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The Black-Scholes Merton Model

—Implications for the Option Delta and the Probability of Exercise

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Abstract

This paper analyzes the implications of the Black-Scholes-Merton model of option pricing, for the deltas of call and put options and their respective probabilities of exercise at expiration. It derives a threshold value of the stock price and shows that in certain cases the options will have a delta in excess of 0.50, and will also have more than a 50% probability of exercise, while other options will have a delta that is lower than 0.50 and a probability of exercise that is lower than 50%. Similar results are obtained for the Garman-Kohlhagen model, which is an extension of the Black-Scholes Merton model, for valuing foreign currency options.

Keywords

Black-Scholes-Merton, Garman-Kohlhagen, Option Delta, Continuous Dividend Yield, Foreign Exchange Options

1. Introduction

Black and Scholes (1973) as we know, obtained exact formulas for valuing call and put options on non-dividend paying stocks, by assuming that stock prices follow a lognormal process. The formulas obtained by them are:

$$C_{E,t} = S_t N(d_1) - X e^{-r(T-t)} N(d_2) \tag{1}$$

and

$$P_{E,t} = X e^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{2}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (4)$$

$N(X)$ is the cumulative probability distribution function for a standard normal variable, and σ is the standard deviation of the rate of return on the stock.

2. Symbols

- S_t is the stock price.
- r is the risk-less rate of interest per annum.
- X is the exercise price of the option.
- σ is the standard deviation of the rate of return on the stock.
- $C_{E,t}$ is the premium of a European call option.
- $P_{E,t}$ is the premium of a European put option.
- $T-t$ is the time until expiration.
- N is the cumulative probability distribution function for a standard normal variable.

3. Significance

Black and Scholes were the first to develop a closed form solution for the valuation of European call and put options. It was a significant step forward from the no-arbitrage properties for options, which had been derived by Merton (1973). Merton (1973) extended the Black-Scholes model to value European options on a stock that pays a continuous dividend yield. Garman and Kohlhagen (1983) adapted the Merton model to value European options on foreign currencies.

4. Interpretation of $N(d_1)$ and $N(d_2)$

The Black-Scholes formula for call options states that

$$C_{E,t} = S_t N(d_1) - X\mathrm{e}^{-r(T-t)} N(d_2)$$

The option premium is invariant to the risk preferences of investors and is consequently valid in a world characterized by risk neutrality. A risk neutral investor would value the option as the discounted value of its expected payoff. Consequently, since

$$C_{E,t} = \mathrm{e}^{-r(T-t)} \left[S_t \mathrm{e}^{r(T-t)} N(d_1) - XN(d_2) \right] \quad (5)$$

it is obvious that

$$\left[S_t \mathrm{e}^{r(T-t)} N(d_1) - XN(d_2) \right]$$

is the expected payoff from the option from the perspective of a risk neutral investor.

$S_t \mathrm{e}^{r(T-t)} N(d_1)$ is the expected value of a variable in a risk neutral world, that is equal to S_T if the option is exercised, and is equal to zero otherwise. $N(d_2)$ is the probability that the option will be exercised in a risk neutral world. If the option is exercised there will be an outflow of X else the outflow will be zero. Consequently, $XN(d_2)$ is the expected outflow on account of the exercise price.

The formula for puts states that

$$\begin{aligned} P_{E,t} &= X\mathrm{e}^{-r(T-t)}N(-d_2) - S_t N(-d_1) \\ &= \mathrm{e}^{-r(T-t)}\left[XN(-d_2) - S_t\mathrm{e}^{r(T-t)}N(-d_1)\right] \end{aligned} \tag{6}$$

If $N(d_2)$ is the probability that a call option with an exercise price of X is exercised in a risk neutral world then $1-N(d_2)$ or $N(-d_2)$ is the probability that a put with the same exercise price will be exercised. Thus $XN(-d_2)$ is the expected inflow on account of the exercise price. $S_t\mathrm{e}^{r(T-t)}N(-d_1)$ is the expected value of a variable that will be equal to S_T if the option is exercised and is equal to zero otherwise.

5. Delta

Delta represents the rate of change of the option premium with respect to the price of the underlying asset, keeping all the other variables constant. In other words, delta is the partial derivative of the option price with respect to the price of the underlying asset. The delta of a call option will always lie between zero and one. For deep out of the money options, delta will be close to zero, whereas for options that are very deep in the money, the delta will be close to one. It must be remembered that delta represents the change in the option price for an infinitesimal change in the asset price. Consequently, as the asset price changes, so will the delta. Thus, while we may interpret a delta value of say 0.45 to mean that for a one dollar move in the asset price, the option premium will move by 45 cents, it must be remembered that is accurate only for an infinitesimal change in the asset price.

Even if the asset price were to remain constant, the delta of an option will change with the sheer passage of time. As the call option approaches maturity, the delta will tend towards one if the option were to be in the money, whereas it will tend towards zero, if the option were to be out of the money.

For put options, quite obviously delta will be between 0 and −1.

By differentiating the Black-Scholes formula with respect to the asset price, we can show that the delta for a European call option is equal to $N(d_1)$ and that for a European put option is equal to $-N(-d_1)$. As we would expect the delta for a call is positive, while that for a put is negative.

6. Properties of $N(d_1)$

Let us consider d_1.

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

For $\ln\left(\frac{S_t}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)$

to be equal to zero, we need that $\ln\left(\frac{S_t}{X}\right)=-\left(r+\frac{\sigma^2}{2}\right)(T-t)$ Thus

$$S_t^* = Xe^{-\left(r+\frac{\sigma^2}{2}\right)(T-t)} < X \tag{7}$$

For lower values of the stock price d_1 will be less than zero and the delta of the option will be less than 0.50. Thus all at the money and in the money calls will have a delta in excess of 0.50 as will come out of the money calls. For stocks whose price is less than the critical value of S_t^*, delta will be less than 0.50 and all such options will be out of the money.

7. Properties of $N(d_2)$

$$d_2 = \frac{\ln\left(\frac{S_t}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Consider the case where $\left(r-\frac{\sigma^2}{2}\right)(T-t)>0$. The probability of exercise being exactly 50% is given by:

$$S_t^* = Xe^{-\left(r-\frac{\sigma^2}{2}\right)(T-t)} < X \tag{8}$$

Thus all in the money and at the money calls, as well as some out of the money calls have a probability of exercise in excess of 50%. For puts the probability of exercise is $N(-d_2)=1-N(d_2)$. Hence in this case only in the money puts have a probability of exercise in excess of 50%.

If $\left(r-\frac{\sigma^2}{2}\right)(T-t)=0$, only at the money and in the money calls have a probability of exercise in excess of 50%. All out of the money calls will have a probability of exercise than is less than 50%.

If r is less than $\frac{\sigma^2}{2}$, then

$$S_t^* = Xe^{-\left(r-\frac{\sigma^2}{2}\right)(T-t)} > X \tag{9}$$

Thus only in the money options with a stock price in excess of the threshold will have a probability of exercise in excess of 50%. All other in the money calls, and at the money as well as out of the money calls, will have a probability of exercise that is less than 50%. In this case all in the money and at the money puts, as well as some out of the money puts will have a probability of exercise in excess of 50%.

8. The Merton Extension and Foreign Exchange Options

Before we go on to analyze options on foreign currencies, let us first derive an equivalent of the Black-Scholes formula for a stock that pays a continuous dividend yield. This model was derived by Merton and has implications for option pricing models for other financial assets such as stock indexes and foreign currencies.

Take the case of a stock that evolves from a current price of S_t to a value of S_T by time T. If this stock were to pay a continuous dividend yield at the rate of δ, the dividend can be construed as a leakage of value from it. Thus, if it were to pay such a dividend it would evolve to a value of $S_T e^{-\delta(T-t)}$ by time T. This price movement is identical to what a non-dividend paying stock which is currently price at $S_t e^{-\delta(T-t)}$ would experience. Thus the Black-Scholes formula can be applied to the stock paying a continuous yield of δ, if we replace its price S_t with $S_t e^{-\delta(T-t)}$.

In the Black-Scholes model, if we substitute S_t with $S_t e^{-\delta(T-t)}$ we get

$$C_{E,t} = S_t e^{-\delta(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{10}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{11}$$

The corresponding formula for European puts on a stock that pays a continuous dividend yield is

$$P_{E,t} = X e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1)$$

where d_1 and d_2 are as defined for the call options.

9. Delta

For a European call: Delta=

$$e^{-\delta(T-t)} N(d_1) \tag{12}$$

For a European put: Delta=

$$-e^{-\delta(T-t)} N(-d_1) \tag{13}$$

10. The Garman Kohlhagen Model

This model is an extension of the Black-Scholes model for foreign currency options. According to it

$$C_E = S_t e^{-r_f(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \tag{14}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{15}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{16}$$

This is the equivalent of the Merton model where the risk-less rate in the foreign country plays the role of the dividend yield.

11. Properties of $N(d_1)$

Let us consider d_1.

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

For $\ln\left(\frac{S_t}{X}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)$

to be equal to zero, we need that $\ln\left(\frac{S_t}{X}\right) = -\left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)$ Thus

$$S_t^* = X\mathrm{e}^{-\left(r - r_f + \frac{\sigma^2}{2}\right)(T-t)} \tag{17}$$

If $r \ge r_f$, that is the domestic interest rate is greater than or equal to the foreign interest rate, then $S_t^* < X$. Thus all in the money calls, at the money calls, and some out of the money calls will have a delta in excess of 0.50.

If $r < r_f$ then setting $\left(r - r_f + \frac{\sigma^2}{2}\right)(T-t) = 0$ we get

$$r^* = r_f - \frac{\sigma^2}{2} \tag{18}$$

For values of r that are greater than or equal to r^* all in the money calls, at the money calls, and some out of the money calls will have a delta in excess of 0.50. However, if $r < r^*$ only in the money calls will have a delta is excess of 0.50.

12. Properties of $N(d_2)$

$$d_2 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - r_f - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Consider the case where $\left(r - r_f - \frac{\sigma^2}{2}\right)(T-t) > 0$. The probability of exercise being exactly 50% is given by:

$$S_t^* = X\mathrm{e}^{-\left(r - r_f - \frac{\sigma^2}{2}\right)(T-t)} \tag{19}$$

If $r \ge r_f + \frac{\sigma^2}{2}$ then $S_t^* < X$.

Thus all in the money and at the money calls, as well as some out of the money calls have a probability of exercise in excess of 50%. However if the domestic risk-less rate is less than this value, then only in the money calls will have a probability of exercise that is 50% or greater.

For puts the probability of exercise is $N(-d_2) = 1 - N(d_2)$. Hence if $r \geq r_f + \frac{\sigma^2}{2}$ then $S_t^* < X$, and only certain in the money puts have a probability of exercise in excess of 50%.

If r is less than $r_f + \frac{\sigma^2}{2}$ then

$$S_t^* > X$$

and all in the money and at the money puts, as well as some out of the money puts will have a probability of exercise in excess of 50%.

13. Conclusion

This paper uses the Black-Scholes-Merton model for stocks, and the equivalent model for foreign exchange options, developed by Garman-Kohlhgen, to make predictions regarding the option delta, and the probability of exercise of the options. Since the underlying models are only valid for European options, so are the deductions of this paper.

The predictions of this paper can be empirically tested. The data can be divided into two segments. The first half can be used to make predictions regarding the probabilities of exercise. The second half of the data can be used to compute the actual exercise percentages for various options and then make a comparison with the predictions. This will be our course of study in a subsequent paper.

This paper has implications for only European options. While American options lack closed-form solutions, studies can possibly be undertaken to assess the probabilities of exercise of such options.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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