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Sasa-Satsuma's Dynamical Equation and Optical Solitary Wave Solutions

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Abstract

This work proposes the construction of a prototype of pulse-kink hybrid solitary waves with a strong Kink dosage of the Sasa-Satsuma equation which describes the dynamics of the wave propagating in an optical fiber where the stimulated Raman scattering effect is bethinking during modeling. The ultimate goal of this work is to propose a plateful of solutions likely to serve as signals during studies on computer or laboratory propagation studies. The resolution of such an equation is not always the easiest thing, and we used the Bogning-Djeumen Tchaho-Kofané method extended to the implicit functions of Bogning to obtain the results. The flexibility of the iB-functions made it possible to deduce the trigonometric solutions, from the obtained solitary wave solutions with a hyperbolic analytical sequence of the studied Sasa-Satsuma equation. A better appreciation of the nature of the solutions obtained is made through the profiles of some solutions obtained during the different analyses.

Keywords

Sasa-Satsuma's Dynamical Equation, Bogning-Djeumen Tchaho-Kofané Method, Optical Soliton, iB-Functions, Stimulated Raman Scattering

1. Introduction

Nowadays, nonlinear optics is a field that increasingly instigates the curiosity of many researchers around the world. This curiosity can be justified by the fact that many telecommunication industries use optical fibers more as a medium for transmitting or transporting large quantities of data over long distances, especially transoceanic, transcontinental and many other distances. This understanding can also be extended to the fact of the wide application of which optical solitons constitute the basic element of data transmission technology. It is undoubtedly for this reason that a good number of works encountered in the literature track down and reveal the optical solitons which coexist with mathematical models such as the nonlinear Schrödinger equation [1] [2] [3] [4], the Fokas-Lenells, Drinfeld-Sokolov-Wilson equations [5], and the generalized Sasa-Satsuma equation [6], to name just a few. Although we are particularly interested in optical solutions, many other relevant previous works have focused on models studied in hydrodynamics [7] [8] and many others. It turns out that these works for the most part only offer exact solutions of the single soliton type and periodic solutions [9]. Thereby, it emerges that very few of the previous works, or almost none, offer solutions of the exact, approximate or forced multiple optical solitary waves type [10] [11] [12], and this comprises hyperbolic functions. This is a problem if we stick to the fact that a new solution of a nonlinear partial differential equation is a new behavior likely to be developed by physical systems whose dynamics are described by the considered mathematical model. As corollaries, we will miss the behaviors which make it possible to analyze and explain new phenomena that occur in physical systems described by the studied Sasa-Satsuma equation, in particular certain propagations and interactions regimes of robust waves of multi-soliton types in nonlinear optical fibers. It is in response to these shortcomings that this work fits and chooses the dynamical model of Sasa-Satsuma.

The aim of this work is to construct new prototypes of optical solitary waves of the Sasa-Satsuma dynamical equation in order to further enrich the literature with new varieties of more robust solitary wave solutions. And at the same time, allow new investigations in laboratories during the propagation tests which will lead to the understanding of new phenomena that occur in the studied model. Owing to all of this, our manuscript is organized as follows: Section 2 gives a brief description of the chosen model while; Section 3 explains the outline of the Bogning-Djeumen Tchaho-Kofané method (BDKm) [13]-[18] used; Section 4 in its content delivers the results obtained; the discussions are carried out in Section 5; a conclusion coupled with perspectives is recorded in Section 6.

2. Chosen Model

The Sasa-Satsuma equation which is the chosen model is written as being [19]-[23]

$$i\frac{\partial \Phi}{\partial t}+\lambda\frac{\partial^2\Phi}{\partial x^2}+\gamma|\Phi|^2\Phi+i\left(\alpha\frac{\partial^3\Phi}{\partial x^3}+\beta|\Phi|^2\frac{\partial\Phi}{\partial x}+\theta\frac{\partial|\Phi|^2}{\partial x}\Phi\right)=0. \tag{1}$$

The first term describes the temporal evolution of the optical soliton molecules while the second term provided the group velocity dispersion (GVD). Then, the third term is the well-known Kerr-law fiber nonlinearity, $\Phi(x,t)$ the optical soliton profile, the factor of the imaginary i through the coefficients β,θ,α sequentially provides the self-steepening, stimulated Raman scattering in additionally third-order dispersion. Two mathematical techniques namely, improved F-expansion and improved auxiliary methods was used in [19] to construct several types of solitons such as dark soliton, bright soliton, periodic soliton, elliptic function and solitary waves solutions of Equation (1), while in [22], it was used to introduce and discussed the Sasa-Satsuma model in berefringent fibers without of four-wave mixing terms (FWM). Equation (1) is used to describe the propagation of femtosecond pulses in optical fibers as well as the propagation and interaction of the ultrashort pulses in the sub-picosecond or femtosecond regime. Let us take a look what it is about the used method.

3. Used Method

The Bogning-Djeumen Tchaho-Kofané method [13] [14] [15] [16] [17] [24]-[33] extended to the implicit Bogning functions (iB-functions) [34] [35] [36] and used within the framework of this work applies to some partial differential equation types in which coexist the nonlinear terms and the dispersive terms (and others) under the form:

$$X\left(\Phi,\Phi_t,\Phi_{xy},\Phi_{xzt},\Phi_{ty},\Phi_{yz},\Phi_{ttz},\Phi_{xxx},\cdots,|\Phi|^2,\left(\Phi|\Phi|^2\right)_t,\cdots\right)=0, \tag{2}$$

where $\Phi(x,y,z,t)$ is an unknown function to be determined, X is some function of Φ and its derivatives with respect to x,y,z,t and X includes the highest order derivatives and the nonlinear terms. Most often, we use the change of variables $\Phi(x,t)=\Omega(\xi)$, $\xi=\sum_{k=0}^{p}\alpha_k x_k$. In the case where Φ is a function of x,y,z and t, ξ becomes $\xi=x+y+z-vt$, where v is the wave speed. In this context, Equation (2) gives rise to the ordinary differential equation (ODE) below:

$$X_{ODE}\left(\Omega,\Omega',\Omega'',\cdots,\Omega'|\Omega|^2,\cdots\right)=0, \tag{3}$$

where Ω',Ω'' represent respectively the first and second derivatives of the envelope Ω with respect to ξ. Then, the solution we are looking for can be expressed under contracted form

$$\Omega(\xi)=\sum_{ij}\mu_{ij}J_{i,j}(\eta\xi), \tag{4}$$

where η is a real constant, μ_{ij} are the unknown constants to be determined and $J_{n,m}(\alpha x)$ are the iB-functions whose explicit hyperbolic form is written as:

$$J_{n,m}\left(\sum_{i=0}^{p}\alpha_i x_i\right)=\frac{\sinh^m\left(\sum_{i=0}^{p}\alpha_i x_i\right)}{\cosh^n\left(\sum_{i=0}^{p}\alpha_i x_i\right)}. \tag{5}$$

where $\alpha_i,(i=0;1;2;\cdots;p)$ are the parameters associated to the independent variables $x_i,(i=0;1;2;\cdots;p)$, m and n are powers of both terms of Equation

(5). For more details, see [34] [35] [36]. So, the combination of Equations (3) and (4) gives rise to the main equation of ranges

$$\sum_{ijn} A_n\left(\mu_{ij},\eta,\nu\right) J_{n,0}\left(\eta\xi\right) + \sum_{ijm} B_m\left(\mu_{ij},\eta,\nu\right) J_{m,1}\left(\eta\xi\right) + \sum_{ijk} C_k\left(\mu_{ij},\eta,\nu\right) J_{-k,0}\left(\eta\xi\right) + \sum_{ijl} D_l\left(\mu_{ij},\eta,\nu\right) J_{-l,1}\left(\eta\xi\right) + \sum_{ij} E\left(\mu_{ij},\eta,\nu\right) J_{0,0}\left(\eta\xi\right) = 0, \quad (6)$$

where i, j, k, l are positive natural integers and n, m the real numbers [30] [31] [34]. It can be noted here that Equation (16) is the one from which all the possible analyzes result. The identification of coefficients $A_n\left(\mu_{ij},\eta,\nu\right), B_m\left(\mu_{ij},\eta,\nu\right)$, $C_k\left(\mu_{ij},\eta,\nu\right), D_l\left(\mu_{ij},\eta,\nu\right), E\left(\mu_{ij},\eta,\nu\right)$ at zero makes it possible to obtain the ranges of equations whose the resolutions could allow to obtain the expressions of the unknown coefficients μ_{ij}. It is important to point out here that, the resolution of this series of equations often leads to exact solutions [4] [34] for certain models and according to the form of the considered ansatz while, for other models and according to the form of the chosen ansatz, it (resolution) leads to approximate or forced solutions. In the case of approximate or forced solutions, the priority in the order of resolution is given to those from the highest clues of $J_{n,0}\left(\eta\xi\right)$, then to those of the highest clues of $J_{m,1}\left(\eta\xi\right)$. But, otherwise we go to those from the coefficients of lowest clues of $J_{-k,0}\left(\eta\xi\right)$ and $J_{-l,1}\left(\eta\xi\right)$. Here, the priority makes reference to the serie that permits to obtain good results or merely that tends more to the sought exact solution. Very often, the series of equations obtained by identify at zero the coefficient of $J_{n,0}\left(\eta\xi\right)$ gives satisfaction.

4. Results

Now, we address the resolution of Equation (1) by applying the BDKm with certain transformations specific to traveling waves.

4.1. Analytical Optical Solitary Wave Solutions

The traveling wave solutions that we seek to construct can be considered in the form below

$$\Phi\left(x,t\right) = \Psi\left(\xi\right) e^{i\phi(x,t)}, \quad (7)$$

where $\phi\left(x,t\right) = -kx + \omega t$; $\xi = x - \nu t$; ω is the angular frequence; k the wave number and ν the wave speed. Thus, the insertion of Equation (7) into Equation (1) yields the following equation

$$m_1\Psi + m_2\frac{\partial^2\Psi}{\partial\xi^2} + m_3\left|\Psi\right|^2\Psi + i\left(m_4\frac{\partial\Psi}{\partial\xi} + \alpha\frac{\partial^3\Psi}{\partial\xi^3} + \beta\left|\Psi\right|^2\frac{\partial\Psi}{\partial\xi} + \theta\frac{\partial\left|\Psi\right|^2}{\partial\xi}\Psi\right) = 0, \quad (8)$$

where $m_1 = \alpha k^3 - \lambda k^2 - \omega$; $m_2 = \lambda + 3\alpha k$; $m_3 = \gamma + \beta k$; $m_4 = -\left(2\lambda k + 3\alpha k^2 + \nu\right)$. Equation (8) describes the dynamics of the amplitude Ψ. We decided to look for $\Psi\left(\xi\right)$ verifying Equation (8) such that

$$\Psi\left(\xi\right) = aJ_{1,0}\left(\eta\xi\right) + ibJ_{1,1}\left(\eta\xi\right) + cJ_{2,1}\left(\eta\xi\right), \quad (9)$$

where $J_{1,0}(\eta\xi), J_{1,1}(\eta\xi)$ and $J_{2,1}(\eta\xi)$ are hyperbolic iB-functions; a,b,c are constant real coefficients, η the inverse of the width at half height of each of the solitons contained in Equation (9), and *i* an imaginary such that $i=\sqrt{-1}$. Equation (9) is a complex multi-soliton whose basic form is represented by the first two terms of coefficient *a* and *b*, respectively. This basic shape is disturbed by the addition of a hybrid soliton of amplitude *c*. We believe that it is this disturbance term that is at the origin of the emergence of new hybrid structures in the propagation media. In this context, inserting of Equation (9) into Equation (8) leads to gamuts main equation below

$$\begin{aligned}&\sum_{l=1}^{6}P_l\left(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha\right)J_{l,0}\left(\eta\xi\right)\\&+\sum_{n=2}^{6}Q_n\left(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha\right)J_{n,1}\left(\eta\xi\right)\\&+i\Bigg[\sum_{j}R_j\left(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha\right)J_{j,0}\left(\eta\xi\right)\\&+\sum_{p=1}^{6}S_p\left(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha\right)J_{p,1}\left(\eta\xi\right)\Bigg]=0,\end{aligned}\tag{10}$$

where $j\in\{1;2;3;4;5;7\}$. Equation (10) is made up of four gamuts of equations, two of which are preponderant in the order of resolution in the case where we want to approach or force the solutions, in particular the ranges of equations resulting from the global coefficients P_l and R_j. In the case $c\neq 0$, we obtain the approximate solution while the case $c=0$ produces exact solutions. Thus, we obtain two families of solutions.

4.1.1. Family I of Solutions: Case: $\beta=-2\theta; m_3=0\Leftrightarrow\gamma=-\beta k; a\neq 0; b\neq 0; c\neq 0$

According to the theory briefly presented in Section 3, only range equations which resulting from the coefficients of the terms, in $J_{l,0}(\eta\xi)$ and in $J_{j,0}(\eta\xi)$ allow to best approach the solutions given by Equation (9) in the case of this Family I. The cape being fixed, one obtains from the identification at zero of the ranges of equations offered by the coefficients P_l and R_j:

- **From the real part**

the term in $J_{6,0}(\eta\xi)$,

$$\eta(\beta+4\theta)bc^2=0,\tag{11}$$

the term in $J_{5,0}(\eta\xi)$,

$$-3m_3ac^2=0,\tag{12}$$

the term in $J_{4,0}(\eta\xi)$,

$$\left[\eta(\beta+2\theta)\left(b^2-a^2\right)-\eta(\beta+6\theta)c^2+6\alpha\eta^3\right]b=0,\tag{13}$$

the term in $J_{3,0}(\eta\xi)$,

$$\left[m_3\left(a^2-b^2+3c^2\right)-2\eta^2 m_2\right]a=0, \tag{14}$$

the term in $J_{2,0}\left(\eta\xi\right)$,

$$\left[2\eta\theta a^2-\eta\left(\beta+2\theta\right)b^2+2\eta\theta c^2-m_4\eta-4\alpha\eta^3\right]b=0, \tag{15}$$

the term in $J_{1,0}\left(\eta\xi\right)$,

$$\left(m_3 b^2+m_2\eta^2+m_1\right)a=0, \tag{16}$$

- **From the imaginary part**

the term in $J_{7,0}\left(\eta\xi\right)$,

$$-2\eta\left(\beta+2\theta\right)c^3=0, \tag{17}$$

the term in $J_{5,0}\left(\eta\xi\right)$,

$$\left[4\eta\left(\beta+2\theta\right)a^2-2\eta\left(\beta+\theta\right)b^2+3\eta\left(\beta-2\theta\right)c^2-24\alpha\eta^3\right]c=0, \tag{18}$$

the term in $J_{4,0}\left(\eta\xi\right)$,

$$-2m_3abc=0, \tag{19}$$

the term in $J_{3,0}\left(\eta\xi\right)$,

$$\left[\eta\left(3\beta+2\theta\right)b^2-3\eta\left(\beta+2\theta\right)a^2-\eta\left(\beta+2\theta\right)c^2+2m_4\eta+20\alpha\eta^3\right]c=0, \tag{20}$$

the term in $J_{2,0}\left(\eta\xi\right)$,

$$2m_3abc=0, \tag{21}$$

the term in $J_{1,0}\left(\eta\xi\right)$,

$$-\left(\eta\beta b^2+\alpha\eta^3+m_4\eta\right)c=0, \tag{22}$$

From the structuring of the above equations, it appears from Equations (11) and (17) that we can have $\beta=-4\theta$ or $b=0$ or $c=0$ and $\beta=-2\theta$ or $c=0$ respectively, while Equations (12), (19), (21) enforce to choose $m_3=0\Leftrightarrow\gamma=-\beta k$ or $a=0$ or $c=0$ or $b=0$. Given all these conditions, only the case $m_3=0\Leftrightarrow\gamma=-\beta k;\beta=-2\theta;a\neq 0;b\neq 0;c\neq 0$ allows to obtain non-trivial approximate solutions. Under these conditions, Equations (12), (17), (19) and (21) are verified. On the other hand, Equation (11) proposes $\beta=-4\theta$ and $b=c=0$, which means in this case that, the contribution of this equation to this order of power of the constitutive hyperbolic functions of the ansatz given by Equation (9) is not significant, and therefore negligible. Continuing our analysis, Equations (14) and (16) lead successively to $m_2=0\Leftrightarrow\lambda=-3\alpha k$ and $m_1=0\Leftrightarrow\omega=\alpha k^3-\lambda k^2$ while the combination of Equations (13) and (15) gives, respectively:

$$c=\pm\eta\sqrt{\frac{3}{2}\frac{\alpha}{\theta}}, \tag{23}$$

and

$$a=\pm\sqrt{\frac{1}{2}\left(\eta^2-3k^2\right)\frac{\alpha}{\theta}-k\frac{\lambda}{\theta}-\frac{1}{2}\frac{\nu}{\theta}}, \tag{24}$$

with $\alpha\theta\succ 0;\lambda\theta\prec 0;\nu\prec 0$ and $\eta\geq k\sqrt{3}$. Then, Equations (18) and (20) give, respectively

$$\theta b^2 - 6\theta c^2 - 12\alpha\eta^2 = 0 \tag{25}$$

and

$$b = \pm\sqrt{\frac{1}{2}\left(10\eta^2 - 3k^2\right)\frac{\alpha}{\theta} - \frac{k}{10}\frac{\lambda}{\theta} - \frac{\nu}{\theta}}. \tag{26}$$

Given Equation (23), the successive combinations of Equations (22), (24) and (25) provides the two respective constraints

$$\nu = 9\alpha\eta^2 + \frac{9}{5}\lambda k \tag{27}$$

and

$$\alpha = -\frac{19\lambda\theta k}{\left(120\eta^2 + 15k^2\right)\theta + 120\eta^2}. \tag{28}$$

Finally, we obtain the approximate expression of **Family I of solutions of Equation (1)** in the form

$$\begin{aligned}\Phi(x,t) = \Bigg[& \pm\sqrt{\frac{1}{2}\left(\eta^2 - 3k^2\right)\frac{\alpha}{\theta} - k\frac{\lambda}{\theta} - \frac{1}{2}\frac{\nu}{\theta}}\, J_{1,0}(\eta\xi) \\ & \pm i\sqrt{\frac{1}{2}\left(10\eta^2 - 3k^2\right)\frac{\alpha}{\theta} - \frac{k}{10}\frac{\lambda}{\theta} - \frac{\nu}{\theta}}\, J_{1,1}(\eta\xi) \\ & \pm \eta\sqrt{\frac{3}{2}\frac{\alpha}{\theta}}\, J_{2,1}(\eta\xi)\Bigg] \mathrm{e}^{i\left(-kx + 4\alpha k^3 t\right)};\end{aligned} \tag{29}$$

with $\alpha\theta \succ 0; \lambda\theta \prec 0; \nu\theta \prec 0; \eta \geq k\sqrt{3}; \beta = -2\theta; \gamma = 2k\theta; \lambda = -3\alpha k; \omega = 4\alpha k^3$ (these expressions of $\omega, \lambda, \gamma, \beta$, are obtained from equations $m_1 = 0, m_2 = 0, m_3 = 0$ and (17) whose are constraints imposed by the studied system.) and the constraints given by Equations (27) and (28) respectively. It should be emphasized here that the third term of Equation (29) is a hybrid soliton of the bright-dark or dark-bright type. By carefully observing this Equation (29), we realize that it is a complex multi-soliton prototype whose third term is at the origin of the appearance in the propagation medium of multiform solitary wave structures, this in relation with the values taken by the parameters of the wave. This can be justified in the sense that the magnitude of the amplitude Ψ stages the squares of the first and the second term, which, in the absence of the third term, generates either a bright soliton or a classic dark soliton. It is necessary to note in this case that, the dynamics of the amplitude given by the Equation (9) is only described by the traveling wave equation provided by the imaginary part of the Equation (8) and which is under the form:

$$m_4\frac{\partial\Psi}{\partial\xi} + \alpha\frac{\partial^3\Psi}{\partial\xi^3} + \beta|\Psi|^2\frac{\partial\Psi}{\partial\xi} + \theta\frac{\partial|\Psi|^2}{\partial\xi}\Psi = 0. \tag{30}$$

Equation (30) compared with Equation (8) suggests that the dynamics of the amplitude given by Equation (29) is in a marginal mode with respect to the GVD and of the Kerr law nonlinearity. Stimulated Raman scattering keeps a great influence on the dynamics of Ψ. This influence is illustrated with the coefficients β,α and θ, respectively.

4.1.2. Family II of Solutions: Case: $c=0$

For $c=0$, all the equations resulting from the coefficients Q_n and R_j (see Equations (17)-(22)) are verified. Then, only some equations provided by the coefficients P_l (see Equations (11) and (12)) and S_p are verified while Equation (10) reduces to

$$\sum_{l=1}^{4} P_l'(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha)\, J_{l,0}(\eta\xi) + i\left[\sum_{p=1}^{4} S_p'(a,b,c,m_1,m_2,m_3,m_4,\beta,\eta,\theta,\alpha)\, J_{p,1}(\eta\xi)\right]=0. \tag{31}$$

Thus, the identification to zero of the equations resulting from the coefficients P_l' and S_p' leads to the following two series of equations.

- **From the real part**

the term in $J_{4,0}(\eta\xi)$,

$$\left[(\beta\eta+2\eta\theta)\left(b^2-a^2\right)+6\alpha\eta^3\right]b=0, \tag{32}$$

the term in $J_{3,0}(\eta\xi)$,

$$\left[m_3\left(a^2-b^2\right)-2\eta^2 m_2\right]a=0, \tag{33}$$

the term in $J_{2,0}(\eta\xi)$,

$$\left[2\eta\theta a^2-(\eta\beta+2\eta\theta)b^2-\eta m_4-4\alpha\eta^3\right]b=0, \tag{34}$$

the term in $J_{1,0}(\eta\xi)$,

$$\left(m_3 b^2+\eta^2 m_2+m_1\right)a=0, \tag{35}$$

- **From the imaginary part**

the term in $J_{4,1}(\eta\xi)$,

$$\left[(\eta\beta+2\eta\theta)\left(b^2-a^2\right)+6\alpha\eta^3\right]a=0, \tag{36}$$

the term in $J_{3,1}(\eta\xi)$,

$$\left[m_3\left(a^2-b^2\right)-2\eta^2 m_2\right]b=0, \tag{37}$$

the term in $J_{2,1}(\eta\xi)$,

$$\left(\eta\beta b^2+\eta m_4+\alpha\eta^3\right)a=0, \tag{38}$$

the term in $J_{1,1}(\eta\xi)$,

$$\left(m_3 b^2+m_1\right)b=0. \tag{39}$$

A thorough observation of the two ranges of equations above shows that, apart from a factor, Equations (32) and (36) are identical. The same is true for Equations (33) and (37). Under this observation including the structuring of the rest of the above equations, and taking into account the fact that we are looking for the non-trivial solutions of Equation (1), this Family II of solutions reveals four subfamilies: Case $a \neq 0, b \neq 0, c = 0, \beta \neq -2\theta, \gamma \neq -2\theta k$; case $m_1 = m_2 = m_3 = 0, a \neq 0, b \neq 0, c = 0$; case $a \neq 0, b = c = 0$; case $a = 0, b \neq 0, c = 0$.

1) Subfamily I of the Family II of solutions: case: $a \neq 0, b \neq 0, c = 0, \beta \neq -2\theta, \gamma \neq 2\theta k$

For this first case, the combination of Equations (32) and (33) or of Equations (36) and (37) results in the same constraint which gives the expression of α as a function of the other parameters as being

$$\alpha = \frac{(\beta + 2\theta)\lambda}{3(\gamma - 2\theta k)} \tag{40}$$

with $\beta \neq -2\theta; \gamma \neq 2\theta k$. Thereafter, the combination of Equations (34) and (35) gives, respectively

$$b = \pm\sqrt{\frac{(\eta^2 - k^2)\lambda + (k^3 + 3\eta^2 k)\alpha - \omega}{\gamma + \beta k}} \tag{41}$$

and

$$a = \pm\sqrt{\frac{\left[2(k^2 - \eta^2)\theta - (k^2 + \eta^2)\beta - 2\gamma k\right]\lambda + \left[(\eta^2 - 3k^2)\gamma + (2\theta - 2\beta)(k^3 + \eta^2 k)\right]\alpha - (\gamma + \beta k)\nu}{2(\gamma + \beta k)\theta}} \tag{42}$$

with $\left\{\left[2(k^2 - \eta^2)\theta - (k^2 + \eta^2)\beta - 2\gamma k\right]\lambda + \left[(\eta^2 - 3k^2)\gamma + (2\theta - 2\beta)(k^3 + \eta^2 k)\right]\alpha - (\gamma + \beta k)\nu\right\}(\gamma + \beta k)\theta \succ 0$ and $\left[(\eta^2 - k^2)\lambda + (k^3 + 3\eta^2 k)\alpha - \omega\right](\gamma + \beta k) \succ 0$. On the other side, the combination of Equations (38) and (39) offers the constraint

$$\omega = \left[(2k^2 - \eta^2)\alpha + \lambda k + \nu\right]k + \left[(3k^2 - \eta^2)\alpha + 2\lambda k + \nu\right]\frac{\gamma}{\beta} \tag{43}$$

where $\beta \neq 0$. So, we obtain **Subfamily I of Family II of the solutions of Equation (1)** and which is represented by the exact solution below

$$\Phi(x,t) = \left[\pm\sqrt{\frac{\left[2(k^2 - \eta^2)\theta - (k^2 + \eta^2)\beta - 2\gamma k\right]\lambda + \left[(\eta^2 - 3k^2)\gamma + (2\theta - 2\beta)(k^3 + \eta^2 k)\right]\alpha - (\gamma + \beta k)\nu}{2(\gamma + \beta k)\theta}} J_{1,0}(\eta\xi) \pm i\sqrt{\frac{(\eta^2 - k^2)\lambda + (k^3 + 3\eta^2 k)\alpha - \omega}{\gamma + \beta k}} J_{1,1}(\eta\xi)\right] \mathrm{e}^{i\left(-kx + \left\{\left[(2k^2 - \eta^2)\alpha + \lambda k + \nu\right]k + \left[(3k^2 - \eta^2)\alpha + 2\lambda k + \nu\right]\frac{\gamma}{\beta}\right\}t\right)}, \tag{44}$$

with constraints obtained for the cause. Equation (44) is also a kind of complex multi-soliton, which due to the squares existing in the modulus of the amplitude Ψ, makes appear in the propagation medium either a bright-soliton, or a dark-soliton, and this according to the values taken by the wave parameters. This gives freedom of choice of the structure that one would like to use during laboratory propagation tests.

2) Subfamily II of the Family II of solutions: case: $\omega = \alpha k^3 - \lambda k^2, a \neq 0, b \neq 0, c = 0, \beta = 2\theta, \gamma = -2\theta k, \lambda = -3\alpha k$

For $m_1 = 0 \Leftrightarrow \omega = \alpha k^3 - \lambda k^2$, Equations (39) and (33), (35), (37) impose successively taking $m_3 = 0 \Leftrightarrow \gamma = -\beta k$ and $m_2 = 0 \Leftrightarrow \lambda = -3\alpha k$ while Equations (32), (34), (36) and (38) remain unchanged. Herein, we also notice that Equations (32) and (36) up to a factor, are identical. Under this ascertainment, the combination of Equations (32), (34) and (38); or the Equations (36), (34) and (38) gives

$$\beta = 2\theta, \tag{45}$$

$$a = \pm\sqrt{\left(2\eta^2 + 3k^2 + 2k\right)\frac{\alpha}{\beta} + \frac{\nu}{\beta}} \tag{46}$$

$$b = \pm\sqrt{\left(3k^2 + 2k - \eta^2\right)\frac{\alpha}{\beta} + \frac{\nu}{\beta}}. \tag{47}$$

So, the **Subfamily II of Family II of the solutions of Equation (1)** is the following exact solution

$$\begin{aligned}\Phi(x,t) = \Bigg[&\pm\sqrt{\left(2\eta^2 + 3k^2 + 2k\right)\frac{\alpha}{\beta} + \frac{\nu}{\beta}} J_{1,0}(\eta\xi) \\ &\pm i\sqrt{\left(3k^2 + 2k - \eta^2\right)\frac{\alpha}{\beta} + \frac{\nu}{\beta}} J_{1,1}(\eta\xi)\Bigg] \mathrm{e}^{i\left[-kx + \left(\alpha k^3 - \lambda k^2\right)t\right]},\end{aligned} \tag{48}$$

with $\alpha\beta \succ 0; \nu\beta \succ 0$ and $3k^2 + 2k \geq \eta^2$. Equation (48) presents the same characteristics noted in the case of Equation (44) with the only difference that its dynamics is described by the reduced traveling wave equation given by Equation (30).

3) Subfamily III of the Family II of solutions: case: $a \neq 0, b = 0, c = 0, \beta \neq -2\theta; \gamma \neq 2\theta k$

For $b = 0; c = 0$, Equations (32), (34), (37) and (39) are verified while the combination of Equations (35) and (38) leads to the explicite expression of ω as being

$$\omega = \frac{1}{2}\left[\left(10\alpha^2 k + 8\lambda\alpha\right)k^2 + \left(2\lambda^2 + 3\alpha\nu\right)k + \lambda\nu\right]. \tag{49}$$

Then, the combined resolution of Equations (33) and (36) gives rise to, respectively

$$\alpha = \frac{(\beta + 2\theta)\lambda}{3(\gamma - 2\theta k)} \tag{50}$$

and

$$a = \pm\sqrt{\frac{2\lambda}{\gamma - 2\theta k}} \tag{51}$$

with $(\gamma - 2\theta k)\lambda \succ 0$. Thus, the **Subfamily III of Family II of solutions of Equation (1)** is reduced to the exact solution below

$$\Phi(x,t)=\left[\pm\sqrt{\frac{2\lambda}{\gamma-2\theta k}}J_{1,0}(\eta\xi)\right]\mathrm{e}^{i\left\{-kx+\frac{1}{2}\left[\left(10\alpha^2k+8\lambda\alpha\right)k^2+\left(2\lambda^2+3\alpha\nu\right)k+\lambda\nu\right]t\right\}} \tag{52}$$

with $(\gamma-2\theta k)\lambda\succ 0$. The solution given Equation (52) shows that the Sasa-Satsuma dynamical equation admits a bright-soliton as an exact pure real solution.

4) Subfamily IV of the Family II of solutions: case: $a=0,b\neq 0,c=0$; $\beta\neq -2\theta;\gamma\neq -\beta k$

When equating $a=c=0$, Equations (33), (35), (36) and (38) are verified while the real part gives, from Equations (32) and (34), respectively

$$b=\pm\eta\sqrt{-\frac{6\alpha}{\beta+2\theta}} \tag{53}$$

and

$$b=\pm\sqrt{\frac{2\lambda k+\left(3k^2-4\eta^2\right)\alpha+\nu}{\beta+2\theta}} \tag{54}$$

with $\alpha(\beta+2\theta)\prec 0;\eta\geq\frac{k}{2}\sqrt{3};\lambda(\beta+2\theta)\succ 0$ and $\nu(\beta+2\theta)\succ 0$. Then, the imaginary part leads, from Equations (37) and (39), successively

$$b=\pm\eta\sqrt{-\frac{2\lambda+6\alpha k}{\gamma+\beta k}} \tag{55}$$

and

$$b=\pm\sqrt{\frac{\lambda k^2+\omega-\alpha k^3}{\gamma+\beta k}} \tag{56}$$

with $(\gamma+\beta k)(2\lambda+6\alpha k)\prec 0$ and $(\gamma+\beta k)\left(\lambda k^2+\omega-\alpha k^3\right)\succ 0$. Since the coefficient b must be unique; on the one hand, Equations (53) and (54) give the equality $b^2=b^2$, which leads to the constraint

$$\alpha=-\frac{2\lambda k+\nu}{2\eta^2+3k^2} \tag{57}$$

with $k\neq 0;\eta\neq 0$. On the other hand, Equations (55) and (56) also give the equality $b^2=b^2$, leading in turn to the constraint

$$\alpha=\frac{\omega+\left(k^2+2\eta^2\right)\lambda}{k\left(k^2-6\eta^2\right)} \tag{58}$$

with $k\neq 0;\eta\neq k\frac{\sqrt{6}}{6}$. If the coefficient b is unique, then the parameter α must also be unique. In this context, the combination of Equations (57) and (58) reveals

$$\omega=\frac{\left[4\left(k^2-1\right)\eta^2-5k^4\right]\lambda+k\left(6\eta^2-k^2\right)\nu}{2\eta^2+3k^2} \tag{59}$$

with $(\gamma+\beta k)\left[\left(3k^2-2\right)\lambda+3k\nu\right]\succ 0$. So, the **Subfamily IV of Family II of**

solutions of Equation (1) can be expressed as

$$\Phi(x,t)=\left[\pm i\eta\sqrt{\frac{2\left(3k^2-2\right)\lambda+6k\nu}{(\gamma+\beta k)\left(2\eta^2+3k^2\right)}}J_{1,1}(\eta\xi)\right]\mathrm{e}^{i\left\{-kx+\frac{\left[4\left(k^2-1\right)\eta^2-5k^4\right]\lambda+k\left(6\eta^2-k^2\right)\nu}{2\eta^2+3k^2}t\right\}} \tag{60}$$

with the constraints given by Equations (57); (58); (59) and $(\gamma+\beta k)\left[\left(3k^2-2\right)\lambda+3k\nu\right]\succ 0$. This last subfamily IV of the Family II of solutions shows that Equation (1) likewise admits a kink-soliton as an exact pure imaginary solution.

4.2. Trigonometric Solutions

The transition from hyperbolic forms to trigonometric forms of iB-functions is done by means of the relation [34] [35] [36]

$$J_{n,m}(\eta x)=i^m T_{n,m}(\eta x) \tag{61}$$

with

$$T_{n,m}(\eta x)=\frac{\sin^m(\eta x)}{\cos^n(\eta x)} \tag{62}$$

such that $T_{n,m}(\eta x)$ is the secondary form of iB-functions related to trigonometric functions. Thus, when we come back to our work, by making the correspondences $\eta\leftarrow i\eta\left(i=\sqrt{-1}\right), x\leftarrow\xi$, we obtain from Equations (29), (44), (48), (52) and (60), the respective trigonometric solutions.

$$\begin{aligned}\Phi(x,t)=\Bigg\{&\pm\sqrt{\frac{1}{2}\left(\eta^2-3k^2\right)\frac{\alpha}{\theta}-k\frac{\lambda}{\theta}-\frac{1}{2}\frac{\nu}{\theta}}\operatorname{cosec}(\eta\xi)\\&\pm\sqrt{\frac{1}{2}\left(10\eta^2-3k^2\right)\frac{\alpha}{\theta}-\frac{k}{10}\frac{\lambda}{\theta}-\frac{\nu}{\theta}}\tan(\eta\xi)\\&\pm i\eta\sqrt{\frac{3}{2}\frac{\alpha}{\theta}}\operatorname{cosec}^2(\eta\xi)\sin(\eta\xi)\Bigg\}\mathrm{e}^{i\left(-kx+4\alpha k^3t\right)},\end{aligned} \tag{63}$$

$$\Phi(x,t)=\left[\pm a\operatorname{cosec}(\eta\xi)\pm b\tan(\eta\xi)\right]\mathrm{e}^{i\left(-kx+\left\{\left[\left(2k^2-\eta^2\right)\alpha+\lambda k+\nu\right]k+\left[\left(3k^2-\eta^2\right)\alpha+2\lambda k+\nu\right]\frac{\gamma}{\beta}\right\}t\right)}, \tag{64}$$

$$\Phi(x,t)=\left[\pm a\operatorname{cosec}(\eta\xi)\pm b\tan(\eta\xi)\right]\mathrm{e}^{i\left[-kx+\left(\alpha k^3-\lambda k^2\right)t\right]}, \tag{65}$$

$$\Phi(x,t)=\pm a\operatorname{cosec}(\eta\xi)\mathrm{e}^{i\left[-kx+\left(\alpha k^3-\lambda k^2\right)t\right]}, \tag{66}$$

and

$$\Phi(x,t)=\pm b\tan(\eta\xi)\mathrm{e}^{i\left[-kx+\left(\alpha k^3-\lambda k^2\right)t\right]}, \tag{67}$$

where *a*, *b* and *c* are expressions obtained in Subsection 4.1, including the accompanying constraints. Equations (63)-(67) constitute the trigonometric versions of the solutions obtained in Subsection 4.1.

4.3. Profile of Some Obtained Traveling Wave Solutions

This subsection is dedicated to the display of the profiles of certain traveling wave solutions given by Equations (29) and (44), respectively. The graphical tool which made it possible to achieve this result is MAPLE. Thus, we have, respectively.

Note here that, the choice of values is linked to constraints given by the obtained analytical expressions. At this level, several values of variables can be defined. This is for example, the case of Figure 1: (A) where we have made the choice of $a = 0.09, b = 0.08, v = 0.2, \eta = 0.02$ which is essential parameters which intervene directly in obtaining profile.

5. Discussions

In this section, it is important to note that the obtained analytical or graphical results corroborate with the theoretical predictions about the multi-soliton characters which consist of the proposed ansatz, and this with a more or less good accuracy. An illustration of this corroboration may be observed through:

- the different profiles of Figure 1 where Figure 1(a) reveals a bright soliton while Figure 1(b) presents a dark soliton. This is a consequence of the fact that the plotted module involves the sum of two squares, one of which is a bright soliton and the other a dark solid. Thus, depending on the values assigned to each of the coefficients a, b of the wave given by Equation (44), one or the other structure is obtained.
- the different profiles of Figure 2 where Figure 2(a) displays a bright-dark soliton structure while Figure 2(b) shows a dark-bright soliton structure. These two structures are all hybrids and have equal bright and dark tendencies respectively. This is the direct result of the fact that the analytic form given by Equation (29) is a package consisting in the order of a bright soliton (first term), a kink soliton (second term) and a hybrid soliton (third term), respectively. As a result, depending on the values assigned to each of the parameters a, b, c, v, η of the solitary wave, the hybrid forms (resulting from

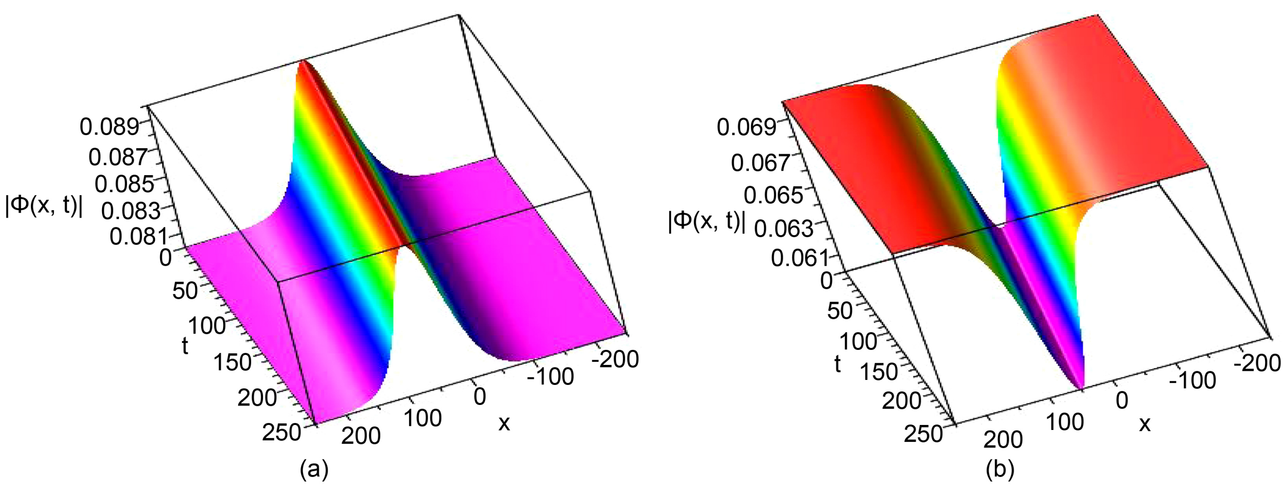

Figure 1. Graphical representation given by Equation (44): Bright and dark dispersive optical solitons: (a) $a = 0.09$; $b = 0.08$; $v = 0.2$; $\eta = 0.02$; (b) $a = 0.06$; $b = 0.07$; $v = 0.2$; $\eta = 0.02$.

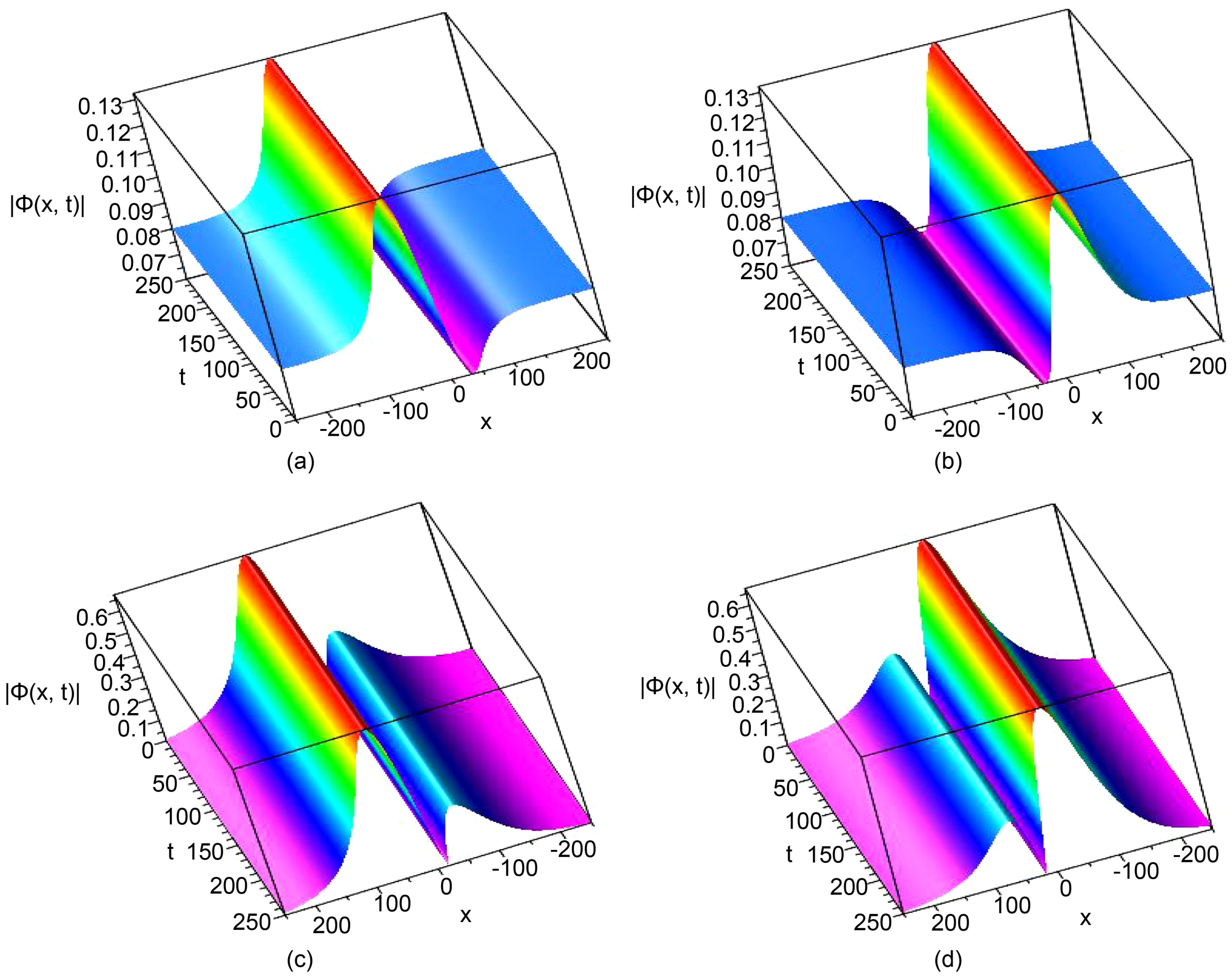

Figure 2. Graphical representation given by Equation (29): Hybrid dispersive optical soliton: (a) $a = 0.1$; $b = 0.08$; $c = -0.009$; $\nu = 0.002$; $\eta = 0.02$; (b) $a = 0.1$; $b = 0.08$; $c = 0.09$; $\nu = 0.002$; $\eta = 0.02$; (c) $a = 0.3$; $b = 0.001$; $c = 0.9$; $\nu = 0.002$; $\eta = 0.02$; (d) $a = 0.3$; $b = 0.001$; $c = -0.9$; $\nu = 0.002$; $\eta = 0.02$.

the third term of Equation (29)), are equals tendencies of bright-dark or dark-bright soliton. On the other hand, Figure 2(c) and Figure 2(d) display two bright-dark soliton structures with a strong bright soliton tendency. This is also due to the different values taken by the wave parameters.

In summary, we note that the values assigned to each of the parameters of the wave in particular, then extended to the parameters of the system studied in general, are fundamental in the formation of the resulting structure. In addition, Figure 1 and Figure 2 confirm one of the a priori ideas that loum during the conception of the ansatz given by Equation (9) and which reported that the disturbance term of amplitude c is at the origin of the emergence of the new hybrid structures displayed by Figure 2(c) and Figure 2(d). It should also be noted here that the results obtained are new and different from those proposed in [19] [20] [21] [22] [23], at least in the mathematical form, and through the displayed profiles. This being the case, one conclusion is in order.

6. Conclusion

In our previous work [20], we constructed the solitary wave multi-solutions of the modified Sasa-Satsuma equation describing the dynamics of sea waves. The method used for this purpose was the BDKm and it had shown its full effectiveness. The transformations associated with the method in its initial form were very cumbersome and required a lot of care in their management when we came to the idea of constructing the solitary wave solutions of the Sasa-Satsuma equation. But, this time, with the one that describes the wave dynamics in optical systems and in particular the optical fiber having the particularity of taking into account the Self-Steeping effect, the third-order dispersion and especially the stimulated Raman scattering effect, we opted to use the BDK method extended to iB-functions. The idea of the method to be adopted has been decided upon. We fixed ourselves the objective of constructing a hybrid solitary wave solution made up of an assembly of solitary waves of the pulse type of order 1, the kink of order 1, and the kink of order 2. In other words, a solitary pulse-kink wave solution with a strong kink tendency can be appreciated in Equation (9). We have successfully carried out analyses and calculations, and the results have been obtained. The advantage of choosing such a form of solution from the start comes from the fact that the other sub-solutions can be obtained via the constraints imposed on the coefficients a, b, c and above all from the properties of the iB-functions. The flexibility linked to iB-functions has also allowed deducing solitary wave solutions with a hyperbolic analytical sequence, which is the trigonometric solution of the Sasa-Satsuma equation concerned in this work. Beyond the mathematical field, the results obtained can find important applications in the theory of solitary waves, physics and especially fiber optic telecommunications.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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