675	112.5220
	AE	0.0001	0.0007	0.0019	0.0027

Then, the computation results of the break-even participation rate for the Black-Scholes model are presented in **Table 2**. In this table, the contract parameters are set to be the same as in Tiong (2000) with interest rate $r = 4\%, 5\%, 6\%$, volatility $\sigma = 0.03, 0.02$, the minimum guaranteed return rate $g = 3\%$, the dividend rate $q = 2\%, 1\%$, maturity time $T = 1$ and cap $c = 10\%, 12\%, 14\%$. Notice that the break-even participation rate α is higher than 1 for some cases. The main reason is that when the minimum guaranteed benefits are relatively lower than dividends provided by q, the discounted payments will be lower than the initial premiums, which makes the break-even participation rate higher than 1. For more information on the situation about the break-even participation rate is higher than 1 see, for example, Tiong (2000) and Jaimungal (2004).

For Heston model, the computation results are presented in **Table 3**. We also set the parameters the same as in Tiong (2000) with the interest rate $r = 4\%, 5\%, 6\%$, the mean of volatility $\overline{v} = 0.03, 0.02$, the variance of volatility $\gamma = 0.2$, the correlation of two Brownian motion $\rho = -0.5$ and $k = 3$, the minimum guaranteed return rate $g = 3\%$, the dividend rate $q = 2\%, 1\%$, maturity time $T = 1$ and cap $c = 10\%, 12\%, 14\%$.

Table 2. The break-even participation rate for Black-Scholes models.

q	Cap	$g = 3\%$					
		$\sigma = 0.03$			$\sigma = 0.02$		
		$r = 4\%$	$r = 5\%$	$r = 6\%$	$r = 4\%$	$r = 5\%$	$r = 6\%$
2%	10%	0.305204	0.651761	3.239664	0.429140	0.809608	2.194496
	12%	0.278614	0.484261	1.195217	0.398136	0.639872	1.065437
	14%	0.265895	0.421771	0.666464	0.383724	0.573124	0.828112
1%	10%	0.293616	0.605957	2.402629	0.407391	0.740946	1.772472
	12%	0.269237	0.459479	0.855894	0.380058	0.598238	0.960387
	14%	0.257560	0.403472	0.625091	0.367383	0.540802	0.766031

Table 3. The break-even participation rate for Heston models.

q	Cap	$g = 3\%$					
		$v = 0.03$			$v = 0.02$		
		$r = 4\%$	$r = 5\%$	$r = 6\%$	$r = 4\%$	$r = 5\%$	$r = 6\%$
2%	10%	0.490737	0.823581	1.645577	0.551115	0.887698	1.678759
	12%	0.464632	0.689075	1.025693	0.524565	0.756356	1.078675
	14%	0.453291	0.635331	0.852352	0.513247	0.703606	0.916962
1%	10%	0.462781	0.752731	1.418702	0.516372	0.806233	1.394885
	12%	0.440448	0.641269	0.927894	0.494186	0.699444	0.971297
	14%	0.430859	0.596108	0.786886	0.484881	0.656086	0.8419131

4.2. Sensitive Analysis for EIAs Contract

In this section, we present the sensitive analysis under the Black-Scholes model and the Heston model. In Figures 1-6, the results of break-even participation rate α^* to six model parameters are presented under the Black-Scholes model.In general, analysis results are intuitive. The break-even participation rate increases with the interest rate r, with dividend rate q, with maturity time T and decreases with volatility σ, with the minimum guaranteed return rate g, with cap c. In Figure 1, the break-even participation rate is 0 when r is from 0% to 3%. The main reason is that when the interest rate r is lower than the minimum

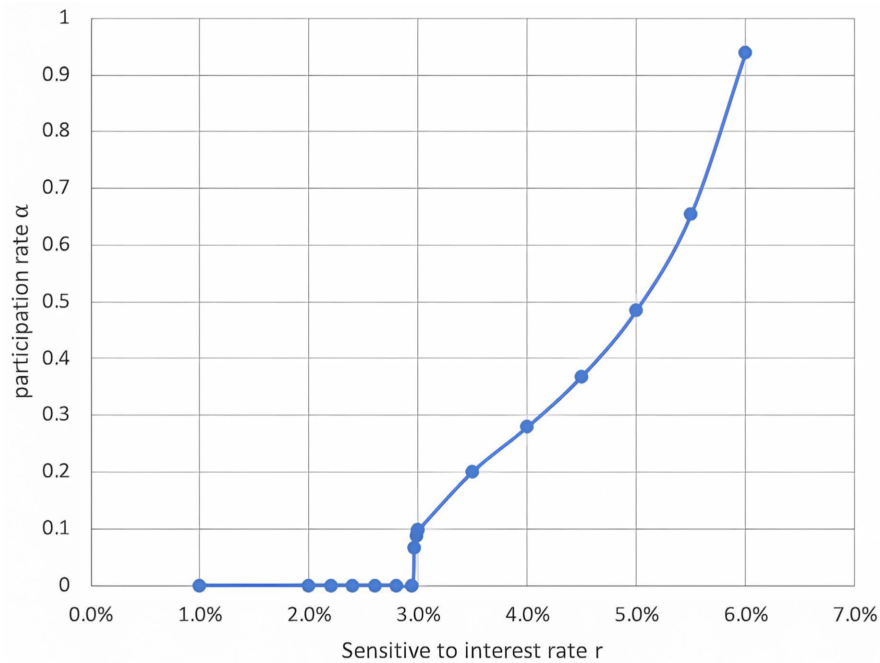

Figure 1. Sensitive of break-even participation rate α^* to interest r under Black-Scholes model.

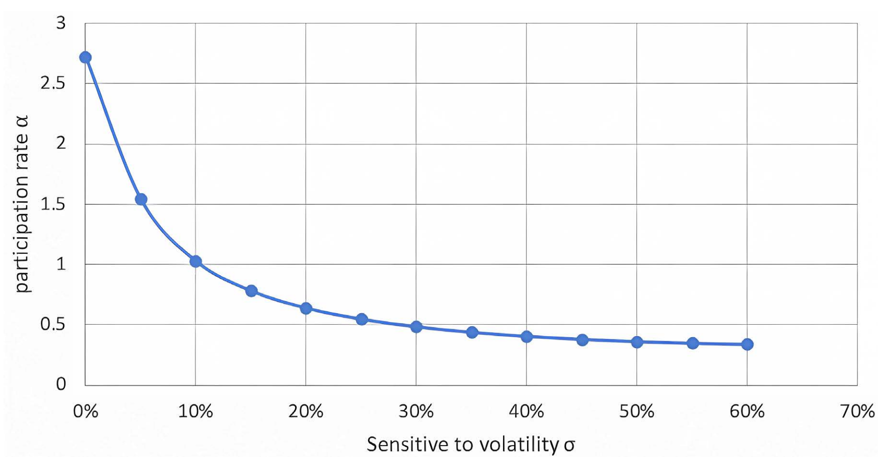

Figure 2. Sensitive of break-even participation rate α^* to volatility σ under Black-Scholes model.

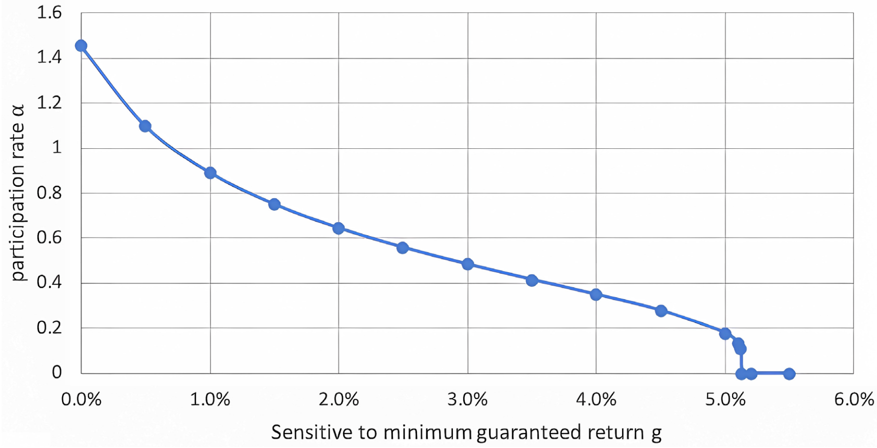

Figure 3. Sensitive of break-even participation rate α^* to minimum guaranteed return g under Black-Scholes model.

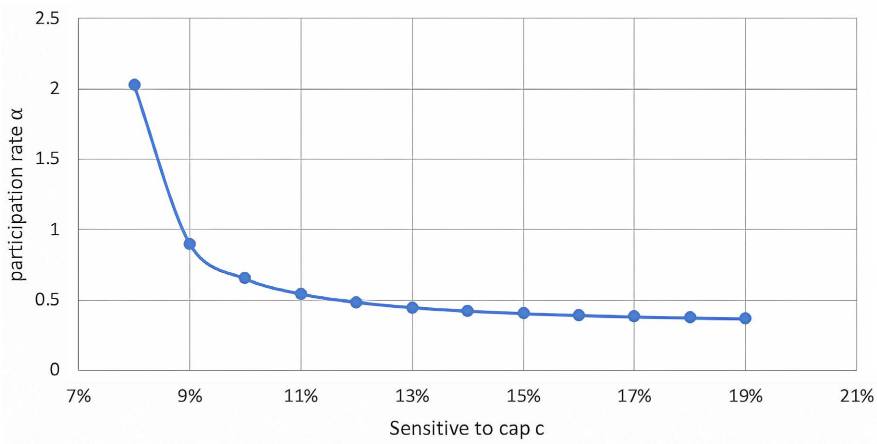

Figure 4. Sensitive of break-even participation rate α^* to cap c under Black-Scholes model.

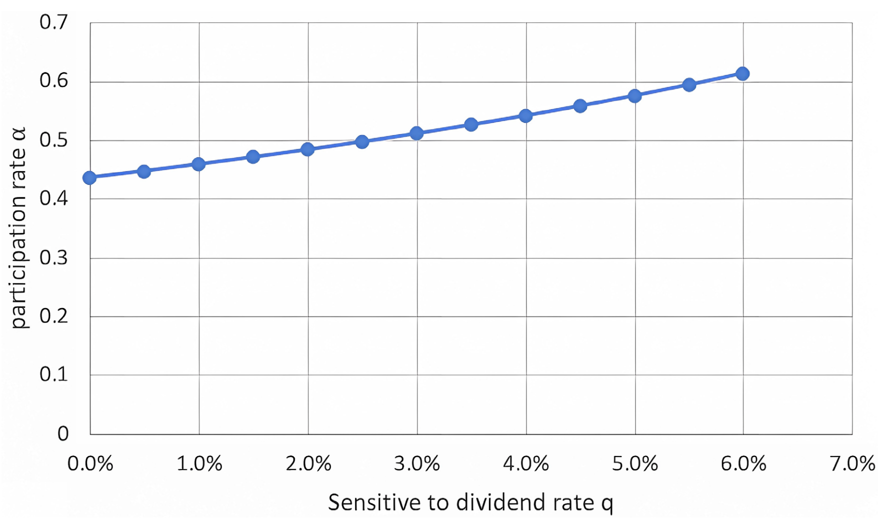

Figure 5. Sensitive of break-even participation rate α^* to dividend rate q under Black-Scholes model.

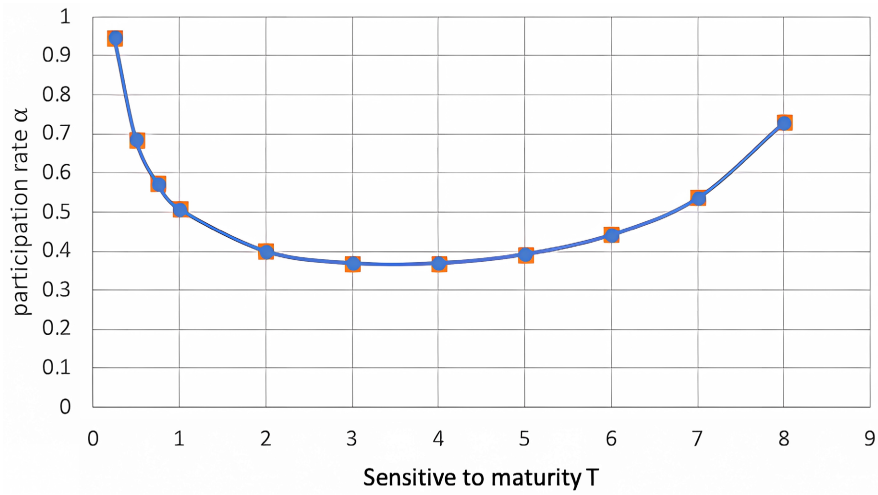

Figure 6. Sensitive of break-even participation rate α^* to maturity time T under Black-Scholes model.

guaranteed return rate g, the policyholder's benefits are all from the minimum guaranteed return, which makes the break-even participation rate 0. And when $r > g$, as the r increases, the insurance company can obtain more benefits in the investment market, then increases the break-even participation rate. But as the volatility σ increases, the investment risk increases, and the costs of hedging will increase, then the break-even participation rate decreases. In **Figure 3**, the increasing g will increase the cost of minimum guaranteed benefits, then the insurance company will decrease the break-even participation rate to reduce the systematic risk. The 0 break-even participation rate appears for the same reason in **Figure 1**. A higher cap will cause fewer benefits from the EIAs product, then the break-even participation rate will decrease. For dividend rate q, a higher dividend means a good investment return in the market, then policyholders can obtain more benefits and thus make a higher break-even participation rate. And a longer maturity time can reduce the cost of guaranteed benefits and increase the investment return, thus the break-even participation rate can increase in **Figure 6**.

In **Figures 7-11**, we also present the sensitive analysis for five model parameters on the Heston model. Notice that the break-even participation rate decreases when the mean of volatility $\bar{v}$ increases in **Figure 7**. The main reason is that the increasing volatility will increase the risk of investment and thus, the value of financial products will be more expensive, which causes the insurance companies to reduce the break-even participation rate. In **Figure 8**, this reason can also explain the decreasing trend of break-even participation rate to initial volatility v_0. In **Figure 9** and **Figure 10**, k and γ have an insignificant influence on the break-even participation rate. In **Figure 11**, the decreasing trend is that the higher correlation will cause a higher systematic risk, and thus cause a lower participation rate.

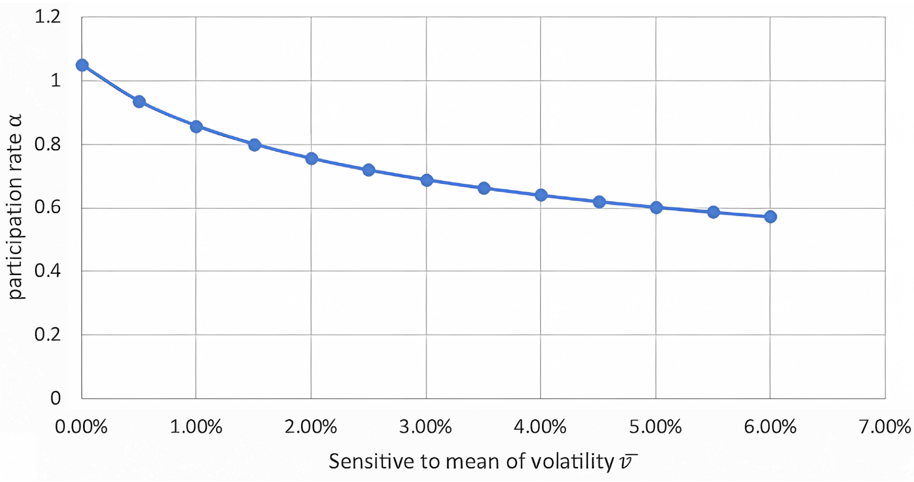

Figure 7. Sensitive of break-even participation rate α^* to mean of volatility $\bar{v}$ under Heston model.

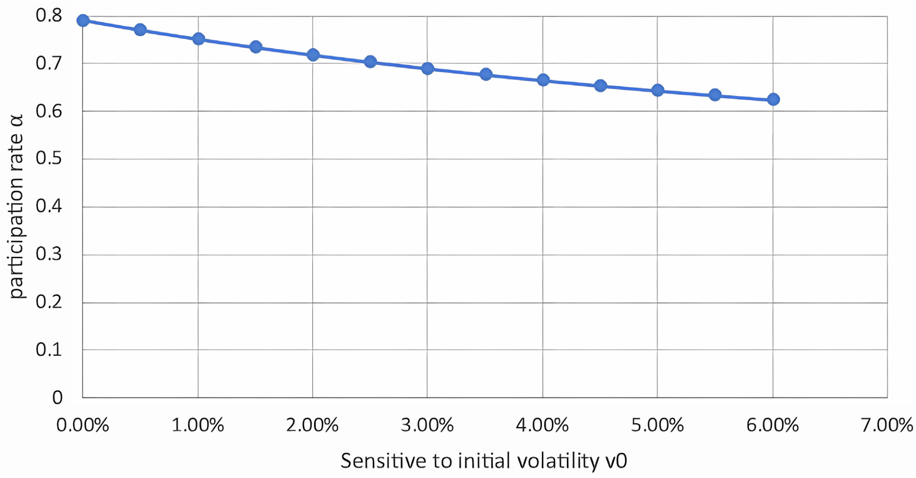

Figure 8. Sensitive of break-even participation rate α^* to initial volatility v_0 under Heston model.

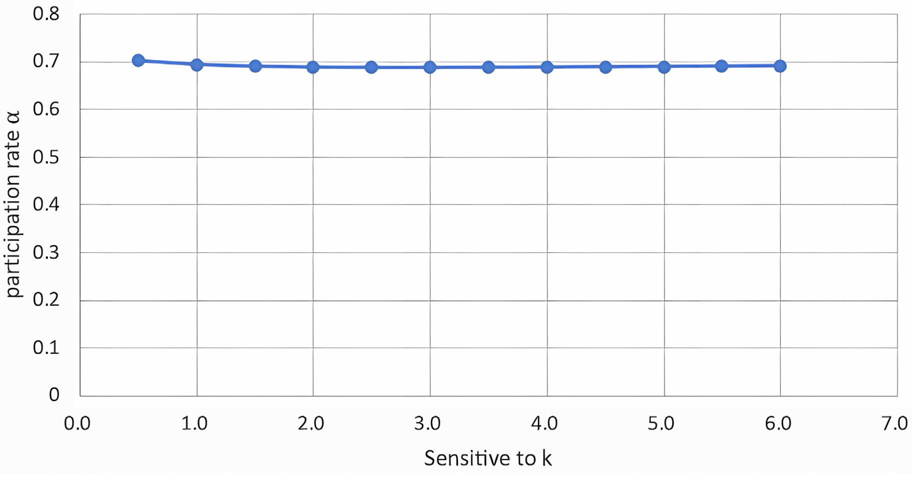

Figure 9. Sensitive of break-even participation rate α^* to k under Heston model.

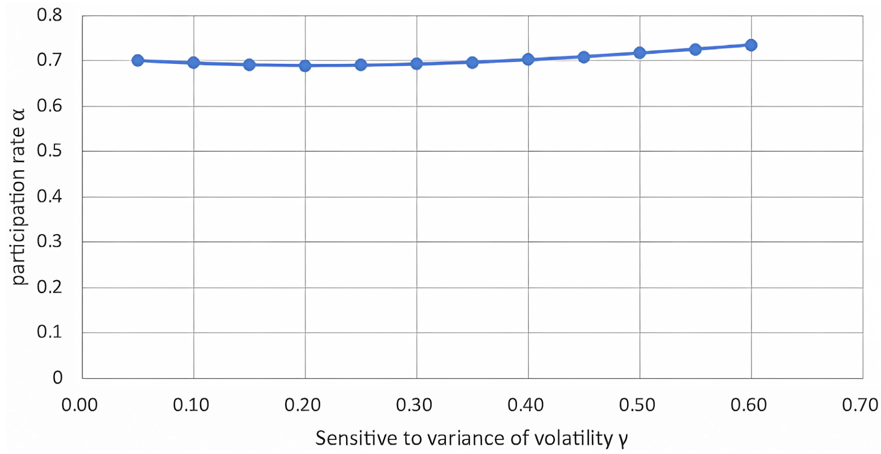

Figure 10. Sensitive of break-even participation rate α^* to variance of volatility γ under Heston model.

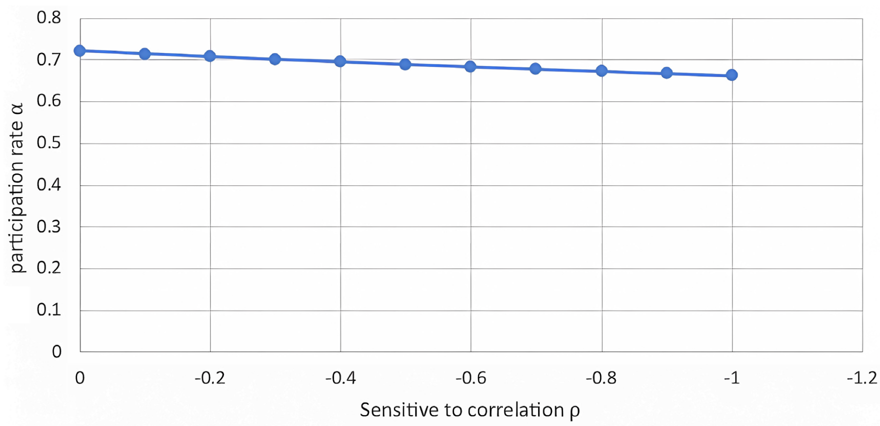

Figure 11. Sensitive of break-even participation rate α^* to correlation ρ under Heston model.

5. Conclusion

Considering the long maturity of the EIAs, the volatility of the equity index might evolute as time going. This paper introduces the Heston volatility model in the pricing of EIAs and applies the Fourier-Cosine expansion method to derive an approximation formula for the price of the EIAs. The numerical test examines the accuracy of the Fourier-Cosine expansion method and analyzes the sensitivity of the break-even rate of the EIAs to the model and product parameters.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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