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The Optimal Wage Share in Adam Smith

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Abstract

Adam Smith explains the origin and causes of the wealth of nations through the operation of a set of institutions within which the growth of average labor productivity is positively related to that of profit. The paper studies this relationship in the simplest case, when it is linear, within a multi-year production model that produces a single good. To simplify, the wage and profit shares of net product are referred to respectively as wage and profit. Since the model doesn't include rents, the sum of these two variables is always equal to one. In addition, real wage is defined as the amount of the good that the average wage per labor unit can buy and optimal wage as the one with the highest real wage. In this context, the percentage increase in average labor productivity obtained when profit increases from zero to a given level above zero, divided by the given level of this variable, gives the same result for all levels of profit above zero. This quotient depends on productivity growth during the reference period and is referred to as the productivity/profit rate. The main result states that if this rate is less than or equal to one, every increase in profit causes a decrease in real wage, while if it is greater than one, the opposite is true for a certain interval of possible values of profit. Moreover, the optimal wage corresponds to a profit equal to zero in the first case, but higher than zero in the second. Consequently, the economic interest of all wage-earners, which consists of obtaining wages with the greatest possible purchasing power, doesn't always coincide with the aim of obtaining the largest possible wage, nor with the equivalent purpose of reducing the exploitation rate to its lowest possible level.

Keywords

Adam Smith, Labor Productivity, Wage Share, Income Distribution, Exploitation Rate

1. Introduction

In Smith (1981: pp. 13-275), the author sets forth the most general aspects of his theory regarding the origin and growth of that part of the wealth of nations consisting of the aggregate of the material goods which they produce annually. To this end, he points to the existence of a set of interrelated social institutions within which, as a rule, the growth of any element results in the growth of wealth. For this reason, we refer to this set as the Wealth Creation System (WCS).

In this paper, we study the effect of profit growth on average labor productivity, which occurs within the WCS mainly, but not only, through the growth of investment and, in particular, investment in science and technology. Since this is an effect that does not always occur immediately, we study it within a model that represents the performance of an economy over a succession of several years. Such an economy produces a single good annually, using as inputs certain quantities of the same good and labor that are consumed entirely within the year. For simplicity's sake, the wage and profit shares of net product are referred to respectively throughout the paper as wage and profit. Since the model doesn't include rents, the sum of these two variables is always equal to one. In addition, real wage is defined as the amount of the good that the average wage per labor unit can buy and optimal wage as the one with the highest real wage.

With regard to the literature on related subjects, it should be noted that the study of income distribution by means of linear production models comprising a single period of production has given rise to numerous publications whose main references are the works of Dmitriev (1974), Leontief (1941), Marx (1990) and Sraffa (1960). These models can be considered more general than ours because they contemplate the production of various goods, while ours can be considered more general than those because it studies a succession of production processes as a whole. In this respect, it is important to say that mono-production is not an essential limitation since the only good produced can be a composite good similar to the standard commodity introduced by Sraffa (1960). Alternatively, net product in our model can roughly represent the GDP of any given economy. For a review of the above-mentioned literature and on the meaning of the standard commodity, the interested reader can consult Kurz and Salvadori (1995) and Benítez Sánchez (1986), respectively.

Within the model introduced here, we approach the simplest case by assuming that the growth of average labor productivity is related to that of profit by means of a linear function. Under this assumption, the percentage increase in the average labor productivity obtained by moving from zero to a given level above zero of profit, divided by the given level of this variable, gives the same result for all levels of profit above zero. To simplify, we refer to this quotient as the productivity/profit rate. The main results are presented in the following three propositions.

Proposition 1. If the productivity/profit rate is less than or equal to one, each increase in profit causes a decrease in the real wage. Hence, the optimal wage corresponds to the zero level of profit.

Proposition 2. If the productivity/profit rate is greater than one, there is an open interval for the wage whose lower bound is between zero and one and whose upper bound is one, such that with each of the wage levels within the interval the real wage is higher than with the zero level of profit. In this case, the optimal wage corresponds to a level of profit greater than zero.

Proposition 3. The productivity/profit rate of a given economy depends on multiple factors. In particular, because technical progress is cumulative, it is usually an increasing function of the number of years included in the reference period.

As far as we know, these results allow us to raise in an original way some problems related to the distribution of income. For example, in the case described in **Proposition 1**, any level of profit above zero represents a net loss to the real wage compared to that corresponding to the zero level of profit. However, in the case described in **Proposition 2**, for each level of profit corresponding to a wage within the specified interval, the real wage is higher than that of the zero level of profit. For this reason, the existence of the profit represents a benefit for the wage-earners, within the WCS, if the number of years that must elapse for this real wage growth to be achieved is considered acceptable. This verifies, to some extent, Smith's optimistic forecast regarding the functioning of the WCS[1].

It should be added that the study of income distribution developed in this paper complements what Benítez Sánchez (2013: p. 386) pointed out with respect to Menger's (2010) thesis on the right of wage-earners to the totality of the product of their work. Indeed, although we do not discuss here what part of the product of their labor is to which wage-earners are entitled, we do contribute to the subject by showing what is the wage that enables them to acquire the highest possible real income within the WCS in a given reference period.

In addition to this introduction, the paper includes six other sections. Section 2 summarizes the WCS. In Section 3, we present the reference model. Section 4 looks at real wage as a function of income distribution. Section 5 develops the formulas that allow calculating the value of certain important variables introduced in the paper, while Section 6 studies three numerical examples to illustrate them. Section 7 presents some concluding remarks.

2. The Wealth Creation System

It is possible to summarize the broad outlines of the WCS using **Diagram 1**. It presents six boxes, each containing a social institution or a salient aspect of the market economy. For the sake of simplicity, we will now refer to the content of all the boxes as institutions. Each arrow indicates the existence of some influence of the institution in the box at the origin of the arrow on the institution of the

[1]"It is the great multiplication of the production of all the different arts, in consequence of the division of labor, which occasions, in a well governed society, that universal opulence that extends itself to the lowest ranks of the people. Every workman has a great quantity of his own work to dispose of beyond what he himself has occasion for; and every other workman being in exactly the same situation, he is enabled to exchange a great quantity of his own goods for a great quantity, or, what comes to the same thing, for the price of a great quantity of theirs. He supplies them abundantly with what they have occasion for, and they accommodate him as amply with what he has occasion for, and a general plenty diffuses itself through all the different ranks of the society" (Smith, 1981: p. 22).

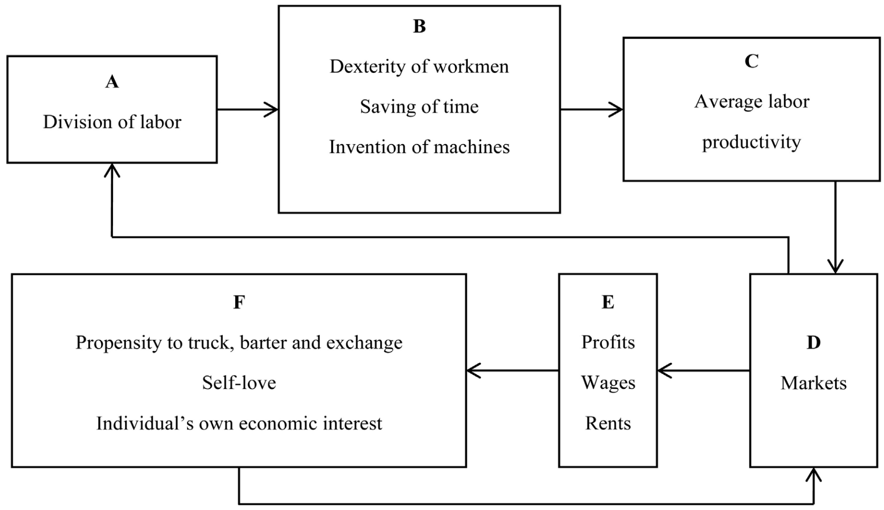

Diagram 1. Some relations between institutions in the WCS.

box that the arrow reaches. The set of these institutions forms a system in which the variation of one element positively affects at least one other element of the whole, either directly or indirectly (through the medium of a third element).

Here are some aspects of the chain of influences that bind the institutions that make up the WCS. We indicate in each description the corresponding quotation in Smith's work.

A → B → C: The growth in average labor productivity is mainly due to the division of labor[2]. The latter institution influences the former through the following three factors: 1) the increase in the skill of the workers, 2) the saving of time spent in moving from one activity to another, and 3) the invention of machines that facilitate work[3].

C → D → E: The growth of average labor productivity increases the quantities produced of each of the goods, which leads to the growth of the volumes of goods exchanged in the different markets. This, in turn, increases the sum of the three types of income: rents, wages, and profits[4].

[2]"The greatest improvement in the productive powers of labor, and the greater part of the skill, dexterity, and judgement with which it is any where directed, or applied, seem to have been the effects of the division of labor" (Smith, 1981: p. 13).

[3]"This great increase of the quantity of work, which, in consequence of the division of labor, the same number of people are capable of performing, is owing to three different circumstances; first, to the increase of dexterity in every particular workman; secondly, to the saving of the time which is commonly lost in passing from one species of work to another; and lastly, to the invention of a great number of machines that facilitate and abridge labor, and enable one man to do the work of many" (Smith, 1981: p. 17).

[4]"It is the great multiplication of the production of all the different arts, in consequence of the division of labor, which occasions, in a well governed society, that universal opulence that extends itself to the lowest ranks of the people. Every workman has a great quantity of his own work to dispose of beyond what he himself has occasion for; and every other workman being in exactly the same situation, he is enabled to exchange a great quantity of his own goods for a great quantity, or, what comes to the same thing, for the price of a great quantity of theirs. He supplies them abundantly with what they have occasion for, and they accommodate him as amply with what he has occasion for, and a general plenty diffuses itself through all the different ranks of the society" (Smith, 1981: p. 22).

D → A: The division of labor is conditioned by the size of commodity exchanges. On the one hand, the growth of trade fosters the development of the division of labor. On the other hand, the magnitude of this growth represents an upper limit for this development[5].

E → F: One of the main goals of an individual participating in economic transactions is to earn an income that allows him to cover his expenses. In particular, the main purpose of an enterprise investing its capital is to obtain a profit. The greater the profit that may be obtained the greater the investment[6].

F → D: The growth of markets is driven by three characteristic traits of individuals: 1) the propensity to exchange goods[7], 2) self-love, and 3) the proclivity of individuals to satisfy their own interests[8]. Sustained by self-love, individuals engage in commercial exchanges seeking to satisfy their own economic interest.

It should be noted that the constituent institutions of the WCS are related to each other in other ways than those already mentioned. However, those given are sufficient to show the existence of a circuit integrated by a set of institutions in which, as a general rule, the growth of any of the elements results in the growth of at least one of the others without any element decreasing. In particular, for the purposes of this paper it is important to note that the growth of average labor productivity is positively related to that of profit, which is an effect that depends on the WCS as a whole[9]. For this reason, when we say that one phenomenon is

[5]"As it is the power of exchanging that gives occasion to the division of labour, so the extent of this division must always be limited by the extent of that power, or, in other words, by the extent of the market. When the market is very small, no person can have any encouragement to dedicate himself entirely to one employment, for want of the power to exchange all that surplus part of the produce of his own labor, which is over and above his own consumption, for such parts of the produce of other men's labor as he has occasion for" (Smith, 1981: p. 31).

[6]"The value which the workmen add to the materials, therefore, resolves itself into two parts, of which the one pays their wages, the other the profits of their employer upon the whole stock of materials and wages which he advanced. He could have no interest to employ them, unless he expected from the sale of their work something more than what was sufficient to replace his stock to him; and he could have no interest to employ a great stock rather than a small one, unless his profit where to bear some proportion to the extent of his stock" (Smith, 1981: p. 66).

[7]"This division of labor, from which so many advantages are derived, is not originally the effect of any human wisdom, which foresees and intends that general opulence to which it gives occasion. It is the necessary tough very slow and gradual consequence of a certain propensity in human nature which has in view no such extensive utility; the propensity to truck, barter, and exchange one thing for another" (Smith, 1981: p. 25).

[8]"But man has almost constant occasion for the help of his brethren, and it is vain for him to expect it from their benevolence only. He will be more likely to prevail if he can interest their self-love in his favor, and show them that it is for their own advantage to do for him what he requires of them. Whoever offers to another a bargain of any kind, proposes to do this. Give that which I want, and you will have this which you want, is the meaning of every such offer; and it is in this manner that we obtain from one another the far greater part of those good offices which we stand in need of. It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest. We address ourselves, not to their humanity but to their self-love, and never talk to them of our own necessities but of their advantages" (Smith, 1981: pp. 26-27).

[9]It also depends on other factors not specified in our brief description of the WCS. For example, in order for certain technical advances to take place, it is essential to have one of the few inventors whose talents make these advances possible.

positively related to another, we resort to a shorthand way of referring to the relation between two phenomena within a system, of which the first has several causes, including the second.

3. The Model

We consider an economy that carries out successively over several years an annual production process in which it obtains a single good by using as inputs certain quantities of the same good and labor that are consumed entirely within the year. The amounts used and produced of the good, as well as the volume of work carried out, may vary in all years. For the purpose of studying the effect of changes in the distribution of income on the amounts indicated, we shall assume that throughout the reference period, at the end of each year, wage-earners receive as labor compensation the same fraction w of the corresponding net product. Since this variable determines the fraction of net product constituting profit $(1-w)$, it also affects the volume of investment and employment, particularly investment in science and technology, and thus average labor productivity. On the basis of these assumptions, we will now define the relevant variables of our analysis as functions of w.

3.1. Basic Concepts

To each year corresponds a particular index t such that $t = 1, 2, \cdots$ For each t, $L_t(w)$ is the labor used, $Q_t(w)$ the quantity produced, $K_t(w)$ the quantity consumed, and $Y_t(w)$ the net product ($Q_t(w) - K_t(w)$) obtained producing the single good in the year t. Therefore, the average labor productivity in that year and in the period $t_1 - t_2$, which runs from the beginning of the year t_1 to the end of the year t_2 are, respectively:

$$ALP_t(w) = \frac{Y_t(w)}{L_t(w)} \tag{1}$$

$$ALP_{t_1-t_2}(w) = \frac{\sum_{t=t_1}^{t_2} Y_t(w)}{\sum_{t=t_1}^{t_2} L_t(w)}. \tag{2}$$

Let $\varkappa_{t_1-t_2} \geq 0$ be the increase in average labor productivity over the period $t_1 - t_2$ above $ALP_{t_1}(1)$ that is attributable to the zero-profit level. That is, the increase that could have occurred if the profit level had been zero during the reference period. We measure the quantities of the good using as a unit of measurement the average net product per labor unit corresponding to the zero level of profit. For this reason, we have:

$$ALP_{t_1}(1) + \varkappa_{t_1-t_2} = 1. \tag{3}$$

It should be noted that, although it is not specified on the right side, on both sides of this equation we have a quantity of the good per unit of labor.

To calculate the real wage, we notice that the total quantity of the only good that can be purchased by the wage-earners in a given year t or in the period t_1 – t_2 results from multiplying by w the numerator in the right side of Equations (1) and (2), respectively. To obtain the real wage in each case we divide the product by the corresponding quantity of labor. From this conclusion, it follows that in a given year t:

$$s_t(w) = wALP_t(w) \tag{4}$$

and, in the period from t_1 to t_2:

$$s_{t_1-t_2}(w) = wALP_{t_1-t_2}(w). \tag{5}$$

3.2. Assumptions

Since the subject of our study is the relation between the growth of profit and that of the average labor productivity, we shall assume that for every couple $(w_1, w_2) \in]0,1]$ such that $w_1 < w_2$ the following two propositions are true:

$$\sum_{t=t_1}^{t_2} Y_t(w_1) \geq \sum_{t=t_1}^{t_2} Y_t(w_2) \tag{6}$$

$$ALP_{t_1-t_2}(w_1) \geq ALP_{t_1-t_2}(w_2). \tag{7}$$

Hence, profit in real terms and average labor productivity are both non-decreasing functions of profit. There are several sets of sufficient conditions for Propositions (6) and (7) to be true that are more or less realistic. For the sake of simplicity, we adopt the following six assumptions of which the first three and the last two are true all along the reference period:

(I) Regardless of the wage level both single good and labor markets are always in equilibrium.

(II) Production plans are chosen following the criterion of profit maximization.

(III) The propensity to consume of the society is a non-increasing function of both income and profit.

(IV) For every $t > t_1$, the investment in the year t ($K_t(w) - K_{t-1}(w)$) is equal to the savings in the year $t-1$.

(V) Net product and employment are both monotonous increasing functions of capital. However, as capital increases, the corresponding percentage change in net product is greater than that which takes place in employment.

(VI) There is no increasing marginal productivity of labor.

The following four comments may be useful for a better understanding of the model.

First, with respect to Assumption (I), it is important to remember that the real wage cannot fall below a certain limit without the supply of labor decreasing enough to cause a diminishing of the net product, so inequality (6), as a rule, ceases to be valid for real wage levels that are below that limit. Nevertheless, our assumption that there is no such limit does not substantially affect the main results of this study, as argued in *Remark* 1 at the end of the next section.

Second, changes in production processes taking place before the reference period are not considered. For this reason, the capital stock of the first year is the same for all levels of income distribution. Therefore, if the employment level is also the same, average labor productivity in this year is constant and independent of income distribution. On the other hand, if the employment level is a decreasing function of the wage, it follows from Assumption (VI) that average labor productivity is never greater than when the wage equals one.

Third, it follows from the conclusions of the second comment and Assumption (III) that the amount of savings in the first year is a non-decreasing function of profit. Then, according to Assumption (IV), $K_2(w)$ is also a non-decreasing function of profit. This result taken together with Assumption (V) implies that both $Y_2(w)$ and $ALP_2(w)$ are non-decreasing functions of profit. Upon this basis, a similar reasoning allows us to conclude successively that $Y_t(w)$ and $ALP_t(w)$ are non-decreasing functions of profit for every $t > 2$. In turn, these conclusions permit us to prove Propositions (6) and (7).

Fourth, if the propensity to consume is 100% and constant the equality holds in both Propositions (6) and (7). The inequality holds if the propensity to consume is a decreasing function of both income and profit.

4. The Real Wage as a Function of Income Distribution

It follows from the second commentary of the previous section that, if the reference period consists of a single year t_1, the next two propositions are true.

Proposition 4. The average product per labor unit is either constant and independent of income distribution or an increasing function of w.

Proposition 5. The real wage is a function of w whose value is either less than or equal to one reaching the equality only when the profit is zero.

In the first of the two cases indicated in **Proposition 4**, we can plot both average labor productivity and real wage as functions of income distribution in **Figure 1**, where $\frac{Q}{L}$ means quantities of the good per unit of labor.

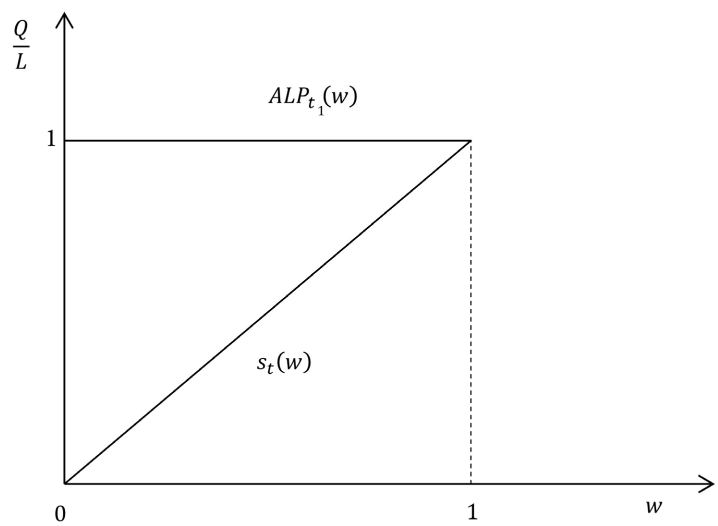

Figure 1. Two functions of the wage in a one-year period.

However, when we consider a sufficiently large number of years for technological advances to be implemented to increase the average labor productivity, **Propositions 4** and **5** are no longer necessarily valid. Furthermore, according to the arguments presented in Section 2, in this case the average labor productivity is an increasing function of profit.

To include the last conclusion in our study, we consider the simplest case, assuming that the function indicated is of linear type. For this reason, to each volume of profit corresponds an increase in the average labor productivity (relative to its level when $w=1$), which is in constant proportion $\mu_{t_1-t_2} \geq 0$ to profit. Thus, given a particular economy and a reference period t_1-t_2:

$$\mu_{t_1-t_2} = \frac{ALP_{t_1-t_2}(w)-1}{1-w} \quad \forall w \in [0,1[\tag{8}$$

⇒

$$ALP_{t_1-t_2}(w) = 1 + \mu_{t_1-t_2}(1-w). \tag{9}$$

As can be seen in Equation (8), the newly defined coefficient, which we refer to as the productivity/profit rate, is expressed in units of the good per unit of labor and indicates the growth of the average labor productivity per unit of profit. It should be added that, because of the unit of measurement selected for the good, the numerator on the right side of (8) is equal to the percentage growth of the average labor productivity when profit goes from zero to a level above zero, which explains the interpretation of the productivity/profit rate offered in the introduction. Given a particular economy and reference period, this rate can vary widely depending on the annual growth of average labor productivity and, if different reference periods with a common start date are considered, it seems normal to expect the rate to be an increasing function of the number of years comprising the periods (see Example 3 in Section 6). This is due to the cumulative nature of the effect of technical progress on average labor productivity in the absence of serious disruptions in the succession of production processes.

Based on Equation (5), to get the real wage, we multiply the right side of Equation (9) by w. It should be noted that in this and the following calculations, the real wage is measured in units of the good per unit of labor:

$$s_{t_1-t_2}(w) = w + w\mu_{t_1-t_2}(1-w). \tag{10}$$

We can distinguish the two terms on the right side of this equation by the fact that the first of them indicates the part of the real wage whose origin is not associated with the profit of the reference period, while the second represents the part that is. Indeed, by multiplying the two sides of Equation (3) by w, it is possible to verify what has been said with respect to the first term, while what has been said about the second term is based on the fact that it is the part resulting from the increase in the average labor productivity associated with profit. The second part is equal to zero when the first part is zero or one, and it is greater than zero when the first part is between zero and one. Furthermore, given any pair $(w_1, w_2) \in [0,1]$, such that $w_1 > w_2$, comparing $s_{t_1-t_2}(w_1)$ with $s_{t_1-t_2}(w_2)$ it is possible to observe that the first real wage is less, equal or greater than the second if the difference between the part not associated with the profit of the first real wage minus that of the second ($w_1 - w_2$) is respectively greater, equal or less than the difference between the part associated with the profit of the second minus that of the first.

For the purpose of studying the real wage as a function of the distribution of income, we derive Equation (10) with respect to w, which results in:

$$s_{t_1-t_2}'(w) = 1 + \mu_{t_1-t_2}(1-w) - w\mu_{t_1-t_2} \tag{11}$$

$$= 1 + \mu_{t_1-t_2} - 2\mu_{t_1-t_2} w \tag{12}$$

$$= 1 + \mu_{t_1-t_2}(1-2w). \tag{13}$$

It is important to distinguish between the two cases characterized respectively by Inequalities (14) and (15) discussed next:

$$\mu_{t_1-t_2} \le 1. \tag{14}$$

In this case, we can observe in Equation (13) that $s_{t_1-t_2}'(w) > 0$ for every $w \in [0,1]$ except if $\mu_{t_1-t_2} = 1$ and $w = 1$ in which case $s_{t_1-t_2}'(w) = 0$. This means that any increase in profit causes a decrease in real wage. Consequently, the real wage peaks when $w = 1$.

$$\mu_{t_1-t_2} > 1. \tag{15}$$

In this case, it is possible to observe in Equation (13), on the one hand, that $s_{t_1-t_2}'(w)$ is a decreasing monotonic function of w. On the other hand, that $s_{t_1-t_2}'(0) > 0$ while $s_{t_1-t_2}'(1) < 0$. It follows from these remarks that there is only one value of $w \in]0,1[$ for which $s_{t_1-t_2}'(w) = 0$ and also that with this value of w the function $s_{t_1-t_2}(w)$ reaches a maximum. Therefore, $s_{t_1-t_2}(w)$ increases when w decreases from $w = 1$ until it reaches a maximum value for a $w_{t_1-t_2}{**} \in]0,1[$ from which it decreases until $s_{t_1-t_2}(w) = 0$ when $w = 0$. When it decreases towards zero, it adopts a value $w_{t_1-t_2}{*} \in]0, w{**}[$ such that $s_{t_1-t_2}(w_{t_1-t_2}{*}) = 1$ (see **Figure 3** in Section 6). We thus come to the following conclusions:

Proposition 6. If $\mu_{t_1-t_2} > 1$ there is a wage level $w_{t_1-t_2}{*} \in]0,1[$, such that for every $w \in]w_{t_1-t_2}{*},1[$, the real wage is higher than the one corresponding to $w = 1$.

Proposition 7. If $\mu_{t_1-t_2} \le 1$ the optimal wage is $w_{t_1-t_2}{**} = 1$. On the other hand, if $\mu_{t_1-t_2} > 1$ the optimal wage belongs to the interval $]w_{t_1-t_2}{*},1[$.

It should be noted that the optimal wage is characterized by the fact that a higher wage implies that both real wage and profit are smaller, while a lower wage implies a smaller real wage but a larger profit. For these reasons, if the income of wage-earners does not include any part of the profit, the situation that arises in the economy when the wage reaches that level is a Pareto optimum. Indeed, in any other situation in which one economic agent receives a higher income, also another agent receives a lower income.. On the other hand, if the income of wage-earners includes part of the profit, the latter does not necessarily occur, so the situation of the economy with an optimal wage is not always a Pareto optimum.

Remark 1. Inequality (6) ensures that the volume of profit increases whenever w decreases which is a necessary condition for the validity of Equation (9). If the volume of profits decreases when w decreases below a certain level $w_1 \in]0,1[$, Equation (9) would be valid only for $w \in [w_1, 1]$. For this reason, $w*$ as much as $w**$ should be selected within this interval.

5. Five Formulas

In this section, we develop five functions allowing calculating the value of as many variables related to the previous analyses. In each of the first four functions, the independent variable is the productivity/profit rate whose possible values are restricted to the interval $[0, +\infty[$ and $[1, +\infty[$ for the first and the second pair of functions, respectively. Table 1 shows some values of this rate as well as those determined by the functions for the other variables. The corresponding graphs are shown in Figure 2 where the segment $[(0,0),(0,1)]$ indicates not only quantities of the good per labor unit but also fractions of the net product.

5.1. The Wage Interval with Real Wages Greater than One

To calculate $w_{t_1-t_2}*$, it is useful to remember what has been said above regarding

Table 1. Tabulation of four functions of the productivity/profit rate.

$\mu_{t_1-t_2}$	$w*(\mu_{t_1-t_2})$	$w**(\mu_{t_1-t_2})$	$ALP**(\mu_{t_1-t_2})$	$s**(\mu_{t_1-t_2})$
1	1	1	1	1
2	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{8}$
3	$\frac{1}{3}$	$\frac{2}{3}$	2	$\frac{16}{12}$
4	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{5}{2}$	$\frac{25}{16}$
5	$\frac{1}{5}$	$\frac{3}{5}$	3	$\frac{18}{10}$
6	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{2}$	$\frac{49}{24}$
7	$\frac{1}{7}$	$\frac{4}{7}$	4	$\frac{16}{7}$
8	$\frac{1}{8}$	$\frac{9}{16}$	$\frac{9}{2}$	$\frac{81}{32}$

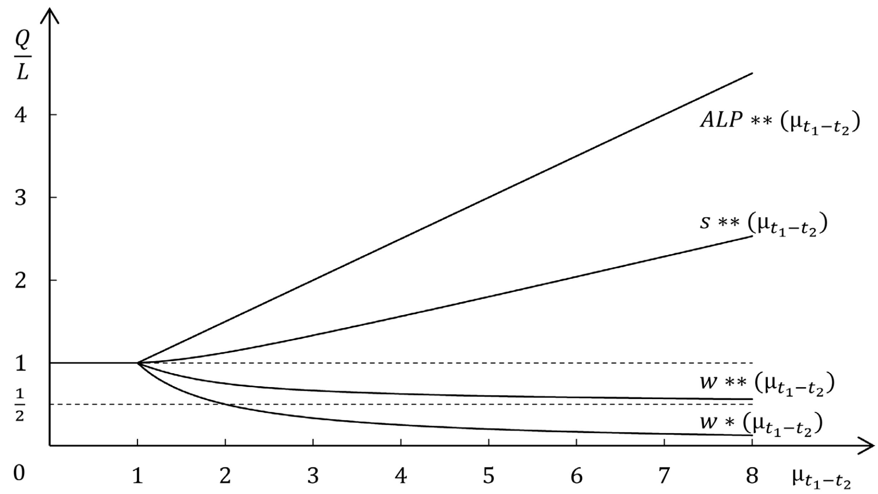

Figure 2. Graphs of the functions in **Table 1**.

the right side of Equation (10). On the basis of this we can see that $w_{t_1-t_2}*$ is the wage for which the decrease in the purchasing power of the first term (with respect to $w=1$) is equal to the increase in the real wage concomitant with this decrease due to the increase in the average labor productivity. Therefore, $w_{t_1-t_2}*$ satisfies the following equation:

$$1-w_{t_1-t_2}*=w_{t_1-t_2}*\mu_{t_1-t_2}\left(1-w_{t_1-t_2}*\right). \tag{16}$$

Dividing the two sides of the equation by $1-w_{t_1-t_2}*$ yields:

$$1=w_{t_1-t_2}*\mu_{t_1-t_2} \tag{17}$$

$\Rightarrow$

$$w_{t_1-t_2}*=\frac{1}{\mu_{t_1-t_2}}. \tag{18}$$

This result shows that $w_{t_1-t_2}*$ is a monotonic decreasing function of $\mu_{t_1-t_2}$ which tends to 1 when $\mu_{t_1-t_2}$ tends to 1 and tends to 0 when $\mu_{t_1-t_2}$ tends to $+\infty$.

5.2. The Optimal Wage

To calculate $w_{t_1-t_2}**$, we equal the right side of Equation (13) to zero:

$$1+\mu_{t_1-t_2}\left(1-2w_{t_1-t_2}**\right)=0 \tag{19}$$

$\Rightarrow$

$$1+\mu_{t_1-t_2}-2\mu_{t_1-t_2}w_{t_1-t_2}**=0 \tag{20}$$

$\Rightarrow$

$$2w_{t_1-t_2}**\mu_{t_1-t_2}=1+\mu_{t_1-t_2} \tag{21}$$

$\Rightarrow$

$$w_{t_1-t_2}**=\frac{1+\mu_{t_1-t_2}}{2\mu_{t_1-t_2}}. \tag{22}$$

This result shows us that $w_{t_1-t_2}**$ is a monotonic decreasing function of $\mu_{t_1-t_2}$, which tends to 1 when $\mu_{t_1-t_2}$ tends to 1 and tends to $\frac{1}{2}$ when $\mu_{t_1-t_2}$ tends to $+\infty$. Thus, when wage-earners receive the optimal wage, as the productivity/profit rate increases, the fraction of income corresponding to profit increases by tending to $\frac{1}{2}$.

5.3. The Average Labor Productivity Corresponding to the Optimal Wage

$ALP_{t_1-t_2}**$ is the value of the average labor productivity corresponding to the optimal wage. Therefore:

$$ALP_{t_1-t_2}** = ALP\left(w_{t_1-t_2}**\right). \tag{23}$$

To calculate this value, we substitute w in the right side of Equation (9) with the right side of Equation (22):

$$ALP_{t_1-t_2}** = 1+\mu_{t_1-t_2}\left(1-\frac{1+\mu_{t_1-t_2}}{2\mu_{t_1-t_2}}\right) \tag{24}$$

$$= 1+\mu_{t_1-t_2}\left(\frac{2\mu_{t_1-t_2}}{2\mu_{t_1-t_2}}-\frac{1+\mu_{t_1-t_2}}{2\mu_{t_1-t_2}}\right) \tag{25}$$

$$= 1+\frac{2\mu_{t_1-t_2}-1-\mu_{t_1-t_2}}{2} \tag{26}$$

$$= \frac{2+\mu_{t_1-t_2}-1}{2} \tag{27}$$

$$= \frac{\mu_{t_1-t_2}+1}{2}. \tag{28}$$

This result shows us that $ALP_{t_1-t_2}**$ is an increasing monotonic function of $\mu_{t_1-t_2}$ which tends to be equal to half the value of this variable as it grows.

5.4. The Real Wage Corresponding to the Optimal Wage

$s_{t_1-t_2}**$ is the real wage that corresponds to the optimal wage. Therefore:

$$s_{t_1-t_2}** = s\left(w_{t_1-t_2}**\right). \tag{29}$$

We can calculate $s_{t_1-t_2}**$ by substituting w and $ALP_{t_1-t_2}(w)$ into the right side of Equation (5) with the right side of Equation (22) and the right side of Equation (28), respectively:

$$s_{t_1-t_2}** = \left(\frac{1+\mu_{t_1-t_2}}{2\mu_{t_1-t_2}}\right)\left(\frac{\mu_{t_1-t_2}+1}{2}\right) \tag{30}$$

$$= \frac{1}{4}\left[\frac{1+2\mu_{t_1-t_2}+\mu_{t_1-t_2}^2}{\mu_{t_1-t_2}}\right]. \tag{31}$$

It follows from this formula that $s_{t_1-t_2}**$ is an increasing monotonic function of $\mu_{t_1-t_2}$ which tends to be equal to a quarter of the value of this variable as it grows.

5.5. Average Labor Productivity as a Function of the Growth Rates of Net Product and Employment

Here, we calculate the average labor productivity over a period t_1-t_2 by knowing the average annual growth rates of net product (g_1) and employment (g_2) over that period. The net product in the first year is $Y_{t_1}(w)$ units, in the second year it is $Y_{t_1}(w)(1+g_1)$ units, in the third year $Y_{t_1}(w)(1+g_1)^2$ units and so on. Likewise, the amount of work used in the first year is $L_{t_1}(w)$ units, in the second year it is $L_{t_1}(w)(1+g_2)$ units, in the third year $L_{t_1}(w)(1+g_2)^2$ units and so on. The average labor productivity over the period is the quotient of the sum of the first sequence of quantities divided by the sum of the second sequence:

$$ALP_{t_1-t_2}(w)=\frac{Y_{t_1}(w)\left[1+(1+g_1)+(1+g_1)^2+\cdots+(1+g_1)^{t_2-t_1}\right]}{L_{t_1}(w)\left[1+(1+g_2)+(1+g_2)^2+\cdots+(1+g_2)^{t_2-t_1}\right]}. \tag{32}$$

$$ALP_{t_1-t_2}(w)=\left[\frac{Y_{t_1}(w)}{L_{t_1}(w)}\right]\left[\frac{1+(1+g_1)+(1+g_1)^2+\cdots+(1+g_1)^{t_2-t_1}}{1+(1+g_2)+(1+g_2)^2+\cdots+(1+g_2)^{t_2-t_1}}\right] \tag{33}$$

$$=\left[ALP_{t_1}(w)\right]\left[\frac{1+(1+g_1)+(1+g_1)^2+\cdots+(1+g_1)^{t_2-t_1}}{1+(1+g_2)+(1+g_2)^2+\cdots+(1+g_2)^{t_2-t_1}}\right]. \tag{34}$$

It is possible to observe that both numerator and denominator in the second factor of the left side of this equation are sums of geometric progressions whose first term is 1 and whose common ratios are $(1+g_1)$ and $(1+g_2)$, respectively. Therefore, applying the formula for this type of sums, we can write the last equation as follows:

$$ALP_{t_1-t_2}(w)=\left[ALP_{t_1}(w)\right]\frac{\left[\dfrac{(1+g_1)^{t_2-t_1+1}-1}{(1+g_1)-1}\right]}{\left[\dfrac{(1+g_2)^{t_2-t_1+1}-1}{(1+g_2)-1}\right]}. \tag{35}$$

Dividing the two sides of this equation by $ALP_{t_1}(w)$, we get the following formula:

$$ALP_{t_1-t_2}(w)=\frac{\left[\dfrac{(1+g_1)^{t_2-t_1+1}-1}{(1+g_1)-1}\right]}{\left[\dfrac{(1+g_2)^{t_2-t_1+1}-1}{(1+g_2)-1}\right]}, \tag{36}$$

in which the average labor productivity during the reference period appears as a multiple of the average labor productivity in the first year of the period.

6. Three Numerical Examples

This section presents three numerical examples to illustrate the results of the preceding sections.

6.1. Example 1

Here, we study both average labor productivity and real wage as functions of income distribution when $\mu_{t_1-t_2} = 2$. Table 2 presents various wage values as well as those corresponding to the two functions. Their graphs are shown in Figure 3.

Substituting $\mu_{t_1-t_2}$ with 2 into Equation (18), we get:

$$w_{t_1-t_2}* = \frac{1}{2}. \tag{37}$$

Therefore, the real wage is higher than 1 for every $w \in \left]\frac{1}{2},1\right[$.

To obtain the optimal wage, we substitute $\mu_{t_1-t_2}$ with 2 into Equation (22):

$$w_{t_1-t_2}** = \frac{1+2}{2(2)} \tag{38}$$

$$= \frac{3}{4}. \tag{39}$$

Now, substituting $\mu_{t_1-t_2}$ with 2 into Equation (31) yields:

$$s_{t_1-t_2}** = \frac{1}{4}\left[\frac{1+2(2)+2^2}{2}\right] \tag{40}$$

Table 2. Tabulation of two functions of the wage when the productivity/profit rate equals two.

w	$ALP_{t_1-t_2}(w)$	$s_{t_1-t_2}(w)$	w	$ALP_{t_1-t_2}(w)$	$s_{t_1-t_2}(w)$
0	3	0	$\frac{5}{8}$	$\frac{7}{4}$	$\frac{35}{32}$
$\frac{1}{8}$	$\frac{11}{4}$	$\frac{11}{32}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{8}$
$\frac{1}{4}$	$\frac{5}{2}$	$\frac{5}{8}$	$\frac{7}{8}$	$\frac{5}{4}$	$\frac{35}{32}$
$\frac{3}{8}$	$\frac{9}{4}$	$\frac{27}{32}$	1	1	1
$\frac{1}{2}$	2	1			

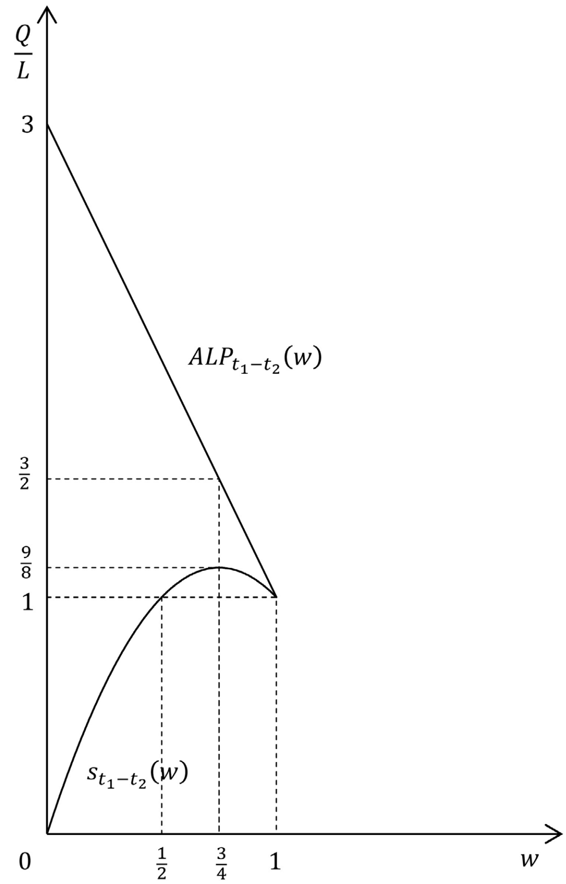

Figure 3. Graphs of the functions in **Table 2**.

$$= \frac{1}{4} \times \frac{9}{2} \tag{41}$$

$$= \frac{9}{8}. \tag{42}$$

This means that with the optimal wage, the real wage is $\frac{1}{8}$ higher than when $w = 1$.

Finally, in order to know the increase in the average labor productivity when $w = w^{**}$, we substitute $\mu_{t_1-t_2}$ with 2 into Equation (28). Then, we have:

$$ALP_{t_1-t_2}{}^{**} = \frac{2+1}{2} \tag{43}$$

$$= \frac{3}{2}. \tag{44}$$

According to this result, when the wage falls from 1 to $\frac{3}{4}$, there is also a 50% increase in the average labor productivity. The increase of $\frac{1}{2}$ unit of the good produced per labor unit increases in $\frac{1}{8}$ units the real wage and in $\frac{3}{8}$ units the profit per unit of labor.

6.2. Example 2

According to Weinstock (2023) in the United States between 1949 and 2021, the average annual growth rate of labor productivity (g_3) in the non-farm business sector was 2.1%, according to FXEMPIRE (2023) the average annual rate of GDP growth (g_1) in the United States from 1947 to 2023 was 3.16% and, according to York (2023), the average share of wages in national income from 1929 to 2023 was 69.9%. Adopting these data for the periods 1950-1985 and 1950-2020, we will roughly calculate successively the average labor productivity and the real wage for each period. With this purpose, to obtain g_2 we use the following formula:

$$g_2 = \frac{1+g_1}{1+g_3} - 1. \tag{45}$$

Substituting into this equation g_1 and g_3 with the corresponding values, we get:

$$g_2 = \frac{1+0.0316}{1+0.021} - 1 \tag{46}$$

$$= 0.0103 \tag{47}$$

Of particular note in this example is the difference in real wage between the two periods, which is due to the cumulative nature of technological progress.

6.2.1. Average Labor Productivity and Real Wage in the US Economy in the Period 1950-1985

Substituting each variable in Equation (36) with its value during the period 1950-1985, we get:

$$ALP_{1950\text{-}1985}(0.699) = \frac{\left[\dfrac{1.0316^{36}-1}{1.0316-1}\right]}{\left[\dfrac{1.0103^{36}-1}{1.0103-1}\right]} \tag{48}$$

$$= \frac{\left[\dfrac{2.0648}{0.0316}\right]}{\left[\dfrac{0.4461}{0.0103}\right]} \tag{49}$$

$$= \frac{65.3417}{43.3106} \tag{50}$$

$$= 1.5086 \tag{51}$$

This result means that the average labor productivity during the period was 1.5086 times the average labor productivity of the first year of the period. Now, substituting each variable in Equation (5) with its value during the period 1950-1985 we get the real wage:

$$s_{1950\text{-}1985}(0.699) = 0.699 \times 1.5086 \quad (52)$$

$$= 1.0541 \quad (53)$$

Hence, the average real wage over the period was 5.41% higher than the average labor productivity in the first year. During the reference period, average labor productivity increased 50.86% with respect to its initial value, of which 35.51% was added to wages and 15.35% to profits. These percentages result from computing $(1.0541 - 0.699) \times 100$ and $[0.5086 - (1.0541 - 0.699)] \times 100$, respectively.

6.2.2. Average Labor Productivity and Real Wage in the US Economy in the Period 1950-2020

Substituting each variable in Equation (36) with its value in the period 1950-2020, we get:

$$ALP_{1950\text{-}2020}(0.699) = \frac{\left[\dfrac{1.0316^{71} - 1}{1.0316 - 1}\right]}{\left[\dfrac{1.0103^{71} - 1}{1.0103 - 1}\right]} \quad (54)$$

$$= \frac{\left[\dfrac{8.1055}{0.0316}\right]}{\left[\dfrac{1.07}{0.0103}\right]} \quad (55)$$

$$= \frac{256.5031}{103.8834} \quad (56)$$

$$= 2.4691 \quad (57)$$

This result means that the average labor productivity of the period was 2.4691 times the average labor productivity of the first year of the period. Now, substituting each variable in Equation (5) with its value during the period 1950-2020, we get the real wage:

$$s_{1950\text{-}2020}(0.699) = 0.699 \times 2.4691 \quad (58)$$

$$= 1.7259 \quad (59)$$

Therefore, the average real wage during the period was 72.59% higher than average labor productivity in the first year. During the reference period, average labor productivity increased 146.91% with respect to its initial value, of which 102.69% was added to wages and 44.22% to profits. These percentages result from computing $(1.7259 - 0.699) \times 100$ and $[1.4691 - (1.7259 - 0.699)] \times 100$, respectively.

6.3. Example 3

As noted in our presentation of the WCS, the increase in average labor productivity when the profit is zero must be very small compared to what can be achieved at other levels of profit. Indeed, if there is no profit, investment loses its greatest attraction[10] and, with it, the growth of markets and the division of labor lose their greatest momentum. For these reasons, for simplicity's sake, in the example studied in this subsection we assume that when the profit is zero the increase in average labor productivity is also zero. In addition, we assume that the change in average labor productivity when the wage increases from 69.9% to 100% in the year 1950 is negligeable for our illustrative purposes. Consequently, according to Equation (3), the unit of measurement of the good is equal to the amount of that good produced on average per labor unit during the first year of the period when the wage is 69.9%. This conclusion implies that it is appropriate to employ in Equation (8) the value of $ALP_{t_1-t_2}(w)$ calculated by means of Equation (36).

Now, we will calculate successively the productivity/profit rate, the wage interval with real wages above one, the optimal wage and the corresponding real wage in the two periods presented in Example 2.

6.3.1. The Productivity/Profit Rate, the Wage Interval with Real Wages Greater than One, the Optimal Wage and the Corresponding Real Wage in the US Economy in the Period 1950-1985

First, substituting each variable in Equation (8) with its value for the period 1950-1985 we calculate the productivity/profit rate:

$$\mu_{1950\text{-}1985} = \frac{1.5086-1}{1-0.699} \tag{60}$$

$$=1.6897 \tag{61}$$

Substituting $\mu_{t_1-t_2}$ with 1.6897 into Equation (18), we get:

$$w_{1950\text{-}1985}* = \frac{1}{1.6897} \tag{62}$$

$$=0.5918 \tag{63}$$

Therefore, the real wage is higher than 1 for every $w \in \left]0.5918, 1\right[$.

Now, we calculate the optimal wage for the period 1950-1985 by substituting $\mu_{t_1-t_2}$ with 1.6897 into Equation (22):

$$w_{1950\text{-}1985}** = \frac{1+1.6897}{2(1.6897)} \tag{64}$$

$$=0.7959 \tag{65}$$

Finally, we get the real wage corresponding to the optimal wage for the period 1950-1985 by substituting $\mu_{t_1-t_2}$ with 1.6897 into Equation (31):

$$s_{1950\text{-}1985}** = \frac{1}{4}\left[\frac{1+2(1.6897)+1.6897^2}{1.6897}\right] \tag{66}$$

[10]"The value which the workmen add to the materials, therefore, resolves itself into two parts, of which the one pays their wages, the other the profits of their employer upon the whole stock of materials and wages which he advanced. He could have no interest to employ them, unless he expected from the sale of their work something more than what was sufficient to replace his stock to him; and he could have no interest to employ a great stock rather than a small one, unless his profit where to bear some proportion to the extent of his stock" (Smith, 1981: p. 66).

$$= \frac{1+3.3794+2.855}{6.7588} \tag{67}$$

$$= 1.0703 \tag{68}$$

Comparing this result with Equation (53), it can be seen that the real wage corresponding to the optimal wage is barely 1.53% higher than the real wage for the period.

6.3.2. The Productivity/Profit Rate, the Wage Interval with Real Wages Greater than One, the Optimal Wage and the Corresponding Real Wage in the US Economy in the Period 1950-2020

First, substituting each variable in Equation (8) with its value for the period 1950-2020, we calculate the productivity/profit rate:

$$\mu_{1950\text{-}2020} = \frac{2.4691-1}{1-0.699} \tag{69}$$

$$= 4.8807 \tag{70}$$

Substituting $\mu_{t_1-t_2}$ with 4.8807 into Equation (18), we get:

$$w_{1950\text{-}2020}{}^{*} = \frac{1}{4.8807} \tag{71}$$

$$= 0.2048 \tag{72}$$

Therefore, the real wage is higher than 1 for every $w \in \left]0.2048, 1\right[$.

Now, we calculate the optimal wage for the period 1950-2020 by substituting $\mu_{t_1-t_2}$ with 4.8807 into Equation (22):

$$w_{1950\text{-}2020}{}^{**} = \frac{1+4.8807}{2(4.8807)} \tag{73}$$

$$= 0.6024 \tag{74}$$

Finally, we get the real wage corresponding to the optimal wage for the period 1950-2020 by substituting $\mu_{t_1-t_2}$ with 4.8807 into Equation (31):

$$s_{1950\text{-}2020}{}^{**} = \frac{1}{4}\left[\frac{1+2(4.8807)+4.8807^2}{4.8807}\right] \tag{75}$$

$$= \frac{1+9.7614+23.8212}{19.5228} \tag{76}$$

$$= 1.7713 \tag{77}$$

Comparing this result with Equation (59), it is possible to see that the real wage that corresponds to the optimal wage is slightly higher (2.63%) than the real wage for the period.

7. Conclusion

The main results of the paper are presented in Section 1 in **Propositions 1, 2**, and **3** and, as indicated there, they allow us to raise in an original way, as far as we know, some problems related to the distribution of income. In particular, the paper shows that the economic interest of wage-earners, which consists of obtaining wages with the greatest possible purchasing power, depends on the reference period[11]. Furthermore, it is shown that, if the productivity/profit rate is greater than one, this economic interest does not coincide with the aim of obtaining the largest possible fraction of the net product, nor with the equivalent aim of keeping the rate of exploitation as low as possible[12]. In addition, the economic interests of wage-earners and capital owners are opposed when $w \in]0, w**[$ but not when $w \in]w**,1]$.

It should also be recalled that, as indicated at the beginning of Section 4, the study of income distribution does not yield the same results when carried out in an annual production model as in a multi-year production model. The reason for the differences is that the former does not usually include the effects of changes in income distribution on average labor productivity. Indeed, these effects take several years to occur, so the multi-year production model is naturally the most appropriate to study them.

Finally, it should be added that a full evaluation of the results of the paper must await the outcome of the study of the consequences of relaxing the assumptions adopted in our model as well as that of the empirical data concerning the relation between profit and average labor productivity.

Conflicts of Interest

The author declares no conflict of interest regarding the publication of this paper.

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[11]According to Example 3, the optimal wage for the workers active during the period 1950-1985 is not the same if we define it in relation to that period or to the 1950-2020 period.

[12]Profit is higher than zero only if the rate of exploitation is also higher. This proposition, known as the Fundamental Marxist Theorem, was originally proved by Okisho (1963).

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