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Multiplicative Normal Noise and Nonconcavity in the Value of Information

Stefan Behringer

Sciences Po, Department of Economics, 28 rue des Saints Pères, Paris, France
Email: stefan.behringer@sciencespo.fr

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Abstract

This paper investigates a Bayesian inverse problem of a price setting monopolist facing a random demand. In contrast to previous investigations an unknown true market potential of demand is distorted by two independent Gaussian errors, a zero-mean additive and a unity-mean multiplicative one. The multi-period game allows for learning from realized market demands (signals). Interestingly increasing the level of noise of a multiplicative error in this dynamic setting can actually improve the Value of Information of signals to the firm, a result that cannot hold for a single additive error or in a static context.

Keywords

Value of Information, Inverse Problems, Bayesian Learning, Monopoly, Multiplicative Noise, Nonconcavity

1. Introduction

This present paper investigates a specific Bayesian inverse problem in the context of a multi-stage monopoly stochastic pricing game that allows for learning between periods and the implied *Value of Information* (*VoI*) of signals. Learning takes place via Bayesian updating of priors about an uncertain market potential. The true market potential is reflected in the demand for the product, which the firm knows to be normally distributed. A related learning process allowing for external signals and advertising concerns but additive noise only has been employed in Weber (2019) but the motivation is also closely linked to the experimentation literature, c.f. Mirman et al. (1993).

In contrast to the above, the present model allows for two Normal observational errors, one mean-zero additive which is frequently encountered in learning settings and well known from standard econometric methods and one mean-one multiplicative as for example investigated in Porter (1983). Within the industrial organization literature both types of errors have been allowed for in e.g. Slade's (1989) duopoly model and Saloner's (1984) model of market entry, but the formulations neither allow for varying the noises independently nor do they focus on the implied *Value of Information*.

The idea of investigating the conjunction of the two independent Gaussian normal errors is motivated by the observation in Dunlop (2019) that having a mixture of both additive and multiplicative noises, even if independent and therefore uncorrelated, weakens conditions that have to be imposed to generate posteriors that are continuous in the pertubations.[1]

In the context of a multi-period price setting monopoly game it is shown that the *Value of Information* derived from per-period signals can increase when also a multiplicative error is allowed for and is no longer monotonic in the confidence parameters. In contrast to the well known nonconcavity in the Value of Information investigated by Radner & Stiglitz (1984) (c.f. Chade & Schlee (2002) or De Lara & Gilotte (2007)) in the present model both the signal and the state spaces are unbounded.

2. The Model

Let the demand for the firm's product be given by a linearized random demand

$$q = \theta - \varepsilon p + \gamma \tag{1}$$

at price *p*, with multiplicative Normal noise $\varepsilon \sim N\left(1, \sigma_\varepsilon^2\right)$ as e.g. in Porter (1983), additive Normal noise $\gamma \sim N\left(0, \sigma_\gamma^2\right)$ and market potential $\theta \sim N\left(\mu, \sigma^2\right)$ that is drawn only once. All random variables are independent and identically distributed (i.i.d.) and the firm's prior mean and variance for θ are denoted by μ_1, σ_1^2. The firm's profit at unit cost c is $\Pi = \left(p - c\right)q$ with optimal prices being found as

$$p^* = \frac{\theta + \gamma + c\varepsilon}{2\varepsilon} \tag{2}$$

and optimal profits as

$$\Pi^* = \frac{\left(\theta + \gamma - c\varepsilon\right)^2}{4\varepsilon}. \tag{3}$$

When the firm sets optimal prices, estimated first and second moments for θ, which is drawn once, and for ε and γ, which are drawn each period, have to be considered. Hence the firm will set price as

$$p^* = \frac{\mu + c}{2} \tag{4}$$

[1]Generalizing for non-zero correlation among the additive and the multiplicative error does not add to our findings qualitatively. In a fully dynamic setting with a stochatically growing market potential drawn at each stage and a noisy Bayesian updating process as investigated in Behringer (2021), the filtering process is known to accommodate such correlations and even non-Gaussian noise (see Stratonovich (1970)).

which is fix and given when the realization of the demand potential determines quantity, so that expected profits are

$$E\left\{\Pi^*\right\} = \frac{(\mu - c)^2}{4}. \tag{5}$$

2.1. Timing of the Game

The firm has initial priors over the time-invariant market potential θ given by $\left(\mu_1, \sigma_1^2\right)$. The timing of the game is:

$t = 1$ The firm sets price p_1 according to μ_1, generates profit Π_1 and observes demand $q_1\left(\theta, \mu_1, \gamma_1, \varepsilon_1\right)$ which acts as a noisy signal about the true θ. The firm uses the signal for *learning* to find updated prior moments $\left(\mu_2, \sigma_2^2\right)$.

$t = 2$ The firm sets price p_2 according to μ_2, generates profit Π_2 and observes demand $q_2\left(\theta, \mu_2, \gamma_2, \varepsilon_2\right)$ which acts as a noisy signal about the true θ. The firm uses the signal for *learning* to find updated prior moments $\left(\mu_3, \sigma_3^2\right)$.

$t = 3$ The firm sets price p_3 according to μ_3, generates profit Π_3. The game is solved by backward induction. In the last period, for given priors, the game is simply the maximization problem without any learning process as presented above.

2.2. The Learning Process

The *learning process* itself proceeds as follows: Setting prices as from (4) with $\theta \sim N\left(\mu_1, \sigma_1^2\right)$ in the first stage leads to realized demand $q_1\left(\theta, \mu_1, \gamma_1, \varepsilon_1\right)$ which can be turned into an unbiased signal conditional on the market potential as

$$\tilde{S}_1 \middle| \theta = q_1 + \frac{\mu_1 + c}{2} = \theta + \gamma_1 + \left(1 - \varepsilon_1\right)\frac{\mu_1 + c}{2} \tag{6}$$

$$\tilde{S}_1 \middle| \theta \sim N\left(\theta, \sigma_\gamma^2 + \frac{\left(\mu_1 + c\right)^2}{4}\sigma_\varepsilon^2\right).$$

In contrast to having an additive error γ only, the variance of this signal increases in the price components μ_1, c and the multiplicative error variance σ_ε^2.

From the perspective of the agent (once the random variables are realized but signals not yet observed) an unbiased conditional signal can be constructed as

$$\tilde{S}_1 \middle| \varepsilon, \gamma = q_1 + \frac{\mu_1 + c}{2} = \theta + \gamma + (1 - \varepsilon)\frac{\mu_1 + c}{2} \sim N\left(\theta, \sigma_1^2\right) \tag{7}$$

Bayesian updating of the prior in the next period yields

$$\mu_2 = \lambda_1 S_1 + \left(1 - \lambda_1\right)\mu_1 \tag{8}$$

where

$$\lambda_1 = \frac{\sigma_1^2}{\sigma_\gamma^2 + \frac{\left(\mu_1 + c\right)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2} \tag{9}$$

determines the weight the agent puts on the signal relative to the prior. From the agent's view prior to signal revelation:

$$\tilde{\mu}_2 = \lambda_1 \tilde{S}_1 + (1-\lambda_1)\mu_1. \tag{10}$$

So expected next period profit given signal (7) is

$$\begin{aligned} E\{\Pi_2^*\} &= E\left\{\frac{(\theta+\gamma-c\varepsilon)^2}{4\varepsilon}\right\} = E\left\{\frac{(\tilde{\mu}_2 - c)^2}{4}\right\} \\ &= \frac{1}{4}\left(\lambda_1\tilde{S}_1 + (1-\lambda_1)\mu_1 - c\right)^2 + \frac{1}{4}Var\left(\lambda_1\tilde{S}_1\right) \\ &= \frac{1}{4}(\mu_1 - c)^2 + \underbrace{\frac{1}{4}\left(\frac{\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right)^2\left(\sigma_1^2\right)}_{\text{VoI1}} \end{aligned} \tag{11}$$

where μ_1 is fix and given as it is employed in the set price.

It appears that the signal has an expected pecuniary value to the monopolist that is related to its moments which elegantly overcomes the traditional problem of agreeing on the correct units with which to measure information. This *Value of Information,* denoted by the horizontal curly bracket VoI1 in (11), is strictly positive, increasing and *convex* in the initial variance of the firm's estimation of the potential as the signal is more valuable the higher is the initial uncertainty about θ.

The signal however is plagued with two independent errors, one being multiplicative in the chosen price. Hence this signal has a lower Value of Information the higher its variance and the price, for which it is decreasing and *convex*, i.e. decreasing at a decreasing rate. If only an additive error is employed, i.e. $\sigma_\varepsilon^2 = 0$, VoI1 is unambiguously higher.

In the next period, given μ_2 is used in the next price $p_2^* = (\mu_2 + c)/2$, from the realized demand $q_2(\varepsilon_2, \gamma_2)$ again a conditional signal can be constructed as

$$\tilde{S}_2 \big| \theta \sim N\left(\theta, \sigma_\gamma^2 + \frac{(\mu_2+c)^2}{4}\sigma_\varepsilon^2\right). \tag{12}$$

The updated variance of the potential is then

$$\sigma_2^2 = \frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2} < \sigma_1^2 \tag{13}$$

increasing and concave in σ_1^2.

This updated variance can now be used to find the next period profit as

$$\Pi_3^* = \frac{1}{4}(\mu_2 - c)^2 + \frac{1}{4}\left(\frac{\sigma_2^2}{\sigma_\gamma^2 + \frac{(\mu_2+c)^2}{4}\sigma_\varepsilon^2 + \sigma_2^2}\right)^2 (\sigma_2^2)$$

so that expected profits are

$$E\{\Pi_3^*\} = E\left\{\frac{1}{4}(\mu_2 - c)^2\right\} + \underbrace{\frac{1}{4}\frac{\left(\frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right)^3}{E\left\{\left(\sigma_\gamma^2 + \frac{(\mu_2+c)^2}{4}\sigma_\varepsilon^2 + \frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right)^2\right\}}}_{E\{\mathrm{VoI2}\}} \quad (14)$$

from the perspective of the first period. This expected two period *Value of Information*, denoted by the horizontal curly bracket $E\{\mathrm{VoI2}\}$ in (14).

For this expected profit term one has that

$$E\left\{\frac{1}{4}(\mu_2 - c)^2\right\} = \frac{1}{4}(\mu_1 - c)^2 + \frac{1}{4}\left(\frac{\sigma_1^2}{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right) + \sigma_1^2}\right)^2 (\sigma_1^2). \quad (15)$$

The expectation of the VoI2 term is more involved due to the expected μ_2 prior, which needs to be evaluated from the first period where $\mu_2 \sim N(\mu_1, \sigma_1^2)$.

Taking this expectation one finds

$$E\{\mathrm{VoI2}\} = \frac{1}{4}\frac{\left(\frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right)^3}{\begin{gathered}\left(\frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right)^2 + 2\left(\left(\frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right) + \frac{\sigma_\varepsilon^2}{4}\sigma_1^2 + \sigma_\gamma^2\right)\left(\frac{\left(\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_1^2}{\sigma_\gamma^2 + \frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2 + \sigma_1^2}\right) \\ + \left(\frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)^2 + 2\left(\frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right)\sigma_\gamma^2 + (\sigma_\gamma^2)^2 \\ + \frac{3}{16}(\sigma_\varepsilon^2)^2(\sigma_1^2)^2 + \frac{3}{2}\sigma_\varepsilon^2\left(\left(\frac{(\mu_1+c)^2}{4}\sigma_\varepsilon^2\right) + \frac{1}{3}\sigma_\gamma^2\right)\sigma_1^2\end{gathered}} \quad (16)$$

making use of results for non-central moments of higher-order Gaussian random variables (c.f. Holmquist, 1988).

3. Investigations of the Value of Information

Again it appears that the signal, now being based on updated priors, has an expected value to the agent. As with VoI1, the signal contains two independent errors and for large error variances the VoI2 eventually goes to zero, i.e.

$$\lim_{\sigma_\gamma^2 \to \infty} (\mathrm{VoI2}) = \lim_{\sigma_\varepsilon^2 \to \infty} (\mathrm{VoI2}) = 0 \tag{17}$$

Also, if only an additive error matters or is employed (i.e. $\sigma_\varepsilon^2 = 0$) a high enough initial prior variance leads to

$$\lim_{\sigma_1^2 \to \infty} \left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 = 0 \right) = \frac{1}{16} \sigma_\gamma^2 \tag{18}$$

because

$$\left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 = 0 \right) = \frac{1}{4} \frac{\left(\dfrac{\sigma_\gamma^2 \sigma_1^2}{\sigma_\gamma^2 + \sigma_1^2} \right)^3}{\left(\sigma_\gamma^2 + \dfrac{\sigma_\gamma^2 \sigma_1^2}{\sigma_\gamma^2 + \sigma_1^2} \right)^2} \tag{19}$$

and the updated priors converge to

$$\lim_{\sigma_1^2 \to \infty} \left(\sigma_2^2 \middle| \sigma_\varepsilon^2 = 0 \right) = \lim_{\sigma_1^2 \to \infty} \left(\frac{\sigma_\gamma^2 \sigma_1^2}{\sigma_\gamma^2 + \sigma_1^2} \right) = \sigma_\gamma^2 . \tag{20}$$

whereas if also a multiplicative error is used in the model

$$\lim_{\sigma_1^2 \to \infty} (\mathrm{VoI2}) = 0 \tag{21}$$

because

$$\lim_{\sigma_1^2 \to \infty} \left(\sigma_2^2 \right) = \lim_{\sigma_1^2 \to \infty} \left(\frac{\left(\sigma_\gamma^2 + \dfrac{(c+\mu_1)^2}{4} \sigma_\varepsilon^2 \right) \sigma_1^2}{\sigma_\gamma^2 + \dfrac{(c+\mu_1)^2}{4} \sigma_\varepsilon^2 + \sigma_1^2} \right) = \sigma_\gamma^2 + \frac{(c+\mu_1)^2}{4} \sigma_\varepsilon^2 \tag{22}$$

i.e. a constant with the remaining σ_1^2 appearing only in the denominator of the VoI2 term resulting from the initial expectations about the second period price.

It can be seen that the numerator of $\left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 > 0 \right)$ is always larger than the numerator of $\left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 = 0 \right)$ because the updated variance of the potential is increased in the presence of an additional error in the signal but it can also be shown that the denominator of $\left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 > 0 \right)$ comprising the total noise in the system is larger than the denominator of $\left(\mathrm{VoI2} \middle| \sigma_\varepsilon^2 = 0 \right)$ leading to an ambiguous net effect.

As a result and most interestingly, contrary to the one-shot case, a model in which only an additive error matters, i.e. $\sigma_\varepsilon^2 = 0$, does no longer always yield a strictly higher Value of Information. A complete analytical analysis suffers from the high dimensionality of VoI2 and this finding will be determined by example that allows to focus on the relevant non-negative orthant of the system.

Proposition 1. *A multi-stage monopoly pricing game with multiplicative and additive uncertainty can generate a Value of Information that is non-monotonic in the confidence parameters* σ_ε^2 *and* σ_γ^2.

Proof by example:

Let $\sigma_1^2 = 15, \sigma_\gamma^2 = 1$, the VoI2 graph is:

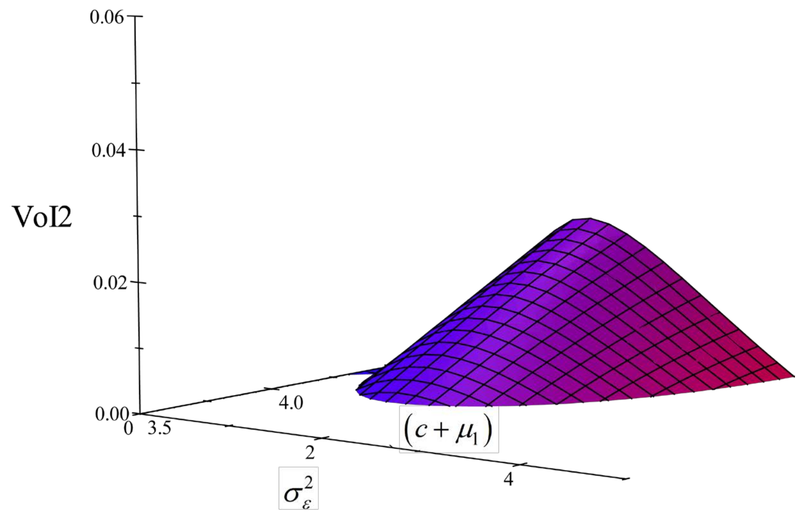

Figure 1. Value of information under uncertainty 1.

Where VoI2 gives the difference of the Value of Information in a two-stage model with and without a multiplicative error. Hence from **Figure 1** one observes that the VoI difference can be positive but that there is a minimal level of $(c+\mu_1)$ which is necessary to have such positive VoI2 difference.

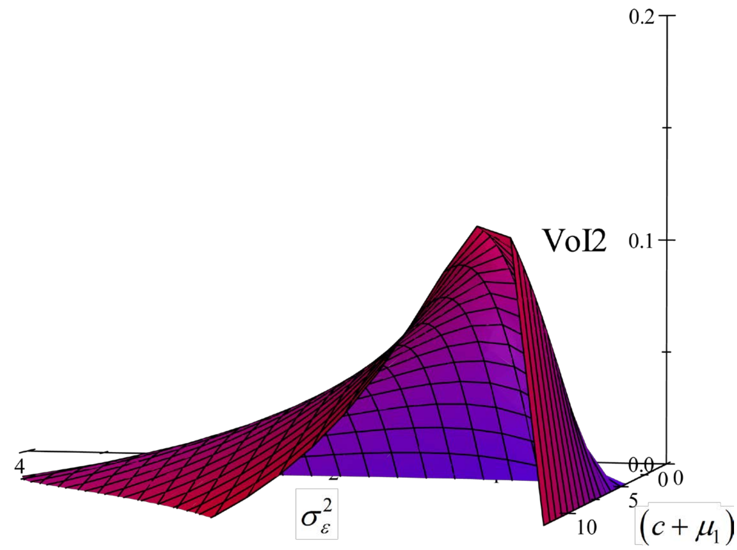

Figure 2. Value of information under uncertainty 2.

A different perspective of the VoI2 graph of the example shows the local concavity of the VoI difference in the noise of the multiplicative error term in the non-negative orthant for given $(c+\mu_1)$ (**Figure 2**) and also the strict inferiority of having $\sigma_\varepsilon^2 = 0$ over some $\sigma_\varepsilon^2 > 0$. Further examples varying the σ_1^2 and σ_γ^2 parameters reveal that the necessary level of $(c+\mu_1)$ and the local concavity in σ_ε^2 are qualitatively robust results.■

4. Conclusion

In the analysis of the period one sub-game or a one-period game, the *Value of Information* of a signal to the firm decreases monotonically in the multiplicative and the additive error noise. Contrary to this, the proposition reveals that in a model with multiple stages firms may benefit from additional signal noise in a multiplicative error term, contrary to additional additive error term noise.

This is the case if the multiplicative noise is moderate whereas the chosen action of the firm has a significant impact on the signal. Whence models that also accommodate a multiplicative error may yield higher *Values of Information* than models which do not once the interaction lasts more than one period. This non-monotonicity implies nonconcavities also in related settings where information is used as an input for economic production.

Further research investigating the structural origins and generalizing the assumptions for this finding is warranted. The blossoming recent literature extending the foundational work of Radner & Stiglitz (1984) indicates that nonconcavities in the *Value of Information* are of interest to the economic audience.

As noted in Dunlop (2019), allowing for a mixture of both, additive and multiplicative noises may improve the general requirements for Bayesian updating. Multiplicative noise may alternatively also be viewed as state-dependent additive noise thereby accounting for model error. On the other hand present textbooks such as Vives (2010) or Veldkamp (2011) reveal a corresponding lacuna in the economics literature compared to other disciplines. Allowing for additive and multiplicative noise always increases realism and robustness in economic settings where the origins and impacts of uncertainty may not be comfortably pinned down in a purely additive fashion, or the latter is chosen for mere mathematical convenience.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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