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On a Compound Poisson Risk Model Perturbed by Brownian Motion with Variable Premium and Tail Dependence between Claims Amounts and Inter-Claim Time

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Abstract

This paper considers the compound Poisson risk model perturbed by Brownian motion with variable premium and dependence between claims amounts and inter-claim times via Spearman copula. It is assumed that the insurance company's portfolio is governed by two classes of policyholders. On the one hand, the first class where the amount of claims is high, and on the other hand, the second class where the amount of claims is low, this difference in claim amounts has significant implications for the insurance company's pricing and risk management strategies. When policyholders are in the first class, they pay an insurance premium of a constant amount c_1 and when they are in the second class, the premium paid is a constant amount c_2 such that $c_1 > c_2$. The nature of claims (low or high) is measured via random thresholds $\{\Theta_i, i = 1, 2, \cdots\}$. The study in this work will focus on the determination of the integro-differential equations satisfied by Gerber-Shiu functions and their Laplace transforms in the risk model perturbed by Brownian motion with variable premium and dependence between claims amounts and inter-claim times via Spearman copula.

Keywords

Gerber-Shiu Function, Copula, Integro-Differential Equation, Laplace Transform, Brownian Motion

1. Introduction

For a long time, risk modeling in actuarial science was based on the compound Poisson risk model introduced by the Swedish actuary Filip Lundberg (1903) and relayed by Harald Cramér (1930). This model in its original version is based on the independence between the random variables involved in the risk model (see for example [1]-[6]). However, in certain practical contexts, this assumption is inadequate and too restrictive. For instance, in flood insurance, the occurrence of multiple floods in a short period can lead to significant damages and claim amounts due to the accumulation of water. In earthquake insurance, it is the opposite, as in a high-risk area, the longer the time between two earthquakes, the more significant the second earthquake due to the accumulation of energy. This observation will prompt actuarial investigations in order to take this dependence into account. Numerous research studies have considered a dependence between claims amounts and inter-claim time via Farlie-Gumbel-Morgenstern copula and have produced interesting results (see for example [7]-[16]). This copula although commonly used in the literature, has certain limitations. It fails to model tail dependencies because it has an upper and lower tail dependence coefficient of zero. Based on this observation, authors will explore other copulas to express tail dependencies. Many works based on Spearman copula have produced satisfactory results on this subject (see for example [17]-[22]). In this work, this copula is retained as a tool for the dependence structure.

The actuarial industry is increasingly confronted with multiple, complex and varied situations. This has led many authors to incorporate a perturbation component via Brownian motion into the risk model in order to better reflect the growth pattern of insurance (see for example [23]-[31]).

In order to enable insurance companies to face the challenges raised above and remain competitive in the insurance market, scholars have found that it's important to vary the premium according to the intensity of claims in the portfolio. Thus, Zhong Li and Kristina P. Sendova studied the classical risk model with variable premium and dependence between claims amounts and inter-claim times via Farlie-Gumbel-Morgenstern copula and without a disturbance component (see [32]). Ying Shen studied the same model by adding a disturbance component through Brownian motion (see [33]). This work is intended to be a continuation of the approach adopted by Ying Shen but taking into account the tail dependence via the Spearman copula. In this approach, two states govern the policyholders in the portfolio: the first class where the amount of claims is high, and the second where the claim amounts accumulate slowly, which motivates the use of the Spearman copula to capture tail dependencies. The premiums paid by policyholders in a given class are homogeneous but different from one class to another class and depend on random thresholds $\{\Theta_i, i=1,2,\cdots\}$. It is assumed that in the first class, the premium paid per unit of time is a constant denoted $c_1 > 0$ and in the second, the premium paid per unit of time is a constant denoted $c_2 > 0$ and $c_1 > c_2$.

Denoting by $U(t)$ the surplus process (with $U(0)=u$) and by $C(t)$ the premium collected by the insurer until time *t*, in the Compound Poisson perturbed by Brownian motion with variable premium and dependence between claims amounts and inter-claim times via Spearman copula, the model follows the following dynamic:

$$\begin{aligned} U(t) &= u + C(t) - S(t) + \sigma B(t) \\ &= u + c_1 \int_0^t I_{\{J(s)=1\}} \mathrm{d}s + c_2 \int_0^t I_{\{J(s)=2\}} \mathrm{d}s - S(t) + \sigma B(t) \end{aligned} \tag{1.1}$$

where:

- $U(t)$ is the surplus process (with $U(0)=u$ the initial surplus and $u>0$);
- $C(t)$ represents the premium collected by the insurer until time t and is a piecewise premium with $c_i, i=1,2$, the constant premium rate collected by the insurer per unit of time for each class for claims;
- $S(t)=\sum_{i=1}^{N(t)} X_i$ is the aggregated loss process with a compound Poisson distribution where:

* $\{N(t), t \geq 0\}$ is the total number of recorded claims up to time *t*, following a Poisson process (Note that $S(t)=0$ if $N(t)=0$);

* $\{X_i, i \geq 1\}$ is a sequence of random variables representing the individual claim amount with a common density function f and cumulative distribution function F assumed to follow an exponential distribution with parameter β and mean μ;

*It is assumed that the distribution of the waiting time until the next claim depends on the amount of the previous claim via random threshold Θ_i, $i=1,2,\cdots$. It is also assumed that the sequence $\{\Theta_i, i=1,2,\cdots\}$ is a set of independent identical distribution random thresholds with common cumulative distribution $L(x)$, probability density $l(x)$ and independent of the amount X_i of claims. $J(t)$ represents two classes of policyholders so that if the amount of claims X_j is such that $X_j > \Theta_j$, the policyholders are placed in the first class and the waiting time V_1 until the next claim follows an exponential distribution with parameter $\lambda_1 > 0$ and probability density $k_1(t) = \lambda_1 \mathrm{e}^{-\lambda_1 t}$. If $X_j < \Theta_j$, the policyholders are placed in the second class and the waiting time V_2 until the next claim follows an exponential distribution with parameter $\lambda_2 > 0$ and probability density $k_2(t) = \lambda_2 \mathrm{e}^{-\lambda_2 t}$; $\lambda_1 \neq \lambda_2$.

- $B(t)$ is a standard Brownian motion independent with the aggregate claims process $S(t)$ and $\sigma > 0$ is the diffusion volatility;
- I_A is an indicator function, which equals 1 if event A occurs and 0 otherwise.

The structure of dependence between premium and the number of claims mentioned in the risk model defined by the relation (1.1) was introduced by Albrecher and Boxman (2004) where the ultimate-survival probabilities are considered (voir [32]).

The purpose of this work is to determine the integro-differential equations satisfied by Gerber-Shiu function and their Laplace transform in the risk model defined by Equation (1.1). To achieve this, the rest of the article is organized as follows: In Section 2, the preliminaries related to the risk model defined by Equation (1.1) will be presented. In Section 3, the integro-differential equations satisfied by the Gerber-Shiu function in the risk model defined by Equation (1.1), are determined. Section 4 is devoted to the study of the Laplace transform for Gerber-Shiu function of the risk model defined by the relation (1.1).

2. Preliminaries

2.1. Instant of Ruin and Ruin Probability

Let T be the instant of ruin of the insurance company. T is defined by:

$$T = \inf\left\{t \geq 0, U(t) < 0\right\} \tag{2.1}$$

When the probability of ruin is always zero, by convention, we denote $T = \infty$ and in this case,

$$U(t) \geq 0 \quad \forall t \geq 0.$$

The probability of ruin other a finite time horizon t is defined by:

$$\psi(u,t) = \Pr\left[T \in [0,t], U(t) < 0 \mid U(0) = u\right] \tag{2.2}$$

Similarly, the ultimate ruin probability is defined by:

$$\psi(u) = \psi(u,\infty) = \Pr\left[T < \infty, U(t) < 0 \mid U(0) = u\right] \tag{2.3}$$

2.2. Expected Discounted Penalty Function of Gerber-Shiu

The expected discounted penalty function of Gerber-Shiu, first appeared in the work of Gerber and Shiu in 1998 (see [5]). Nowadays, this function is of significant interest in research.

Its analysis remains a central question both in insurance and finance, as it is a valuable tool not only for studying the probability of ruin but also calculating retirement and reinsurance premiums, option pricing, and more. In the risk model defined by the relation (1.1), this function is defined by:

$$\phi_i(u) = E\left[\mathrm{e}^{-\delta T_i} w\left(U\left(T_i^-\right), \left|U\left(T_i\right)\right|\right) I\left(T_i < \infty\right) \mid U(0) = u\right], i = 1,2 \tag{2.4}$$

where

- T_i is the instant of ruin defined by Equation (2.1);
- T_i^- is the instant just before ruin;
- δ is a interest force;
- The penalty function $w(x,y)$ is a positive function of the surplus just before ruin, $U\left(T_i^-\right)$ and the deficit at ruin $\left|U\left(T_i\right)\right|, \forall x, y \geq 0$;
- I_A is a indicator function, wich equal 1 if event A occurs and 0 otherwise.

Because of the perturbation term, the expected discounted penalty function of the Gerber-Shiu is decomposed according to wether ruin is caused by claims or oscillation, *i.e.*

$$\phi_i(u) = \phi_{w,i}(u) + \phi_{d,i}(u), i = 1,2 \tag{2.5}$$

Where:

- $\phi_{w,i}(u)$ is the Gerber-Shiu when ruin is caused by claims and is defined by:

$$\phi_{w,i}(u) = E\left[e^{-\delta T_i} w\left(U\left(T_i^-\right), \left|U\left(T_i\right)\right|\right) I\left(T_i < \infty, U\left(T_i\right) < 0\right) \mid U(0) = u\right], i = 1,2 \quad (2.6)$$

- $\phi_{d,i}(u)$ is the Gerber-Shiu when ruin is due to oscillation and is defined by:

$$\phi_{d,i}(u) = E\left[e^{-\delta T_i} w\left(U\left(T_i^-\right), \left|U\left(T_i\right)\right|\right) I\left(T_i < \infty, U\left(T_i\right) = 0\right) \mid U(0) = u\right]$$

$$\phi_{d,i}(u) = w(0,0)\left[e^{-\delta T_i} I\left(T_i < \infty, U\left(T_i\right) = 0\right) \mid U(0) = u\right], i = 1,2 \quad (2.7)$$

with $w(0,0) = 1$, where: $T_i, T_i^-, \delta, I_A,, U\left(T_i^-\right), \left|U\left(T_i\right)\right|$ and $w(x,y)$ are defined in the relation (2.4).

To guarantee that ruin will not be a certain event, the model must verify the following solvency condition:

$$\frac{c_1}{\lambda_1}\mathbb{P}\{X > \theta\} + \frac{c_2}{\lambda_2}\mathbb{P}\{X < \theta\} > \mu \quad (2.8)$$

2.3. Dependence Structure

2.3.1. Copulas

Copulas introduced by Abe Sklar in 1998, are an innovative and relevant tool for introducing dependence between multiples random variables. Given the marginal distribution functions of several random variables, copulas allow us to establish their joint distribution function. Nowadays, they are fundamental in modeling multivariate distributions in finance, insurance and hydrology. Key references on copulas theory include Joe [7] and Nelsen [13].

Definition 2.1. A bivariate copula C is a non-decreasing, right-continuous function defined from $[0,1]^2$ into $[0,1]$ and satisfying the following properties:

1) $\lim_{u_1 \to 0} C(u_1,u_2) = 0^2$ and $\lim_{u_2 \to 0} C(u_1,u_2) = 0 \quad \forall u_1,u_2 \in [0,1]$;
2) $\lim_{u_1 \to 1} C(u_1,u_2) = u_2$ and $\lim_{u_2 \to 1} C(u_1,u_2) = u_1 \quad \forall u_1,u_2 \in [0,1]$;
3) $\forall (u_1,u_2) \in [0,1]^2, (v_1,v_2) \in [0,1]^2$ such that $u_1 \le v_1$ and $u_2 \le v_2$, C verifies $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \ge 0$.

Theorem 2.1. (Sklar's theorem). Let two random variables U_1 and U_2 and F their joint distribution function with F_1 and F_2 their marginal. Then, there exists a copula C defined from $[0,1]^2$ into $[0,1]$ such that for all u_1 and u_2 in $\mathbb{R}$, $F(u_1,u_2) = C(F_1(u_1), F_2(u_2))$.

2.3.2. Tail Dependence Concept

The concept of tail dependence relates to the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. It is a concept that is relevant for the study of dependence between extreme value.

Definition 2.2. If C is a bivariate copula such that the limit:

- $\lim_{u \to 1^-} \dfrac{\bar{C}(u,u)}{1-u} = \lim_{u \to 1^-} \dfrac{1-2u+C(u,u)}{1-u} = \lambda_U$ exists, then C has an upper tail dependence if $\lambda_U \in (0,1]$ and has upper tail independence if $\lambda_U = 0$.

- $\lim_{u\to 0^+}\frac{C(u,u)}{1-u}=\lambda_L$ exists, then C has a lower tail dependence if $\lambda_L\in(0,1]$ and has lower tail independence if $\lambda_L=0$.

The real numbers λ_U and λ_L are called tail dependence coefficients.

Remark 2.1.

- From a probabilistic point of view,

$$\lambda_U=\lim_{u\to 1^-}\mathbb{P}(U_2>u|U_1>u);\ \lambda_L=\lim_{u\to 0^+}\mathbb{P}(U_2<u|U_1<u).$$

- The Farlie-Gumbel-Morgenstern copula defined for all $(u_1,u_2)\in[0,1]^2$ by

$$C_{FGM}(u_1,u_2)=u_1u_2+\theta u_1u_2(1-u_1)(1-u_2),$$

where $\theta\in[-1,1]$ is a dependency parameter, has the tail dependence coefficients $\lambda_U=0$ and $\lambda_L=0$, hence its inability to measure tail dependencies.

2.3.3. Model of Dependence Based on Spearman Copula

In this article, the structure of dependence is ensured by the Spearman copula. Il is defined for all $(u,v)\in[0,1]^2$ and $\alpha\in[0,1]$ as follow:

$$C_\alpha(u,v)=(1-\alpha)C_I(u,v)+\alpha C_M(u,v) \tag{2.9}$$

where: $C_I(u,v)=uv$; $C_M(u,v)=\min(u,v)$ and α is a dependency parameter.

The Spearman copula allows for the introduction of positive dependence as well as tail dependencies in many situations. It is suitable for modelling dependence on extreme values because $\lambda_L=\alpha$ and $\lambda_U=\alpha$. It also includes independence when $\alpha=0$. Using Formula (2.9), the random vectors of claim amounts and inter-claim times (X,V_i) has the joint distribution function given by

$$\begin{aligned}F_{X,V_i}(x,t)&=C_\alpha\left(F_X(x),F_{V_i}(t)\right)\\&=(1-\alpha)C_I\left(F_X(x),F_{V_i}(t)\right)+\alpha C_M\left(F_X(x),F_{V_i}(t)\right)\\F_{X,V_i}(x,t)&=(1-\alpha)F_{I,i}(x,t)+\alpha F_{M,i}(x,t)\end{aligned} \tag{2.10}$$

where F_X,F_{V_i} are the marginal distributions of the random variables X and V_i, respectively.

The support of the copula C_M is $D=\left\{(u,v)\in[0,1]^2:u=v\right\}$ (see Nelsen [13]). On $[0,1]^2\setminus D$, $\frac{\partial^2 C_M}{\partial u\partial v}(u,v)=0$ and on D, $C_M(u,v)$ is the uniform distribution.

Since the dependence structure between the random variables X and V_i ($i=1,2$) is described by the copula C_M, they are monotonic and there is almost certainly an increasing function l such that $X=l(V_i)$ (see Nelsen [13], Page 27). The distribution function of X is

$$\begin{aligned}&F_X(x)=F_{V_i}\left(l^{-1}(x)\right)\\&\Leftrightarrow 1-e^{-\beta x}=1-e^{-\lambda_i l^{-1}(x)}\\&\Leftrightarrow e^{-\beta x}=e^{-\lambda_i l^{-1}(x)}\\&\Leftrightarrow \int_0^\infty e^{-\beta x}dx=\int_0^\infty e^{-\lambda_i l^{-1}(x)}dx\end{aligned}$$

$$\Leftrightarrow \frac{1}{\beta} = \int_0^\infty e^{-\lambda_i l^{-1}(x)} dx .$$

We deduce that:

$$l(t) = \frac{\lambda_i}{\beta} t \tag{2.11}$$

The joint distribution $F_{X,V_i}(x,t)$ of the random vector (X, V_i) is singular, whose support is the set $D' = \{(x,t) : F_X(x) = F_{V_i}(t)\} = \{(x,t) : x = l(t)\}$. Its distribution is:

$$G_i(t) = F_{M,i}(l(t),t) = 1 - e^{-\lambda_i t} \quad \text{on the set} \quad D' = \left\{(x,t) : x = \frac{\lambda_i}{\beta} t\right\}.$$

2.3.4. Some Functions an Operators

Before moving on the next section, let's introduce some functions and operators that will be used in this work.

- The function $\varphi_f(u)$ is defined by:

$$\varphi_f(u) = \int_u^\infty \omega(u, y-u) f(y) dy \tag{2.12}$$

- The Laplace transform of a function f is defined by:

$$\hat{f}(s) = \int_0^\infty e^{-sy} f(y) dy, s \geq 0 \tag{2.13}$$

- The Dickson Hipp operator $T_r, r \geq 0$ and some of its properties are given by:

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) dy \tag{2.14}$$

$$T_r f(0) = \hat{f}(r), r \geq 0 \tag{2.15}$$

$$T_{r_1} T_{r_2} f(x) = T_{r_2} T_{r_1} f(x) = \frac{T_{r_1} f(x) - T_{r_2} f(x)}{r_2 - r_1} \tag{2.16}$$

$$T_{r_1} T_{r_2} f(0) = T_{r_2} T_{r_1} f(0) = \frac{\hat{f}(r_1) - \hat{f}(r_2)}{r_2 - r_1} \tag{2.16}$$

where $r_2, r_1 \geq 0$, $r_2 \neq r_1$.

Many properties on Dickson Hipp operator can be consulted in [34] [35] [36] [37].

3. Integro-Differential Equations Satisfied by the Gerber-Shiu Function $\phi_{w,i}(u)$ and $\phi_{d,i}(u)$, $i = 1, 2$

The main goal of this section is to show that the Gerber-Shiu function $\phi_{w,i}(u)$ and $\phi_{d,i}(u), i = 1, 2$ satisfy some integro-differential equation. To achieve that, some notations and preliminaries are introduced.

3.1. Results and Preliminary

Let's put:

- $W_i(t) = -c_i t - \sigma B(t), i = 1,2$ a Brownian motion starting from zero with drift $-c_i$ and variance σ^2;
- $\overline{W}_i(t) = \sup_{0 \le s \le t} W_i(s)$ is the running supremum of $W_i(t)$;
- $\tau_i = \inf\{t \ge 0 : W_i(t) = u\}$ is the first hitting time of the value $u > 0$.

By Formula (2.0.2) of ([34], pp. 295), we have for $\delta \ge 0$,

$$E\left[e^{-\delta\tau_i}\right] = e^{-\eta_i u} \quad \text{or} \quad \eta_i = \frac{c_i}{\sigma^2} + \sqrt{\frac{2\delta}{\sigma^2} + \frac{c_i^2}{\sigma^4}};\ i = 1,2. \tag{3.1}$$

For $\delta \ge 0$, we define the following potential measure

$$\mathcal{P}_i(u, dy, dx) = E\left[e^{-\delta V_i} I\left(\overline{W}_i(V_i) < u, W_i(V_i) \in dy, X \in dx\right)\right], \tag{3.2}$$

where:

- $W_i(t), i = 1,2$ is a Brownian motion starting from zero with drift $-c_i$ and variance σ^2;
- X is a random variable representing the amount of claim when a disaster occurs.
- $V_i, i = 1,2$ is a random variable representing the inter-claim time of exponential law with parameter λ_i;
- I_A is an indicator function, which equals 1 if event A occurs and 0 otherwise;
- u is the initial surplus with: $u > 0$; $x > 0$; $y < u$.

The potential measure $\mathcal{P}_i(u, dy, dx)$ plays an important role in analyzing the Gerber-Shiu functions $\phi_{w,i}(u)$ and $\phi_{d,i}(u), i = 1,2$.

In order to determine the potential measure in relation (3.2), let's first calculate the following measure:

$$\mathcal{U}_{q,i}(u, dy) = \mathbb{P}\left(\overline{W}_i(e_q) < u, W_i(e_q) \in dy\right) \tag{3.3}$$

where:

- e_q is a random variable of exponential law with parameter q;
- u is defined in relation (3.2), with $u > 0$; $y < u$; $i = 1,2$.

Lemma 3.1 ([34]). Assume that e_q is independent of $\{W_i(t)\}, i = 1,2$. Then the following variables $\overline{W}_i(e_q)$ and $\overline{W}_i(e_q) - W_i(e_q)$ are independent and exponentially distributed with respective rates:

$$\nu_{1,i} = \frac{c_i}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{c_i^2}{\sigma^4}} \quad \text{et} \quad \nu_{2,i} = -\frac{c_i}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{c_i^2}{\sigma^4}} \tag{3.4}$$

For $0 \le y < u$, $i = 1,2$,

$$\begin{aligned}\mathcal{U}_{q,i}(u, dy) &= \int_{x \in [y,u)} \mathbb{P}\left[\overline{W}_i\left(e_q \in dx, W_i(e_q) - \overline{W}_i(e_q) + \overline{W}_i(e_q) \in dy\right)\right] \\ &= \int_{x \in [y,u)} \nu_{1,i} e^{-\nu_{1,i}x} \nu_{2,i} e^{-\nu_{2,i}(x-y)} dx dy\end{aligned}$$

$$\mathcal{U}_{q,i}(u, dy) = \frac{\nu_{1,i}\nu_{2,i}}{\nu_{1,i} + \nu_{2,i}}\left(e^{-\nu_{1,i}y} - e^{-(\nu_{1,i}+\nu_{2,i})u + \nu_{2,i}y}\right) dy \tag{3.5}$$

For $y < 0$, $i = 1,2$,

$$\begin{aligned}\mathcal{U}_{q,i}(u, dy) &= \int_{x \in [0,u)} \mathbb{P}\left[\overline{W}_i\left(e_q \in dx, W_i(e_q) - \overline{W}_i(e_q) + \overline{W}_i(e_q) \in dy\right)\right] \\ &= \int_{x \in [0,u)} \nu_{1,i} e^{-\nu_{1,i}x} \nu_{2,i} e^{-\nu_{2,i}(x-y)} dx dy\end{aligned}$$

$$\mathcal{U}_{q,i}(u,dy)=\frac{\nu_{1,i}\nu_{2,i}}{\nu_{1,i}+\nu_{2,i}}\left(\mathrm{e}^{\nu_{2,i}y}-\mathrm{e}^{-(\nu_{1,i}+\nu_{2,i})u+\nu_{2,i}y}\right)dy \tag{3.6}$$

Relations (3.5) and (3.6) show that for $u>0$, the potential measure $\mathcal{U}_{q,i}(u,\mathrm{d}y)$ is absolutely continuous w.r.t Lebesgue measure.

From (3.5) and (3.6), we can obtain the following result:

Lemma 3.2: *For* $0\le y<u$, *the measure* $\mathcal{P}_i(u,\mathrm{d}y,\mathrm{d}x)$ *has a density given by*

$$p_i(u,y,x)=\frac{\lambda_i\eta_{1,i}\eta_{2,i}}{(\lambda_i+\delta)(\eta_{1,i}+\eta_{2,i})}\left(\mathrm{e}^{-\eta_{1,i}y}-\mathrm{e}^{-(\eta_{1,i}+\eta_{2,i})u+\eta_{2,i}y}\right)f(x) \tag{3.7}$$

For $y<0$, *the measure* $\mathcal{P}_i(u,b,\mathrm{d}y,\mathrm{d}x)$ *has a density given by*

$$p_i(u,y,x)=\frac{\lambda_i\eta_{1,i}\eta_{2,i}}{(\lambda_i+\delta)(\eta_{1,i}+\eta_{2,i})}\left(\mathrm{e}^{\eta_{2,i}y}-\mathrm{e}^{-(\eta_{1,i}+\eta_{2,i})u+\eta_{2,i}y}\right)f(x) \tag{3.8}$$

where:

$$\eta_{1,i}=\frac{c_i}{\sigma^2}+\sqrt{\frac{2(\lambda_i+\delta)}{\sigma^2}+\frac{c_i^2}{\sigma^4}};\ \eta_{2,i}=-\frac{c_i}{\sigma^2}+\sqrt{\frac{2(\lambda_i+\delta)}{\sigma^2}+\frac{c_i^2}{\sigma^4}} \tag{3.9}$$

Proof. En conditioning on the value of V_i, we have:

$$\begin{aligned}\mathcal{P}_i(u,\mathrm{d}y,\mathrm{d}x)&=\int_0^\infty \mathrm{e}^{-\delta t}f_{V_i}(t)f_{X|V_i=t}(x)\mathbb{P}\left(\bar{W}_i(t)<u,W_i(t)\in\mathrm{d}y\right)\mathrm{d}x\mathrm{d}t\\&=\lambda_i\int_0^\infty \mathrm{e}^{-(\lambda_i+\delta)t}\mathbb{P}\left(\bar{W}_i(t)<u,W_i(t)\in\mathrm{d}y\right)f(x)\mathrm{d}x\mathrm{d}t\\&=\lambda_i\int_0^\infty \mathrm{e}^{-(\lambda_i+\delta)t}\mathbb{P}\left(\bar{W}_i(t)<u,W_i(t)\in\mathrm{d}y\right)f(x)\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$\mathcal{P}_i(u,\mathrm{d}y,\mathrm{d}x)=\frac{\lambda_i}{\lambda_i+\delta}f(x)\mathcal{U}_{\lambda_i+\delta,i}(u,\mathrm{d}y)\mathrm{d}x \tag{3.10}$$

By combining this last relation with relations (3.5) and (3.6), the expected result is obtained.

3.2. Integro-Differential Equations Satisfied by the Functions $\phi_{w,i}(u),\ i=1,2$

Theorem 3.1. *The Gerber-Shiu functions* $\phi_{w,i}(u), i=1,2$ *in the risk model defined by relation* (1.1) *verify the following integro-diffferential equations.*

$$\begin{aligned}\phi_{w,1}(u)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m_{1,1}(u)+T_{\eta_{2,1}}\Upsilon_{1,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}_{1,1}(\eta_{2,1})\right)\\&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m_{2,1}(u)+T_{\eta_{2,1}}\Upsilon_{2,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}_{2,1}(\eta_{2,1})\right)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\phi_{w,2}(u)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(m_{1,2}(u)+T_{\eta_{2,2}}\Upsilon_{1,2}(v)-\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{1,2}(\eta_{2,2})\right)\\&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(m_{2,2}(u)+T_{\eta_{2,2}}\Upsilon_{2,2}(v)-\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{2,2}(\eta_{2,2})\right)\end{aligned} \tag{3.12}$$

where

$$L(x)=\mathbb{P}(x>\Theta_1);\ \bar{L}(x)=\mathbb{P}(x<\Theta_1) \tag{3.13}$$

$$\chi_{1,1}(x)=L(x)f(x);\ \chi_{2,1}(x)=\bar{L}(x)f(x);\\ \chi_{1,2}(x)=\chi_{1,1}(x);\ \chi_{2,2}(x)=\chi_{2,1}(x) \tag{3.14}$$

$$h_1(x)=\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}};\ h_2(x)=\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}} \tag{3.15}$$

$$\zeta_{1,1}(x)=h_1(x)L(x);\ \zeta_{2,1}(x)=h_1(x)\bar{L}(x);\\ \zeta_{1,2}(x)=h_2(x)L(x);\ \zeta_{2,2}(x)=h_2(x)\bar{L}(x) \tag{3.16}$$

$$\Upsilon_{1,1}(y)=\int_0^y\left(\chi_{1,1}(x)\phi_{w,1}(y-x)+\chi_{2,1}(x)\phi_{w,2}(y-x)\right)\mathrm{d}x+\varphi_f(y) \tag{3.17}$$

$$\Upsilon_{2,1}(y)=\int_0^y\left(\zeta_{1,1}(x)\phi_{w,1}(y-x)+\zeta_{2,1}(x)\phi_{w,2}(y-x)\right)\mathrm{d}x+\varphi_{h_1}(y) \tag{3.18}$$

$$\Upsilon_{1,2}(y)=\int_{x=0}^y\left(\chi_{1,2}(x)\phi_{w,1}(y-x)+\chi_{2,2}(x)\phi_{w,2}(y-x)\right)\mathrm{d}x+\varphi_f(y) \tag{3.19}$$

$$\Upsilon_{2,2}(y)=\int_0^y\left(\zeta_{1,2}(x)\phi_{w,1}(y-x)+\zeta_{2,2}(x)\phi_{w,2}(y-x)\right)\mathrm{d}x+\varphi_{h_2}(y) \tag{3.20}$$

$$m_{1,1}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{1,1}(v)\mathrm{d}v;\ m_{2,1}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{2,1}(v)\mathrm{d}v \tag{3.21}$$

$$m_{1,2}(u)=\int_0^u\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}(v)\mathrm{d}v;\ m_{2,2}(u)=\int_0^u\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{2,2}(v)\mathrm{d}v \tag{3.22}$$

- **Proof of relation (3.1)**

By conditioning on the time and the amount of the first claim and taking into account the fact that ruin may occur or not, he follows:

$$\begin{aligned}\phi_{w,1}(u)&=\mathbb{E}\left[\mathrm{e}^{-\delta V_1}\mathbb{E}\left[\phi_{w,1}(u-W_1-X_1)I_{\{X_1<u-W_1,\bar{W}_1<u,X_1<\Theta_1\}}\mid(V_1,X_1)\right]\right]\\&\quad+\mathbb{E}\left[\mathrm{e}^{-\delta V_1}\mathbb{E}\left[\phi_{w,2}(u-W_1-X_1)I_{\{X_1<u-W_1,\bar{W}_1<u,X_1<\Theta_1\}}\mid(V_1,X_1)\right]\right]\\&\quad+\mathbb{E}\left[\mathrm{e}^{-\delta V_1}\mathbb{E}\left[w(u-W_1,X_1-(u-W_1))I_{\{X_1>u-W_1,\bar{W}_1<u\}}\mid(V_1,X_1)\right]\right]\\&=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\\&\quad\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_1(x,t)\\&\quad+\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\\&\quad\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\mathrm{d}F_1(x,t)\end{aligned}$$

$$\begin{aligned}\phi_{w,1}(u)&=(1-\alpha)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\\&\quad\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_{I,1}(x,t)\\&\quad+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathbb{P}\left[\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\\&\quad\times\int_{x=0}^{u-y}\mathrm{e}^{-\delta t}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)\right.\\&\quad\left.+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_{M,1}(x,t)\\&\quad+(1-\alpha)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\\&\quad\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\mathrm{d}F_{I,1}(x,t)\\&\quad+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\\&\quad\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\mathrm{d}F_{M,1}(x,t)\end{aligned} \tag{3.23}$$

By setting:

$$I_{1,1}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_{I,1}(x,t)$$

$$I_{2,1}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\times\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)\mathrm{d}F_{I,1}(x,t)$$

$$I_{3,1}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_{M,1}(x,t)$$

$$I_{4,1}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\times\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)\mathrm{d}F_{M,1}(x,t)$$

Relation (3.23) becomes:

$$\phi_{w,1}(u)=(1-\alpha)\left(I_{1,1}+I_{2,1}\right)+\alpha\left(I_{3,1}+I_{4,1}\right)\tag{3.24}$$

Let's calculate $I_{1,1}+I_{2,1}$,

$$\begin{aligned}I_{1,1}+I_{2,1}&=\lambda_1\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-(\delta+\lambda_1)t}\mathbb{P}\left[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\\&\quad\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)\right.\\&\quad\left.+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]f(x)\mathrm{d}x\mathrm{d}t\\&\quad+\lambda_1\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-(\delta+\lambda_1)t}\mathbb{P}\left(\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\\&\quad\times\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)f(x)\mathrm{d}x\mathrm{d}t\\&=\lambda_1\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)\right.\\&\quad\left.+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]f(x)\\&\quad\times\left(\int_{t=0}^{\infty}\mathrm{e}^{-(\delta+\lambda_1)t}\mathbb{P}\left[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right]\mathrm{d}t\right)\mathrm{d}x\\&\quad+\lambda_1\int_{y=-\infty}^{u}\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)f(x)\\&\quad\times\left(\int_{t=0}^{\infty}\mathrm{e}^{-(\delta+\lambda_1)t}\mathbb{P}\left(\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\right)\mathrm{d}t\right)\mathrm{d}x\end{aligned}$$

$$\begin{aligned}I_{1,1}+I_{2,1}=\frac{\lambda_1}{\delta+\lambda_1}&\left(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left(\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)\right.\right.\\&\left.+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right)f(x)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x\\&+\int_{y=-\infty}^{u}\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)f(x)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x\end{aligned}\tag{3.25}$$

By relation (3.13), relation (3.25) becomes:

$$\begin{aligned}I_{1,1}+I_{2,1}=\frac{\lambda_1}{\delta+\lambda_1}&\left(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left(L(x)\phi_{w,1}(u-y-x)\right.\right.\\&\left.+\overline{L}(x)\phi_{w,2}(u-y-x)\right)f(x)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x\\&\left.+\int_{y=-\infty}^{u}\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)f(x)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x\right)\end{aligned}\tag{3.26}$$

By relation (3.14), relation (3.26) becomes:

$$I_{1,1}+I_{2,1}=\frac{\lambda_1}{\delta+\lambda_1}\Big(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)$$
$$+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x \tag{3.27}$$
$$+\int_{y=-\infty}^{u}\int_{x=u-y}^{\infty}w\big(u-y,x-(u-y)\big)f(x)\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y)\mathrm{d}x$$

By lemma 3.2, relation (3.27) can be put in the form:

$$I_{1,1}+I_{2,1}=\frac{\lambda_1\eta_{1,1}\eta_{2,1}}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\times\Big(\int_{y=0}^{0}\int_{x=0}^{u-y}\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)$$
$$+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\Big(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\Big)\mathrm{d}y\mathrm{d}x$$
$$+\int_{y=-\infty}^{0}\int_{x=0}^{u-y}\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)$$
$$\times\Big(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\Big)\mathrm{d}y\mathrm{d}x$$
$$+\int_{y=0}^{0}\int_{x=u-y}^{\infty}w\big(u-y,x-(u-y)\big)\Big(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\Big)f(x)\mathrm{d}y\mathrm{d}x$$
$$+\int_{y=-\infty}^{0}\int_{x=u-y}^{\infty}w\big(u-y,x-(u-y)\big)\Big(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\Big)f(x)\mathrm{d}y\mathrm{d}x\Big) \tag{3.28}$$

Let's calculate $I_{3,1}$ and $I_{4,1}$.

The distribution $F_{M,1}(x,t)$ of the random vector (X,V_1) is a comonotonic distribution whose support is the set:

$$D_1'=\{(x,t):F_X(x)=F_{V_1}(t)\}=\{(x,t):x=l(t)\}=\Big\{(x,t):x=\frac{\lambda_1}{\beta}t\Big\}.$$

The distribution $F_{M,1}(x,t)$ is $G_1(t)=F_{M,1}(l(t),t)=1-\mathrm{e}^{-\lambda_1 t}$ on $D_1'=\Big\{(x,t):x=\frac{\lambda_1}{\beta}t\Big\}$.

Let's calculate $I_{3,1}$,

$$I_{3,1}=\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\big[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\big]$$
$$\times\int_{t=0}^{\infty}\int_{x=0}^{u-y}\mathbb{P}\big[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\big]\mathrm{d}F_{M,1}(x,t)$$

$$I_{3,1}=\int_{y=-\infty}^{u}\int_{K_1}\mathrm{e}^{-\delta t}\mathbb{P}\big[\overline{W}_1(t)<u,W_1(t)\in\mathrm{d}y\big]$$
$$\times\big[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\big]\mathrm{d}G_1(t) \tag{3.29}$$

where:

$$K_1=\Big\{t\in\mathbb{R}_+ \mid 0\le x=\frac{\lambda_1}{\beta}t\le u-y\Big\}$$

$$K_1=\Big\{t\in\mathbb{R}_+ \mid 0\le t\le\frac{\beta}{\lambda_1}(u-y)\Big\} \tag{3.30}$$

By injecting relation (3.30) into relation (3.29), he follows:

$$
\begin{aligned}
I_{3,1} &= \lambda_1 \int_{y=-\infty}^{u} \int_0^{\frac{\beta}{\lambda_1}(u-y)} \mathrm{e}^{-(\delta+\lambda_1)t} \mathbb{P}\left[\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y\right] \\
&\quad \times \left[\mathbb{P}(x > \Theta_1)\phi_{w,1}\left(u - y - \frac{\lambda_1}{\beta}t\right) + \mathbb{P}(x < \Theta_1)\phi_{w,2}\left(u - y - \frac{\lambda_1}{\beta}t\right)\right]\mathrm{d}t \\
&= \lambda_1 \int_{y=-\infty}^{u} \int_0^{\infty} \mathrm{e}^{-(\delta+\lambda_1)t} \mathbb{P}\left[\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y\right] \\
&\quad \times \left[\mathbb{P}(x > \Theta_1)\phi_{w,1}\left(u - y - \frac{\lambda_1}{\beta}t\right) + \mathbb{P}(x < \Theta_1)\phi_{w,2}\left(u - y - \frac{\lambda_1}{\beta}t\right)\right] I_{\left\{0 \le t \le \frac{\beta}{\lambda_1}(u-y)\right\}}\mathrm{d}t \\
&= \frac{\lambda_1}{\delta+\lambda_1} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\mathbb{P}\left[\bar{W}_1\left(e_{(\delta+\lambda_1)}\right) < u, W_1\left(e_{(\delta+\lambda_1)}\right) \in \mathrm{d}y\right] \\
&\quad \times \Bigg(\mathbb{P}(x > \Theta_1)\phi_{w,1}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right) \\
&\quad + \mathbb{P}(x < \Theta_1)\phi_{w,2}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right)\Bigg) I_{\left\{0 \le e_{(\delta+\lambda_1)} \le \frac{\beta}{\lambda_1}(u-y)\right\}}\Bigg] \\
&= \frac{\lambda_1}{\delta+\lambda_1} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\Bigg(\mathbb{P}(x > \Theta_1)\phi_{w,1}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right) \\
&\quad + \mathbb{P}(x < \Theta_1)\phi_{w,2}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right)\Bigg) I_{\left\{0 \le e_{(\delta+\lambda_1)} \le \frac{\beta}{\lambda_1}(u-y)\right\}}\Bigg] \mathcal{U}_{\delta+\lambda_1,1}(u, \mathrm{d}y)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,1} &= \frac{\lambda_1}{\delta+\lambda_1} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\Bigg(L(x)\phi_{w,1}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right) \\
&\quad + \bar{L}(x)\phi_{w,2}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right)\Bigg) I_{\left\{0 \le e_{(\delta+\lambda_1)} \le \frac{\beta}{\lambda_1}(u-y)\right\}}\Bigg] \mathcal{U}_{\delta+\lambda_1,1}(u, \mathrm{d}y)
\end{aligned}
\tag{3.31}
$$

By lemma 3.2, relation (3.31) can be written:

$$
\begin{aligned}
I_{3,1} &= \frac{\lambda_1 \eta_{1,1}\eta_{2,1}}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})} \\
&\quad \times \Bigg(\int_{y=0}^{u} \left(\mathrm{e}^{-\eta_{1,1}y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \times \mathbb{E}\Bigg[\Bigg(L(x)\phi_{w,1}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right) \\
&\quad + \bar{L}(x)\phi_{w,2}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right)\Bigg) I_{\left\{0 \le e_{(\delta+\lambda_1)} \le \frac{\beta}{\lambda_1}(u-y)\right\}}\Bigg]\mathrm{d}y \\
&\quad + \int_{y=-\infty}^{0} \left(\mathrm{e}^{\eta_{2,1}y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \times \mathbb{E}\Bigg[\Bigg(L(x)\phi_{w,1}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right) \\
&\quad + \bar{L}(x)\phi_{w,2}\left(u - y - \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}\right)\Bigg) I_{\left\{0 \le e_{(\delta+\lambda_1)} \le \frac{\beta}{\lambda_1}(u-y)\right\}}\Bigg]\mathrm{d}y\Bigg)
\end{aligned}
$$

$$= \frac{\lambda_1 \eta_{1,1} \eta_{2,1}}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})} \left(\int_{y=0}^{u} \left(\mathrm{e}^{-\eta_{1,1} y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1} y} \right) \right.$$

$$\times \int_0^{\frac{\beta}{\lambda_1}(u-y)} (\delta+\lambda_1) \mathrm{e}^{-(\delta+\lambda_1)t} \left[L(x) \phi_{w,1} \left(u - y - \frac{\lambda_1}{\beta} t \right) \right.$$

$$\left. + \bar{L}(x) \phi_{w,2} \left(u - y - \frac{\lambda_1 t}{\beta} \right) \right] \mathrm{d}t \mathrm{d}y + \int_{y=-\infty}^{0} \left(\mathrm{e}^{\eta_{2,1} y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1} y} \right)$$

$$\times \int_0^{\frac{\beta}{\lambda_1}(u-y)} (\delta+\lambda_1) \mathrm{e}^{-(\delta+\lambda_1)t} \left[L(x) \phi_{w,1} \left(u - y - \frac{\lambda_1}{\beta} t \right) \right.$$

$$\left. \left. + \bar{L}(x) \phi_{w,2} \left(u - y - \frac{\lambda_1 t}{\beta} \right) \right] \mathrm{d}t \mathrm{d}y \right)$$

$$I_{3,1} = \frac{\lambda_1 \eta_{1,1} \eta_{2,1}}{\eta_{1,1}+\eta_{2,1}} \frac{\beta}{\lambda_1} \left(\int_{y=0}^{u} \left(\mathrm{e}^{-\eta_{1,1} y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1} y} \right) \right.$$
$$\times \int_0^{u-y} \mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}} \left[L(x) \phi_{w,1}(u-y-x) + \bar{L}(x) \phi_{w,2}(u-y-x) \right] \mathrm{d}x \mathrm{d}y$$
$$+ \int_{y=-\infty}^{0} \left(\mathrm{e}^{\eta_{2,1} y} - \mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1} y} \right)$$
$$\left. \times \int_0^{u-y} \mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}} \left[L(x) \phi_{w,1}(u-y-x) + \bar{L}(x) \phi_{w,2}(u-y-x) \right] \mathrm{d}x \mathrm{d}y \right) \quad (3.32)$$

Let's calculate $I_{4,1}$:

$$I_{4,1} = \int_{t=0}^{\infty} \int_{y=-\infty}^{u} \mathrm{e}^{-\delta t} \mathbb{P}\left(\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y \right)$$
$$\times \int_{x=u-y}^{\infty} w(u-y, x-(u-y)) \mathrm{d}F_{M,1}(x,t)$$
$$= \int_{y=-\infty}^{u} \int_{\mathbb{R}_+ \setminus K_1} \mathrm{e}^{-\delta t} \mathbb{P}\left[\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y \right]$$
$$\times w\left(u-y, \frac{\lambda_1}{\beta} t - (u-y) \right) \mathrm{d}G_1(t)$$
$$= \lambda_1 \int_{y=-\infty}^{u} \int_{\frac{\beta}{\lambda}(u-y)}^{\infty} \mathrm{e}^{-(\delta+\lambda_1)t} \mathbb{P}\left[\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y \right]$$
$$\times w\left(u-y, \frac{\lambda_1}{\beta} t - (u-y) \right) \mathrm{d}t$$
$$= \lambda_1 \int_{y=-\infty}^{u} \int_0^{\infty} \mathrm{e}^{-(\delta+\lambda_1)t} \mathbb{P}\left[\bar{W}_1(t) < u, W_1(t) \in \mathrm{d}y \right]$$
$$\times w\left(u-y, \frac{\lambda_1}{\beta} t - (u-y) \right) I_{\left\{ t \ge \frac{\beta}{\lambda_1}(u-y) \right\}} \mathrm{d}t$$

$$I_{4,1} = \frac{\lambda_1}{\delta+\lambda_1} \int_{y=-\infty}^{u} \mathbb{E}\left[\mathbb{P}\left[\bar{W}_1\left(e_{(\delta+\lambda_1)}\right) < u, W_1\left(e_{(\delta+\lambda_1)}\right) \in \mathrm{d}y \right] \right.$$
$$\left. \times w\left(u-y, \frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta} - (u-y) \right) I_{\left\{ e_{(\delta+\lambda_1)} \ge \frac{\beta}{\lambda_1}(u-y) \right\}} \right]$$

$$I_{4,1}=\frac{\lambda_1}{\delta+\lambda_1}\int_{y=-\infty}^{u}\mathbb{E}\left[w\left(u-y,\frac{\lambda_1 e_{(\delta+\lambda_1)}}{\beta}-(u-y)\right)I_{\left\{e_{(\delta+\lambda_1)}\geq\frac{\beta}{\lambda_1}(u-y)\right\}}\right]\mathcal{U}_{\delta+\lambda_1,1}(u,\mathrm{d}y) \tag{3.33}$$

By lemma 3.2, relation (3.33) becomes:

$$\begin{aligned}I_{4,1}=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\right.\\&\times\int_{\frac{\beta}{\lambda_1}(u-y)}^{\infty}(\delta+\lambda_1)\mathrm{e}^{-(\delta+\lambda_1)t}w\left(u-y,\frac{\lambda_1 t}{\beta}-(u-y)\right)\mathrm{d}t\\&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\&\left.\times\int_{\frac{\beta}{\lambda_1}(u-y)}^{\infty}(\delta+\lambda_1)\mathrm{e}^{-(\delta+\lambda_1)t}w\left(u-y,\frac{\lambda_1 t}{\beta}-(u-y)\right)\mathrm{d}t\right)\end{aligned}$$

$$\begin{aligned}I_{4,1}=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\frac{\beta}{\lambda_1}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\right.\\&\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}w(u-y,x-(u-y))\mathrm{d}x\\&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\&\left.\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}w(u-y,x-(u-y))\mathrm{d}x\right)\end{aligned} \tag{3.34}$$

Let's calculate $I_{3,1}+I_{4,1}$.

Using relations (3.28) and (3.34), we have:

$$\begin{aligned}I_{3,1}+I_{4,1}=&\frac{\beta\lambda_1\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\right.\\&\times\int_{0}^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\&\left.\times\int_{0}^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\right)\\&+\frac{\beta\lambda_1\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\right.\\&\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}w(u-y,x-(u-y))\mathrm{d}x\\&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\&\left.\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_1)x}{\lambda_1}}w(u-y,x-(u-y))\mathrm{d}x\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_1\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\frac{\beta}{\lambda_1}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_0^{u-y}h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_0^{u-y}h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\Bigg)
\end{aligned}
\tag{3.35}
$$

By Formulas (3.28) and (3.35) the function $\phi_{w,1}(u)$ can be put in the form:

$$
\begin{aligned}
\phi_{w,1}(u) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\Bigg(\int_{y=0}^{u}\int_{x=0}^{u-y}\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)\\
&+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\mathrm{d}y\mathrm{d}x\\
&+\int_{y=0}^{u}\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)f(x)\mathrm{d}y\mathrm{d}x\\
&+\int_{y=-\infty}^{0}\int_{x=0}^{u-y}\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\\
&\times\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\mathrm{d}y\mathrm{d}x\\
&+\int_{y=-\infty}^{0}\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)f(x)\mathrm{d}y\mathrm{d}x\Bigg)\\
&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_0^{u-y}h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_0^{u-y}h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{w,1}(u) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\\
&\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\Bigg)\\
&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\int_0^{u-y} h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y \\
&+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\mathrm{d}y \\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \\
&\times\int_0^{u-y} h_1(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y \\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right)\times\int_{u-y}^{\infty}h_1(x)w(u-y,x-(u-y))\mathrm{d}x\Big)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{w,1}(u)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \\
&\times\int_{x=0}^{u-y}\left(\left(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\right)\mathrm{d}x+\varphi_f(u-y)\right)\mathrm{d}y \\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \\
&\times\int_{x=0}^{u-y}\left(\left(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,1}(x)\phi_{w,2}(u-y-x)\right)\mathrm{d}x+\varphi_f(u-y)\right)\mathrm{d}y\Bigg) \\
&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \\
&\times\int_0^{u-y}\left(\left(\zeta_{1,1}(x)\phi_{w,1}(u-y-x)+\zeta_{2,2}(x)\phi_{w,2}(u-y-x)\right)\mathrm{d}x+\varphi_{h_1}(u-y)\right)\mathrm{d}y \\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,1}y}-\mathrm{e}^{-(\eta_{1,1}+\eta_{2,1})u+\eta_{2,1}y}\right) \\
&\times\int_0^{u-y}\left(\left(\zeta_{1,1}(x)\phi_{w,1}(u-y-x)+\zeta_{2,1}(x)\phi_{w,2}(u-y-x)\right)\mathrm{d}x+\varphi_{h_1}(u-y)\right)\mathrm{d}y\Bigg)
\end{aligned}
\tag{3.36}
$$

where $\zeta_{1,1}(x)$ and $\zeta_{2,1}(x)$ are defined in relation (3.16).

A change of variable $v=u-y$, brings (3.36) into:

$$
\begin{aligned}
\phi_{w,1}(u)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\Bigg(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right) \\
&\times\int_{x=0}^{v}\left(\left(\chi_{1,1}(x)\phi_{w,1}(v-x)+\chi_{2,1}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_f(v)\right)\mathrm{d}v \\
&+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right) \\
&\times\int_{x=0}^{v}\left(\left(\chi_{1,1}(x)\phi_{w,1}(v-x)+\chi_{2,1}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_f(v)\right)\mathrm{d}v\Bigg) \\
&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\Bigg(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right) \\
&\times\int_{x=0}^{v}\left(\left(\zeta_{1,1}(x)\phi_{w,1}(v-x)+\zeta_{2,1}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_{h_1}(v)\right)\mathrm{d}v \\
&+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right) \\
&\times\int_{x=u}^{v}\left(\left(\zeta_{1,1}(x)\phi_{w,1}(v-x)+\zeta_{2,1}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_{h_1}(v)\right)\mathrm{d}v\Bigg)
\end{aligned}
\tag{3.37}
$$

By relations (3.17) and (3.18), relation (3.37) can be put in the form:

$$
\begin{aligned}
\phi_{w,1}(u)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\Bigg(\int_0^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\Upsilon_{1,1}(v)\mathrm{d}v \\
&+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\Upsilon_{1,1}(v)\mathrm{d}v\Bigg)
\end{aligned}
$$

$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_{v=0}^{u}\left(e^{-\eta_{1,1}(u-v)}-e^{-\eta_{1,1}u-\eta_{2,1}v}\right)\Upsilon_{2,1}(y)dv\right.$$
$$\left.+\int_{v=u}^{\infty}\left(e^{\eta_{2,1}(u-v)}-e^{-\eta_{1,1}u-\eta_{2,1}v}\right)\Upsilon_{2,1}(y)dv\right) \quad (3.38)$$

Relation (3.38) can be written:

$$\phi_{w,1}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\int_0^u e^{-\eta_{1,1}(u-v)}\Upsilon_{1,1}(v)dv+\int_u^\infty e^{\eta_{2,1}(u-v)}\Upsilon_{1,1}(v)dv\right.$$
$$\left.-\int_0^\infty e^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon_{1,1}(v)dv\right)+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_0^u e^{-\eta_{1,1}(u-v)}\Upsilon_{2,1}(v)dv\right.$$
$$\left.+\int_u^\infty e^{\eta_{2,1}(u-v)}\Upsilon_{2,1}(v)dv-\int_0^\infty e^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon_{2,1}(y)dv\right)$$

$$\phi_{w,1}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m_{1,1}(u)+T_{\eta_{2,1}}\Upsilon_{1,1}(v)-e^{-\eta_{1,1}u}\hat{\Upsilon}_{1,1}(\eta_{2,1})\right)$$
$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m_{2,1}(u)+T_{\eta_{2,1}}\Upsilon_{2,1}(v)-e^{-\eta_{1,1}u}\hat{\Upsilon}_{2,1}(\eta_{2,1})\right)$$

- **Proof of relation (3.12):**

$$\phi_{w,2}(u)=\mathbb{E}\left[e^{-\delta V_2}\mathbb{E}\left[\phi_{w,2}(u-W_2-X_1)I_{\{X_1<u-W_2,\bar{W}_1<u,X_1>\Theta_1\}}\mid(V_2,X_1)\right]\right]$$
$$+\mathbb{E}\left[e^{-\delta V_2}\mathbb{E}\left[\phi_{w,2}(u-W_2-X_1)I_{\{X_1<u-W_2,\bar{W}_2<u,X_1<\Theta_1\}}\mid(V_2,X_1)\right]\right]$$
$$+\mathbb{E}\left[e^{-\delta V_2}\mathbb{E}\left[w(u-W_2,X_1-(u-W_2))I_{\{X_1>u-W_2,\bar{W}_2<u\}}\mid(V_2,X_1)\right]\right]$$
$$=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]dF_2(x,t)$$
$$+\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left(\bar{W}_2(t)<u,W_2(t)\in dy\right)$$
$$\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))dF_2(x,t)$$

$$\phi_{w,2}(u)=(1-\alpha)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]dF_{I,2}(x,t)$$
$$+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\int_{x=0}^{u-y}e^{-\delta t}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]dF_{M,2}(x,t)$$
$$+(1-\alpha)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left(\bar{W}_2(t)<u,W_2(t)\in dy\right)$$
$$\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))dF_{I,2}(x,t) \quad (3.39)$$
$$+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left(\bar{W}_2(t)<u,W_2(t)\in dy\right)$$
$$\times\int_{x=u-y}^{\infty}w(u-y,x-(u-y))dF_{M,2}(x,t)$$

By setting:

$$I_{1,2}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}e^{-\delta t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]dF_{I,2}(x,t)$$

$$I_{2,2} = \int_{t=0}^{\infty}\int_{y=-\infty}^{u} \mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right) \times \int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right)\mathrm{d}F_{I,2}(x,t)$$

$$I_{3,2} = \int_{t=0}^{\infty}\int_{y=-\infty}^{u} \mathrm{e}^{-\delta t}\mathbb{P}\left[\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right] \times \int_{x=0}^{u-y}\left[\mathbb{P}(x > \Theta_1)\phi_{w,1}(u-y-x) + \mathbb{P}(x < \Theta_1)\phi_{w,2}(u-y-x)\right]\mathrm{d}F_{M,2}(x,t)$$

$$I_{4,2} = \int_{t=0}^{\infty}\int_{y=-\infty}^{u} \mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right) \times \int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right)\mathrm{d}F_{M,2}(x,t)$$

Relation (3.39) becomes:

$$\phi_{w,2}(u) = (1-\alpha)\left(I_{1,2} + I_{2,2}\right) + \alpha\left(I_{3,2} + I_{4,2}\right) \tag{3.40}$$

Let's calculate $I_{1,2} + I_{2,2}$.

$$\begin{aligned} I_{1,2} + I_{2,2} &= \lambda_2 \int_{t=0}^{\infty}\int_{y=-\infty}^{u} \mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left[\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right] \\ &\quad \times \int_{x=0}^{u-y}\left[\mathbb{P}(x > \Theta_1)\phi_{w,1}(u-y-x) \right. \\ &\quad \left. + \mathbb{P}(x < \Theta_1)\phi_{w,2}(u-y-x)\right] f(x)\mathrm{d}x\mathrm{d}t \\ &\quad + \lambda_2 \int_{t=0}^{\infty}\int_{y=-\infty}^{u} \mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left(\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right) \\ &\quad \times \int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right) f(x)\mathrm{d}x\mathrm{d}t \\ &= \lambda_2 \int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left[\mathbb{P}(x > \Theta_1)\phi_{w,1}(u-y-x) + \mathbb{P}(x < \Theta_1)\phi_{w,2}(u-y-x)\right] f(x) \\ &\quad \times \left(\int_{t=0}^{\infty} \mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left[\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right]\mathrm{d}t\right)\mathrm{d}x \\ &\quad + \lambda_2 \int_{y=-\infty}^{u}\int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right) f(x) \\ &\quad \times \left(\int_{t=0}^{\infty} \mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left(\bar{W}_2(t) < u, W_2(t) \in \mathrm{d}y\right)\mathrm{d}t\right)\mathrm{d}x \end{aligned}$$

$$\begin{aligned} I_{1,2} + I_{2,2} = \frac{\lambda_2}{\delta+\lambda_2}&\left(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left(\mathbb{P}(x > \Theta_1)\phi_{w,1}(u-y-x)\right.\right. \\ &\left. + \mathbb{P}(x < \Theta_1)\phi_{w,2}(u-y-x)\right) f(x)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x \\ &\left. + \int_{y=-\infty}^{u}\int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right) f(x)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x\right) \end{aligned} \tag{3.41}$$

By relation (3.13), relation (3.41) can be written:

$$\begin{aligned} I_{1,2} + I_{2,2} = \frac{\lambda_2}{\delta+\lambda_2}&\left(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left(L(x)\phi_{w,1}(u-y-x)\right.\right. \\ &\left. + \bar{L}(x)\phi_{w,2}(u-y-x)\right) f(x)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x \\ &\left. + \int_{y=-\infty}^{u}\int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right) f(x)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x\right) \end{aligned} \tag{3.42}$$

By relation (3.14), relation (3.42) becomes:

$$\begin{aligned} I_{1,2} + I_{2,2} = \frac{\lambda_2}{\delta+\lambda_2}&\left(\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\left(\chi_{1,2}(x)\phi_{w,1}(u-y-x)\right.\right. \\ &\left. + \chi_{2,2}(x)\phi_{w,2}(u-y-x)\right)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x \\ &\left. + \int_{y=-\infty}^{u}\int_{x=u-y}^{\infty} w\left(u-y, x-(u-y)\right) f(x)\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y)\mathrm{d}x\right) \end{aligned} \tag{3.43}$$

By lemma 3.2, relation (3.43) can be put in the form:

$$I_{1,2}+I_{2,2}=\frac{\lambda_2\eta_{1,2}\eta_{2,2}}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}$$
$$\times\Big(\int_{y=0}^{u}\int_{x=0}^{u-y}\left(\chi_{1,2}(x)\phi_{w,1}(u-y-x)+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\right)\left(e^{-\eta_{1,2}y}-e^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)dydx$$
$$+\int_{y=-\infty}^{0}\int_{x=0}^{u-y}\left(\chi_{1,1}(x)\phi_{w,1}(u-y-x)+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\right)\left(e^{\eta_{2,2}y}-e^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)dydx \quad (3.44)$$
$$+\int_{y=0}^{u}\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\left(e^{-\eta_{1,2}y}-e^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)f(x)dydx$$
$$+\int_{y=-\infty}^{0}\int_{x=u-y}^{\infty}w(u-y,x-(u-y))\left(e^{\eta_{2,2}y}-e^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)f(x)dydx\Big)$$

Let's calculate $I_{3,2}$ and $I_{4,2}$.

The distribution $F_{M,2}(x,t)$ random vector (X,V_2) is a comonotonic distribution whose support is the set:

$$D_2'=\{(x,t):F_X(x)=F_{V_2}(t)\}=\{(x,t):x=l(t)\}=\left\{(x,t):x=\frac{\lambda_2}{\beta}t\right\}.$$

The distribution $F_{M,2}(x,t)$ is $G_2(t)=F_{M,2}(l(t),t)=1-e^{-\lambda_2 t}$ on $D_2'=\left\{(x,t):x=\frac{\lambda_2}{\beta}t\right\}$.

Let's calculate $I_{3,2}$.

$$I_{3,2}=\int_{y=-\infty}^{u}e^{-\delta t}\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]$$
$$\times\int_{t=0}^{\infty}\int_{x=0}^{u-y}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in dy\right]dF_{M,2}(x,t)$$

$$I_{3,2}=\int_{y=-\infty}^{u}\int_{K_2}e^{-\delta t}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{w,2}(u-y-x)\right]dG_2(t) \quad (3.45)$$

where:

$$K_2=\left\{t\in\mathbb{R}_+\mid 0\le x=\frac{\lambda_2}{\beta}t\le u-y\right\}$$
$$K_2=\left\{t\in\mathbb{R}_+\mid 0\le t\le\frac{\beta}{\lambda_2}(u-y)\right\} \quad (3.46)$$

By injecting relation (3.46) into relation (3.45), he follows:

$$I_{3,2}=\lambda_2\int_{y=-\infty}^{u}\int_{0}^{\frac{\beta}{\lambda_2}(u-y)}e^{-(\delta+\lambda_2)t}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}\left(u-y-\frac{\lambda_2}{\beta}t\right)+\mathbb{P}(x<\Theta_1)\phi_{w,2}\left(u-y-\frac{\lambda_2}{\beta}t\right)\right]dt$$
$$=\lambda_2\int_{y=-\infty}^{u}\int_{0}^{\infty}e^{-(\delta+\lambda_2)t}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in dy\right]$$
$$\times\left[\mathbb{P}(x>\Theta_1)\phi_{w,1}\left(u-y-\frac{\lambda_2}{\beta}t\right)+\mathbb{P}(x<\Theta_1)\phi_{w,2}\left(u-y-\frac{\lambda_2}{\beta}t\right)\right]I_{\left\{0\le t\le\frac{\beta}{\lambda_2}(u-y)\right\}}dt$$

$$
\begin{aligned}
&= \frac{\lambda_2}{\delta+\lambda_2} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\mathbb{P}\left[\bar{W}_2\left(e_{(\delta+\lambda_2)}\right) < u, W_2\left(e_{(\delta+\lambda_2)}\right) \in \mathrm{d}y \right] \\
&\times \left(\mathbb{P}\left(x > \Theta_1\right) \phi_{w,1}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right. \\
&\left. + \mathbb{P}\left(x < \Theta_1\right) \phi_{w,2}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right) I_{\left\{0 \le e_{(\delta+\lambda_2)} \le \frac{\beta}{\lambda_2}(u-y)\right\}} \Bigg] \\
&= \frac{\lambda_2}{\delta+\lambda_2} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\left(\mathbb{P}\left(x > \Theta_1\right) \phi_{w,1}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right. \\
&\left. + \mathbb{P}\left(x < \Theta_1\right) \phi_{w,2}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right) I_{\left\{0 \le e_{(\delta+\lambda_2)} \le \frac{\beta}{\lambda_2}(u-y)\right\}} \Bigg] \mathcal{U}_{\delta+\lambda_2,2}\left(u, \mathrm{d}y\right)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,2} &= \frac{\lambda_2}{\delta+\lambda_2} \int_{y=-\infty}^{u} \mathbb{E}\Bigg[\left(L(x) \phi_{w,1}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right. \\
&\left. + \bar{L}(x) \phi_{w,2}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right) I_{\left\{0 \le e_{(\delta+\lambda_2)} \le \frac{\beta}{\lambda_2}(u-y)\right\}} \Bigg] \mathcal{U}_{\delta+\lambda_2,2}\left(u, \mathrm{d}y\right)
\end{aligned} \tag{3.47}
$$

By lemma 3.2, relation (3.47) can be written:

$$
\begin{aligned}
I_{3,2} &= \frac{\lambda_2 \eta_{1,2} \eta_{2,2}}{\left(\delta+\lambda_2\right)\left(\eta_{1,2}+\eta_{2,2}\right)} \Bigg(\int_{y=0}^{u} \left(\mathrm{e}^{-\eta_{1,2} y} - \mathrm{e}^{-\left(\eta_{1,2}+\eta_{2,2}\right)u + \eta_{2,2} y} \right) \\
&\times \mathbb{E}\Bigg[\left(L(x) \phi_{w,1}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right. \\
&\left. + \bar{L}(x) \phi_{w,2}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right) I_{\left\{0 \le e_{(\delta+\lambda_2)} \le \frac{\beta}{\lambda_2}(u-y)\right\}} \Bigg] \mathrm{d}y \\
&+ \int_{y=-\infty}^{0} \left(\mathrm{e}^{\eta_{2,2} y} - \mathrm{e}^{-\left(\eta_{1,2}+\eta_{2,2}\right)u + \eta_{2,2} y} \right) \times \mathbb{E}\Bigg[\left(L(x) \phi_{w,1}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right. \\
&\left. + \bar{L}(x) \phi_{w,2}\left(u - y - \frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta} \right) \right) I_{\left\{0 \le e_{(\delta+\lambda_2)} \le \frac{\beta}{\lambda_2}(u-y)\right\}} \Bigg] \mathrm{d}y \Bigg) \\
&= \frac{\lambda_2 \eta_{1,2} \eta_{2,2}}{\left(\delta+\lambda_2\right)\left(\eta_{1,2}+\eta_{2,2}\right)} \Bigg(\int_{y=0}^{u} \left(\mathrm{e}^{-\eta_{1,2} y} - \mathrm{e}^{-\left(\eta_{1,2}+\eta_{2,2}\right)u + \eta_{2,2} y} \right) \\
&\times \int_0^{\frac{\beta}{\lambda_2}(u-y)} \left(\delta+\lambda_2\right) \mathrm{e}^{-\left(\delta+\lambda_2\right)t} \left[L(x) \phi_{w,1}\left(u - y - \frac{\lambda_2}{\beta} t \right) \right. \\
&\left. + \bar{L}(x) \phi_{w,2}\left(u - y - \frac{\lambda_2 t}{\beta} \right) \right] \mathrm{d}t\mathrm{d}y + \int_{y=-\infty}^{0} \left(\mathrm{e}^{\eta_{2,2} y} - \mathrm{e}^{-\left(\eta_{1,2}+\eta_{2,2}\right)u + \eta_{2,2} y} \right) \\
&\times \int_0^{\frac{\beta}{\lambda_2}(u-y)} \left(\delta+\lambda_2\right) \mathrm{e}^{-\left(\delta+\lambda_2\right)t} \left[L(x) \phi_{w,1}\left(u - y - \frac{\lambda_2}{\beta} t \right) \right. \\
&\left. + \bar{L}(x) \phi_{w,2}\left(u - y - \frac{\lambda_2 t}{\beta} \right) \right] \mathrm{d}t\mathrm{d}y \Bigg)
\end{aligned}
$$

$$I_{3,2}=\frac{\lambda_2\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\frac{\beta}{\lambda_2}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)$$
$$\times\int_0^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y$$
$$+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)$$
$$\times\int_0^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\Bigg) \quad (3.48)$$

Let's calculate $I_{4,2}$.

$$I_{4,2}=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left(\bar{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right)$$
$$\times\int_{x=u-y}^{\infty}w\left(u-y,x-(u-y)\right)\mathrm{d}F_{M,2}(x,t)$$
$$=\int_{y=-\infty}^{u}\int_{\mathbb{R}_+\setminus K_2}\mathrm{e}^{-\delta t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times w\left(u-y,\frac{\lambda_2}{\beta}t-(u-y)\right)\mathrm{d}G_2(t)$$
$$=\lambda_2\int_{y=-\infty}^{u}\int_{\frac{\beta}{\lambda_2}(u-y)}^{\infty}\mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times w\left(u-y,\frac{\lambda_2}{\beta}t-(u-y)\right)\mathrm{d}t$$
$$=\lambda_2\int_{y=-\infty}^{u}\int_0^{\infty}\mathrm{e}^{-(\delta+\lambda_2)t}\mathbb{P}\left[\bar{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times w\left(u-y,\frac{\lambda_2}{\beta}t-(u-y)\right)I_{\left\{t\geq\frac{\beta}{\lambda_2}(u-y)\right\}}\mathrm{d}t$$

$$I_{4,2}=\frac{\lambda_2}{\delta+\lambda_2}\int_{y=-\infty}^{u}\mathbb{E}\Bigg[\mathbb{P}\left[\bar{W}_2\left(e_{(\delta+\lambda_2)}\right)<u,W_2\left(e_{(\delta+\lambda_2)}\right)\in\mathrm{d}y\right]$$
$$\times w\left(u-y,\frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta}-(u-y)\right)I_{\left\{e_{(\delta+\lambda_2)}\geq\frac{\beta}{\lambda_2}(u-y)\right\}}\Bigg]$$

$$I_{4,2}=\frac{\lambda_2}{\delta+\lambda_2}\int_{y=-\infty}^{u}\mathbb{E}\left[w\left(u-y,\frac{\lambda_2 e_{(\delta+\lambda_2)}}{\beta}-(u-y)\right)I_{\left\{e_{(\delta+\lambda_2)}\geq\frac{\beta}{\lambda_2}(u-y)\right\}}\right]\mathcal{U}_{\delta+\lambda_2,2}(u,\mathrm{d}y) \quad (3.49)$$

By lemma 3.2, relation (3.49) becomes:

$$I_{4,2}=\frac{\lambda_2\eta_{1,2}\eta_{2,2}}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)$$
$$\times\int_{\frac{\beta}{\lambda_2}(u-y)}^{\infty}(\delta+\lambda_2)\mathrm{e}^{-(\delta+\lambda_2)t}w\left(u-y,\frac{\lambda_2 t}{\beta}-(u-y)\right)\mathrm{d}t$$
$$+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)$$
$$\times\int_{\frac{\beta}{\lambda_2}(u-y)}^{\infty}(\delta+\lambda_2)e^{-(\delta+\lambda_2)t}w\left(u-y,\frac{\lambda_2 t}{\beta}-(u-y)\right)\mathrm{d}t\Bigg)$$

$$I_{4,2}=\frac{\lambda_2\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\frac{\beta}{\lambda_2}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}w\big(u-y,x-(u-y)\big)\mathrm{d}x+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}w\big(u-y,x-(u-y)\big)\mathrm{d}x\Bigg) \quad (3.50)$$

Let's calculate $I_{3,2}+I_{4,2}$.

Using relations (3.48) and (3.50), we have:

$$\begin{aligned}I_{3,2}+I_{4,2}&=\frac{\beta\lambda_2\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\&\quad\times\int_0^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\&\quad+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\&\quad\times\int_0^{u-y}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\Bigg)\\&\quad+\frac{\beta\lambda_2\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}w\big(u-y,x-(u-y)\big)\mathrm{d}x\\&\quad+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}\mathrm{e}^{\frac{-\beta(\delta+\lambda_2)x}{\lambda_2}}w\big(u-y,x-(u-y)\big)\mathrm{d}x\Bigg)\\&=\frac{\lambda_2\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\frac{\beta}{\lambda_2}\Bigg(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\&\quad\times\int_0^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\&\quad+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\&\quad\times\int_0^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\&\quad+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\\&\quad+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\Bigg)\end{aligned} \quad (3.51)$$

By Formulas (3.44) and (3.51), the function $\phi_{w,2}(u)$ can be put in the form:

$$\begin{aligned}\phi_{w,2}(u)&=\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\Bigg(\int_{y=0}^{u}\int_{x=0}^{u-y}\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)\\&\quad+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\big)\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\mathrm{d}y\mathrm{d}x\\&\quad+\int_{y=0}^{u}\int_{x=u-y}^{\infty}w\big(u-y,x-(u-y)\big)\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)f(x)\mathrm{d}y\mathrm{d}x\\&\quad+\int_{y=-\infty}^{0}\int_{x=0}^{u-y}\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)\end{aligned}$$

$$+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\Big)\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\mathrm{d}y\mathrm{d}x$$
$$+\int_{y=-\infty}^{0}\int_{x=u-y}^{\infty}w\big(u-y,x-(u-y)\big)\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)f(x)\mathrm{d}y\mathrm{d}x\Big)$$
$$+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\right.$$
$$\times\int_{0}^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y$$
$$+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\mathrm{d}y$$
$$+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)$$
$$\times\int_{0}^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y$$
$$\left.+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\right)$$

$$\begin{aligned}
\phi_{w,2}(u)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\right.\\
&\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)\\
&\left.+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\right)\\
&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\right.\\
&\times\int_{0}^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&+\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\
&\times\int_{0}^{u-y}h_2(x)\left[L(x)\phi_{w,1}(u-y-x)+\bar{L}(x)\phi_{w,2}(u-y-x)\right]\mathrm{d}x\mathrm{d}y\\
&\left.+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\times\int_{u-y}^{\infty}h_2(x)w\big(u-y,x-(u-y)\big)\mathrm{d}x\right)
\end{aligned}\tag{3.52}$$

By relation (3.16), relation (3.52) becomes:

$$\begin{aligned}
\phi_{w,2}(u)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\right.\\
&\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\
&\left.\times\int_{x=0}^{u-y}\Big(\big(\chi_{1,2}(x)\phi_{w,1}(u-y-x)+\chi_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_f(u-y)\Big)\mathrm{d}y\right)\\
&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\int_{y=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\right.\\
&\times\int_{0}^{u-y}\Big(\big(\zeta_{1,2}(x)\phi_{w,1}(u-y-x)+\zeta_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_{h_2}(u-y)\Big)\mathrm{d}y\\
&+\int_{y=-\infty}^{0}\left(\mathrm{e}^{\eta_{2,2}y}-\mathrm{e}^{-(\eta_{1,2}+\eta_{2,2})u+\eta_{2,2}y}\right)\\
&\left.\times\int_{0}^{u-y}\Big(\big(\zeta_{1,2}(x)\phi_{w,1}(u-y-x)+\zeta_{2,2}(x)\phi_{w,2}(u-y-x)\big)\mathrm{d}x+\varphi_{h_2}(u-y)\Big)\mathrm{d}y\right)
\end{aligned}\tag{3.53}$$

A change of variable $v = u - y$, brings relation (3.53) into

$$\begin{aligned}\phi_{w,2}(u) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\Big(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right) \\ & \times\int_{x=0}^{v}\left(\left(\chi_{1,2}(x)\phi_{w,1}(v-x)+\chi_{2,2}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_f(v)\right)\mathrm{d}v \\ & +\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right) \\ & \times\int_{x=0}^{v}\left(\left(\chi_{1,2}(x)\phi_{w,1}(v-x)+\chi_{2,2}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_f(v)\right)\mathrm{d}v\Big) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\Big(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right) \\ & \times\int_{x=0}^{v}\left(\left(\zeta_{1,2}(x)\phi_{w,1}(v-x)+\zeta_{2,2}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_{h_2}(v)\right)\mathrm{d}v \\ & +\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right) \\ & \times\int_{x=u}^{v}\left(\left(\zeta_{1,2}(x)\phi_{w,1}(v-x)+\zeta_{2,2}(x)\phi_{w,2}(v-x)\right)\mathrm{d}x+\varphi_{h_2}(v)\right)\mathrm{d}v\Big)\end{aligned} \quad (3.54)$$

By Formulas (3.19) and (3.20), relation (3.54) becomes:

$$\begin{aligned}\phi_{w,2}(u) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\Big(\int_0^u\left(\mathrm{e}^{-\eta_{1,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right)\Upsilon_{1,2}(v)\mathrm{d}v \\ & +\int_u^{\infty}\left(\mathrm{e}^{\eta_{2,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right)\Upsilon_{1,2}(v)\mathrm{d}v\Big) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\Big(\int_0^u\left(\mathrm{e}^{-\eta_{1,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right)\Upsilon_{2,2}(v)\mathrm{d}v \\ & +\int_u^{\infty}\left(\mathrm{e}^{\eta_{2,2}(u-v)}-\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\right)\Upsilon_{2,2}(v)\mathrm{d}v\Big)\end{aligned} \quad (3.55)$$

Relation (3.55) can be in the form:

$$\begin{aligned}\phi_{w,2}(u) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\Big(\int_0^u\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}(v)\mathrm{d}v \\ & +\int_u^{\infty}\mathrm{e}^{\eta_{2,2}(u-v)}\Upsilon_{1,2}(v)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\Upsilon_{1,2}(v)\mathrm{d}v \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\Big(\int_0^u\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{2,2}(v)\mathrm{d}v \\ & +\int_{v=u}^{\infty}\mathrm{e}^{\eta_{2,2}(u-v)}\Upsilon_{2,2}(v)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,2}u-\eta_{2,2}v}\Upsilon_{2,2}(v)\mathrm{d}v\Big)\end{aligned}$$

$$\begin{aligned}\phi_{w,2}(u) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(m_{1,2}(u)+T_{\eta_{2,2}}\Upsilon_{1,2}(v)-\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{1,2}(\eta_{2,2})\right) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(m_{2,2}(u)+T_{\eta_{2,2}}\Upsilon_{2,2}(v)-\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{2,2}(\eta_{2,2})\right)\end{aligned}$$

3.3. Integro-Differential Equations Satisfied by the Function $\phi_{d,i}(u)$, $i = 1,2$

Theorem 3.2. *The Gerber-Shiu function* $\phi_{d,i}(u), i = 1,2$ *verifies the following integro-differential equations*:

$$\begin{aligned}\phi_{d,1}(u) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m^{\star}_{1,1}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{1,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})\right)\\ &\quad+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m^{\star}_{2,1}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{2,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})\right)+\mathrm{e}^{-\eta_{1,1}u}\end{aligned} \tag{3.56}$$

$$\begin{aligned}\phi_{d,2}(u) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m^{\star}_{1,2}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{1,2}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{1,2}(\eta_{2,1})\right)\\ &\quad+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m^{\star}_{2,2}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{2,2}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{2,2}(\eta_{2,1})\right)+\mathrm{e}^{-\eta_{1,2}u}\end{aligned} \tag{3.57}$$

where:

- $L(x)$ and $\bar{L}(x)$ are defined in relation (3.13);
- $\chi_{1,1}(x);\chi_{2,1}(x);\chi_{1,2}(x);\chi_{2,2}(x)$ are defined in relation (3.14);
- $h_1(x)$ and $h_2(x)$ are defined in relation (3.15);
- $\zeta_{1,1}(x);\zeta_{2,1}(x);\zeta_{1,2}(x)$ and $\zeta_{2,2}(x)$ are defined in relation (3.16).

$$\Upsilon^{\star}_{1,1}(y)=\int_0^y\left(\chi_{1,1}(x)\phi_{d,1}(y-x)+\chi_{2,1}(x)\phi_{d,2}(y-x)\right)\mathrm{d}x \tag{3.58}$$

$$\Upsilon^{\star}_{2,1}(y)=\int_0^y\left(\zeta_1(x)\phi_{d,1}(y-x)+\zeta_{2,1}(x)\phi_{d,2}(y-x)\right)\mathrm{d}x \tag{3.59}$$

$$\Upsilon^{\star}_{1,2}(y)=\int_0^y\left(\chi_{1,2}(x)\phi_{d,1}(y-x)+\chi_{2,2}(x)\phi_{d,2}(y-x)\right)\mathrm{d}x \tag{3.60}$$

$$\Upsilon^{\star}_{2,2}(y)=\int_0^y\left(\zeta_{1,2}(x)\phi_{d,1}(y-x)+\zeta_{2,2}(x)\phi_{d,2}(y-x)\right)\mathrm{d}x \tag{3.61}$$

$$m^{\star}_{1,1}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{1,1}(v)\mathrm{d}v;\ m^{\star}_{2,1}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{2,1}(v)\mathrm{d}v \tag{3.62}$$

$$m^{\star}_{1,2}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{2,2}(v)\mathrm{d}v;\ m^{\star}_{2,2}(u)=\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{2,2}(v)\mathrm{d}v \tag{3.63}$$

- **Proof of relation (3.56)**

By a similar approach as in subsection (3.2), and by contioning on whether or not ruin occurs due to oscillation before the first claim, he follows:

$$\begin{aligned}\phi_{d,i}(u) &= \mathbb{E}\left[\mathrm{e}^{-\delta V_i}\mathbb{E}\left[\phi_{d,1}(u-W_i-X_1)I_{\{X_1<u-W_i,\bar{W}_1<u,X_1>\Theta_1\}}\mid(V_i,X_1)\right]\right]\\ &\quad+\mathbb{E}\left[\mathrm{e}^{-\delta V_i}\mathbb{E}\left[\phi_{d,2}(u-W_i-X_1)I_{\{X_1<u-W_i,\bar{W}_i<u,X_1<\Theta_1\}}\mid(V_i,X_1)\right]\right].\\ &\quad+\mathbb{E}\left[\mathrm{e}^{-\delta\tau_i}I(\tau_i<V_i)\right],\ i=1,2.\end{aligned}$$

where $\tau_i=\inf\{t\geq 0:W_i(t)=u\}$ is the first instant when the process $W_i(t)$ reaches value *u*.

The random variable V_i and the processes $W_i(t), i=1,2$ being independent, he follows:

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\delta\tau_1}I(\tau_1\leq V_1)\right] &= \mathbb{E}\left(\mathbb{E}\left[\mathrm{e}^{-\delta\tau_1}I(\tau_1\leq V_1)\mid W_1(t)\right]\right)\\ &= \mathbb{E}\left(\mathrm{e}^{-(\delta+\lambda_1)\tau_1}\right)\\ &= \mathrm{e}^{-\eta_{1,1}u}\end{aligned} \tag{3.64}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\delta\tau_2}I\left(\tau_2\le V_2\right)\right]&=\mathbb{E}\left(\mathbb{E}\left[\mathrm{e}^{-\delta\tau_2}I\left(\tau_2<V_2\right)\mid W_2\left(t\right)\right]\right)\\&=\mathbb{E}\left(\mathrm{e}^{-(\delta+\lambda_2)\tau_2}\right)\\&=\mathrm{e}^{-\eta_{1,1}u}\end{aligned}\tag{3.65}$$

where $\eta_{1,1};\eta_{1,2}$ are defined in relation (3.9).

So

$$\begin{aligned}\phi_{d,1}\left(u\right)=&\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_1\left(t\right)<u,W_1\left(t\right)\in\mathrm{d}y\right]\\&\times\left[\mathbb{P}\left(x>\Theta_1\right)\phi_{d,1}\left(u-y-x\right)\right.\\&\left.+\mathbb{P}\left(x<\Theta_1\right)\phi_{d,2}\left(u-y-x\right)\right]\mathrm{d}F_1\left(x,t\right)+\mathrm{e}^{-\eta_{1,1}u}\end{aligned}$$

$$\begin{aligned}\phi_{d,1}\left(u\right)=&\left(1-\alpha\right)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_1\left(t\right)<u,W_1\left(t\right)\in\mathrm{d}y\right]\\&\times\int_{x=0}^{u-y}\left[\mathbb{P}\left(x>\Theta_1\right)\phi_{d,1}\left(u-y-x\right)+\mathbb{P}\left(x<\Theta_1\right)\phi_{d,2}\left(u-y-x\right)\right]\mathrm{d}F_{I,1}\left(x,t\right)\\&+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathbb{P}\left[\overline{W}_1\left(t\right)<u,W_1\left(t\right)\in\mathrm{d}y\right]\\&\times\int_{x=0}^{u-y}\mathrm{e}^{-\delta t}\left[\mathbb{P}\left(x>\Theta_1\right)\phi_{d,1}\left(u-y-x\right)\right.\\&\left.+\mathbb{P}\left(x<\Theta_1\right)\phi_{d,2}\left(u-y-x\right)\right]\mathrm{d}F_{M,1}\left(x,t\right)+\mathrm{e}^{-\eta_{1,1}u}\end{aligned}\tag{3.66}$$

By the similar approach as in subsection (3.2), we have:

$$\begin{aligned}\phi_{d,1}\left(u\right)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}\left(1-\alpha\right)}{\left(\delta+\lambda_1\right)\left(\eta_{1,1}+\eta_{2,1}\right)}\left(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\right.\\&\times\int_{x=0}^{v}\left(\left(\chi_{1,1}\left(x\right)\phi_{d,1}\left(v-x\right)+\chi_{2,1}\left(x\right)\phi_{d,2}\left(v-x\right)\right)\mathrm{d}x\right)\mathrm{d}v\\&+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\\&\left.\times\int_{x=0}^{v}\left(\left(\chi_{1,1}\left(x\right)\phi_{d,1}\left(v-x\right)+\chi_{2,1}\left(x\right)\phi_{d,2}\left(v-x\right)\right)\mathrm{d}x\right)\mathrm{d}v\right)\\&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\right.\\&\times\int_{x=0}^{v}\left(\left(\zeta_{1,1}\left(x\right)\phi_{d,1}\left(v-x\right)+\zeta_{2,1}\left(x\right)\phi_{d,2}\left(v-x\right)\right)\mathrm{d}x\right)\mathrm{d}v\\&+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\\&\left.\times\int_{x=u}^{v}\left(\left(\zeta_{1,1}\left(x\right)\phi_{d,1}\left(v-x\right)+\zeta_{2,1}\left(x\right)\phi_{d,2}\left(v-x\right)\right)\mathrm{d}x\right)\mathrm{d}v\right)+\mathrm{e}^{-\eta_{1,1}u}\end{aligned}\tag{3.67}$$

By Formulas (3.58) and (3.59), relation (3.67) becomes:

$$\begin{aligned}\phi_{d,1}\left(u\right)=&\frac{\lambda_1\eta_{1,1}\eta_{2,1}\left(1-\alpha\right)}{\left(\delta+\lambda_1\right)\left(\eta_{1,1}+\eta_{2,1}\right)}\left(\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{1,1}^{\star}\left(v\right)\mathrm{d}v\right.\\&\left.+\int_u^{\infty}\mathrm{e}^{\eta_{2,1}(u-v)}\Upsilon_{1,1}^{\star}\left(v\right)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon_{1,1}^{\star}\left(v\right)\mathrm{d}v\right)\\&+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{2,1}^{\star}\left(v\right)\mathrm{d}v\right.\\&\left.+\int_u^{\infty}\mathrm{e}^{\eta_{2,1}(u-v)}\Upsilon_{2,1}^{\star}\left(v\right)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon_{2,1}^{\star}\left(v\right)\mathrm{d}v\right)+\mathrm{e}^{-\eta_{1,1}u}\end{aligned}\tag{3.68}$$

$$\phi_{d,1}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m^\star_{1,1}(u)+T_{\eta_{2,1}}\Upsilon^\star_{1,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^\star_{1,1}(\eta_{2,1})\right)$$
$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m^\star_{2,1}(u)+T_{\eta_{2,1}}\Upsilon^\star_{2,1}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^\star_{2,1}(\eta_{2,1})\right)+\mathrm{e}^{-\eta_{1,1}u}$$

- **Proof of relation (3.57):**

$$\phi_{d,2}(u)=\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\int_{x=0}^{u-y}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times\left[\mathbb{P}(x>\Theta_1)\phi_{d,1}(u-y-x)\right.$$
$$\left.+\mathbb{P}(x<\Theta_1)\phi_{d,2}(u-y-x)\right]\mathrm{d}F_2(x,t)+\mathrm{e}^{-\eta_{1,2}u}$$

$$\phi_{d,2}(u)=(1-\alpha)\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathrm{e}^{-\delta t}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times\int_{x=0}^{u-y}\left[\mathbb{P}(x>\Theta_1)\phi_{d,1}(u-y-x)+\mathbb{P}(x<\Theta_1)\phi_{d,2}(u-y-x)\right]\mathrm{d}F_{I,2}(x,t)$$
$$+\alpha\int_{t=0}^{\infty}\int_{y=-\infty}^{u}\mathbb{P}\left[\overline{W}_2(t)<u,W_2(t)\in\mathrm{d}y\right]$$
$$\times\int_{x=0}^{u-y}\mathrm{e}^{-\delta t}\left[\mathbb{P}(x>\Theta_1)\phi_{d,1}(u-y-x)\right.$$
$$\left.+\mathbb{P}(x<\Theta_1)\phi_{d,2}(u-y-x)\right]\mathrm{d}F_{M,2}(x,t)+\mathrm{e}^{-\eta_{1,2}u} \quad (3.69)$$

By a similar approach as in subsection (3.2), he follows:

$$\phi_{d,2}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\right.$$
$$\times\int_{x=0}^{v}\left(\left(\chi_{1,1}(x)\phi_{d,1}(v-x)+\chi_{2,1}(x)\phi_{d,2}(v-x)\right)\mathrm{d}x\right)\mathrm{d}v$$
$$+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)$$
$$\left.\times\int_{x=0}^{v}\left(\left(\chi_{1,1}(x)\phi_{d,1}(v-x)+\chi_{2,1}(x)\phi_{d,2}(v-x)\right)\mathrm{d}x\right)\mathrm{d}v\right)$$
$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_{v=0}^{u}\left(\mathrm{e}^{-\eta_{1,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)\right.$$
$$\times\int_{x=0}^{v}\left(\left(\zeta_{1,1}(x)\phi_{d,1}(v-x)+\zeta_{2,1}(x)\phi_{d,2}(v-x)\right)\mathrm{d}x\right)\mathrm{d}v \quad (3.70)$$
$$+\int_{v=u}^{\infty}\left(\mathrm{e}^{\eta_{2,1}(u-v)}-\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\right)$$
$$\left.\times\int_{x=u}^{v}\left(\left(\zeta_{1,1}(x)\phi_{d,1}(v-x)+\zeta_{2,1}(x)\phi_{d,2}(v-x)\right)\mathrm{d}x\right)\mathrm{d}v\right)+\mathrm{e}^{-\eta_{1,2}u}$$

By Formulas (3.60) and (3.61), relation (3.70) becomes:

$$\phi_{d,2}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^\star_{1,2}(v)\mathrm{d}v\right.$$
$$\left.+\int_u^{\infty}\mathrm{e}^{\eta_{2,1}(u-v)}\Upsilon^\star_{1,2}(v)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon^\star_{1,2}(v)\mathrm{d}v\right)$$
$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\int_0^u\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^\star_{2,2}(v)\mathrm{d}v\right. \quad (3.71)$$
$$\left.+\int_u^{\infty}\mathrm{e}^{\eta_{2,1}(u-v)}\Upsilon^\star_{2,2}(v)\mathrm{d}v-\int_0^{\infty}\mathrm{e}^{-\eta_{1,1}u-\eta_{2,1}v}\Upsilon^\star_{2,2}(v)\mathrm{d}v\right)+\mathrm{e}^{-\eta_{1,2}u}$$

$$\phi_{d,2}(u)=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(m^{\star}_{1,2}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{1,2}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{1,2}(\eta_{2,1})\right)$$
$$+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(m^{\star}_{2,2}(u)+T_{\eta_{2,1}}\Upsilon^{\star}_{2,2}(v)-\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{2,2}(\eta_{2,1})\right)+\mathrm{e}^{-\eta_{1,2}u}$$

4. Laplace Transfors of the Functions $\phi_{w,i}(u)$ and $\phi_{d,i}(u)$, $i=1,2$

4.1. Laplace Transfors of the Functions $\phi_{w,i}(u)$, $i=1,2$

Theorem 4.1. *The Laplace transforms of the Gerber-Shiu functions* $\phi_{w,i}(u), i=1,2$ *verify the following relations*:

$$\begin{aligned}&\hat{\phi}_{w,1}(s)+\left(\varepsilon_1\hat{\chi}_{2,1}(s)+\varepsilon_2\hat{\zeta}_{2,1}(s)\right)\hat{\phi}_{w,2}(s)\\&=\varepsilon_1\left(\hat{\chi}_{2,1}(\eta_{2,1})\hat{\phi}_{w,2}(\eta_{2,1})+\hat{\varphi}_f(\eta_{2,1})-\hat{\varphi}_f(s)\right)\\&\quad+\varepsilon_2\left(\hat{\zeta}_{2,1}(\eta_{2,1})\hat{\phi}_{w,2}(\eta_{2,1})+\hat{\varphi}_{h_1}(\eta_{2,1})-\hat{\varphi}_{h_1}(s)\right)\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\hat{\phi}_{w,2}(s)+\left(\varepsilon_3\hat{\chi}_{1,2}(s)+\varepsilon_4\hat{\zeta}_{1,2}(s)\right)\hat{\phi}_{w,1}(s)\\&=\varepsilon_3\left(\hat{\chi}_{1,2}(\eta_{2,2})\hat{\phi}_{w,1}(\eta_{2,2})+\hat{\varphi}_f(\eta_{2,2})-\hat{\varphi}_f(s)\right)\\&\quad+\varepsilon_4\left(\hat{\zeta}_{1,2}(\eta_{2,2})\hat{\phi}_{w,1}(\eta_{2,2})+\hat{\varphi}_{h_1}(\eta_{2,2})-\hat{\varphi}_{h_1}(s)\right)\end{aligned}\tag{4.2}$$

where:

- $\eta_{2,1}$ *and* $\eta_{2,2}$ *are defined in relation* (3.9);
- $\chi_{2,1}(x);\chi_{1,2}(x)$ *are defined in relation* (*3*.14);
- $h_1(x)$ *and* $h_2(x)$ *are defined in relation* (3.15);
- $\zeta_{2,1}(x);\zeta_{1,2}(x)$ *are defined in relation* (3.16):

$$\varepsilon_1=\frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(s+\eta_{1,1})(s-\eta_{2,1})};\ \varepsilon_2=\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{(s+\eta_{1,1})(s-\eta_{2,1})}\tag{4.3}$$

$$\varepsilon_3=\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(s+\eta_{1,2})(s-\eta_{2,2})};\ \varepsilon_4=\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{(s+\eta_{1,2})(s-\eta_{2,2})}\tag{4.4}$$

- **Proof of relation (4.1)**

  Using Equation (3.11), he follows:

$$\begin{aligned}\hat{m}_{1,1}(s)&=\int_0^{\infty}\mathrm{e}^{-su}\int_0^{u}\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{1,1}(v)\mathrm{d}v\mathrm{d}u\\&=\int_0^{\infty}\int_0^{u}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon_{1,1}(v)\mathrm{d}v\mathrm{d}u\\&=\frac{1}{s+\eta_{1,1}}\int_0^{\infty}\mathrm{e}^{\eta_{1,1}v}\mathrm{e}^{-(s+\eta_{1,1})v}\Upsilon_{1,1}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,1}}\int_0^{\infty}\mathrm{e}^{-sv}\Upsilon_{1,1}(v)\mathrm{d}v=\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{1,1}(s)\end{aligned}\tag{4.5}$$

$$\int_0^{\infty}\mathrm{e}^{-su}T_{\eta_{2,1}}\Upsilon_{1,1}(v)\mathrm{d}u=T_sT_{\eta_{2,1}}\Upsilon_{1,1}(0)\tag{4.6}$$

$$\int_0^{\infty}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,1}v}\hat{\Upsilon}_{1,1}(\eta_{2,1})\mathrm{d}u=\hat{\Upsilon}_{1,1}(\eta_{2,1})\int_0^{\infty}\mathrm{e}^{-(s+\eta_{1,1})u}\mathrm{d}u=\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{1,1}(\eta_{2,1})\tag{4.7}$$

$$\begin{aligned}\hat{m}_{2,1}(s) &= \int_0^\infty e^{-su}\int_0^u e^{-\eta_{1,1}(u-v)}\Upsilon_{2,1}(v)\,dv\,du \\ &= \int_0^\infty\int_0^u e^{-su}e^{-\eta_{1,1}(u-v)}\Upsilon_{2,1}(v)\,dv\,du \\ &= \frac{1}{s+\eta_{1,1}}\int_0^\infty e^{\eta_{1,1}v}e^{-(s+\eta_{1,1})v}\Upsilon_{2,1}(v)\,dv \\ &= \frac{1}{s+\eta_{1,1}}\int_0^\infty e^{-sv}\Upsilon_{2,1}(v)\,dv \\ &= \frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{2,1}(s)\end{aligned} \tag{4.8}$$

$$\int_0^\infty e^{-su}T_{\eta_{2,1}}\Upsilon_{2,1}(v)\,du = T_sT_{\eta_{2,1}}\Upsilon_{2,1}(0) \tag{4.9}$$

$$\int_0^\infty e^{-su}e^{-\eta_{1,1}v}\hat{\Upsilon}_{2,1}(\eta_{2,1})\,du = \hat{\Upsilon}_{2,1}(\eta_{2,1})\int_0^\infty e^{-(s+\eta_{1,1})u}\,du = \frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{2,1}(\eta_{2,1}) \tag{4.10}$$

By Formulas (4.5) to (4.10) he follows:

$$\begin{aligned}\hat{\phi}_{w,1}(s) = {} & \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{1,1}(s)+T_sT_{\eta_{2,1}}\Upsilon_{1,1}(0)-\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{1,1}(\eta_{2,1})\right) \\ & +\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{2,1}(s)+T_sT_{\eta_{2,1}}\Upsilon_{2,1}(0)-\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}_{2,1}(\eta_{2,1})\right)\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{w,1}(s) = {} & \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\frac{\hat{\Upsilon}_{1,1}(s)-\hat{\Upsilon}_{1,1}(\eta_{2,1})}{s+\eta_{1,1}}-\frac{\hat{\Upsilon}_{1,1}(s)-\hat{\Upsilon}_{1,1}(\eta_{2,1})}{s-\eta_{2,1}}\right) \\ & +\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\frac{\hat{\Upsilon}_{2,1}(s)-\hat{\Upsilon}_{2,1}(\eta_{2,1})}{s+\eta_{1,1}}-\frac{\hat{\Upsilon}_{2,1}(s)-\hat{\Upsilon}_{2,1}(\eta_{2,1})}{s-\eta_{2,1}}\right)\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{w,1}(s) = {} & \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(s+\eta_{1,1})(s-\eta_{2,1})}\left(\hat{\Upsilon}_{1,1}(\eta_{2,1})-\hat{\Upsilon}_{1,1}(s)\right) \\ & +\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{(s+\eta_{1,1})(s-\eta_{2,1})}\left(\hat{\Upsilon}_{2,1}(\eta_{2,1})-\hat{\Upsilon}_{2,1}(s)\right)\end{aligned} \tag{4.11}$$

Relation (4.11) can be put in the form:

$$\begin{aligned}\hat{\phi}_{w,1}(s) = {} & \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(s+\eta_{1,1})(s-\eta_{2,1})}\Big(\hat{\chi}_{1,1}(\eta_{2,1})\hat{\phi}_{w,1}(\eta_{2,1})+\hat{\chi}_{2,1}(\eta_{2,1})\hat{\phi}_{w,2}(\eta_{2,1}) \\ & +\hat{\varphi}_f(\eta_{2,1})-\hat{\chi}_{1,1}(s)\hat{\phi}_{w,1}(s)-\hat{\chi}_{2,1}(s)\hat{\phi}_{w,2}(s)-\hat{\varphi}_f(s)\Big) \\ & +\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{(s+\eta_{1,1})(s-\eta_{2,1})}\Big(\hat{\zeta}_{1,1}(\eta_{2,1})\hat{\phi}_{w,1}(\eta_{2,1})+\hat{\zeta}_{2,1}(\eta_{2,1})\hat{\phi}_{w,2}(\eta_{2,1}) \\ & +\hat{\varphi}_{h_1}(\eta_{2,1})-\hat{\zeta}_{1,1}(s)\hat{\phi}_{w,1}(s)-\hat{\zeta}_{2,1}(s)\hat{\phi}_{w,2}(s)-\hat{\varphi}_{h_1}(s)\Big)\end{aligned} \tag{4.12}$$

Using relation (4.11), we notice that $\hat{\phi}_{w,1}(\eta_{2,1})=0$. By isolating $\hat{\phi}_{w,1}(s)$ in relation (4.12), the expected result is obtained.

- **Proof of relation (4.2)**

Using Equation (3.12), he follows:

$$\begin{aligned}\hat{m}_{1,2}(s) &= \int_0^\infty \mathrm{e}^{-su}\int_0^u e^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}(v)\,\mathrm{d}v\mathrm{d}u \\ &= \int_0^\infty\int_0^u \mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}(v)\,\mathrm{d}v\mathrm{d}u \\ &= \frac{1}{s+\eta_{1,2}}\int_0^\infty \mathrm{e}^{\eta_{1,2}v}\mathrm{e}^{-(s+\eta_{1,2})v}\Upsilon_{1,2}(v)\,\mathrm{d}v \\ &= \frac{1}{s+\eta_{1,2}}\int_0^\infty \mathrm{e}^{-sv}\Upsilon_{1,2}(v)\,\mathrm{d}v \\ &= \frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}(s)\end{aligned} \tag{4.13}$$

$$\int_0^\infty \mathrm{e}^{-su}T_{\eta_{2,2}}\Upsilon_{1,2}(v)\,\mathrm{d}u = T_sT_{\eta_{2,2}}\Upsilon_{1,2}(0) \tag{4.14}$$

$$\int_0^{\infty \mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{1,2}(\eta_{2,2})\mathrm{d}u} = \hat{\Upsilon}_{1,2}(\eta_{2,2})\int_0^\infty \mathrm{e}^{-(s+\eta_{1,2})u}\mathrm{d}u = \frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}(\eta_{2,2}) \tag{4.15}$$

$$\begin{aligned}\hat{m}_{2,2}(s) &= \int_0^\infty \mathrm{e}^{-su}\int_0^u \mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{2,2}(v)\,\mathrm{d}v\mathrm{d}u \\ &= \int_0^\infty\int_0^u \mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}(u-v)}\mathrm{Y}_{2,2}(v)\,\mathrm{d}v\mathrm{d}u \\ &= \frac{1}{s+\eta_{1,2}}\int_0^\infty \mathrm{e}^{\eta_{1,2}v}\mathrm{e}^{-(s+\eta_{1,2})v}\Upsilon_{2,2}(v)\,\mathrm{d}v \\ &= \frac{1}{s+\eta_{1,2}}\int_0^\infty \mathrm{e}^{-sv}\Upsilon_{2,2}(v)\,\mathrm{d}v \\ &= \frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}(s)\end{aligned} \tag{4.13}$$

$$\int_0^\infty \mathrm{e}^{-su}T_{\eta_{2,2}}\Upsilon_{2,2}(v)\,\mathrm{d}u = T_sT_{\eta_{2,2}}\Upsilon_{2,2}(0) \tag{4.17}$$

$$\int_0^\infty \mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{2,2}(\eta_{2,2})\,\mathrm{d}u = \hat{\Upsilon}_{2,2}(\eta_{2,2})\int_0^\infty \mathrm{e}^{-(s+\eta_{1,2})u}\mathrm{d}u = \frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}(\eta_{2,2}) \tag{4.18}$$

By relations (4.13) to (4.18), he follows:

$$\begin{aligned}\hat{\phi}_{w,2}(s) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}(s)+T_sT_{\eta_{2,2}}\Upsilon_{1,2}(0)-\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}(\eta_{2,2})\right) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}(s)+T_sT_{\eta_{2,2}}\Upsilon_{2,2}(0)-\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}(\eta_{2,2})\right)\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{w,2}(s) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\frac{\hat{\Upsilon}_{1,2}(s)-\hat{\Upsilon}_{1,2}(\eta_{2,2})}{s+\eta_{1,2}}-\frac{\hat{\Upsilon}_{1,2}(s)-\hat{\Upsilon}_{1,2}(\eta_{2,2})}{s-\eta_{2,2}}\right) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\frac{\hat{\Upsilon}_{2,2}(s)-\hat{\Upsilon}_{2,2}(\eta_{2,2})}{s+\eta_{1,2}}-\frac{\hat{\Upsilon}_{2,2}(s)-\hat{\Upsilon}_{2,2}(\eta_{2,2})}{s-\eta_{2,2}}\right)\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{w,2}(s) = {} & \frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(s+\eta_{1,2})(s-\eta_{2,2})}\left(\hat{\Upsilon}_{1,2}(\eta_{2,2})-\hat{\Upsilon}_{1,2}(s)\right) \\ & +\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{(s+\eta_{1,2})(s-\eta_{2,2})}\left(\hat{\Upsilon}_{1,2}(\eta_{2,2})-\hat{\Upsilon}_{2,2}(s)\right)\end{aligned} \tag{4.19}$$

$$\hat{\phi}_{w,2}(s)=\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(s+\eta_{1,2})(s-\eta_{2,2})}\Big(\hat{\chi}_{1,2}(\eta_{2,2})\hat{\phi}_{w,1}(\eta_{2,2})+\hat{\chi}_{2,2}(\eta_{2,2})\hat{\phi}_{w,2}(\eta_{2,2})$$
$$+\hat{\varphi}_f(\eta_{2,2})-\hat{\chi}_{1,2}(s)\hat{\phi}_{w,1}(s)-\hat{\chi}_{2,2}(s)\hat{\phi}_{w,2}(s)-\hat{\varphi}_f(s)\Big)$$
$$+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{(s+\eta_{1,2})(s-\eta_{2,2})}\Big(\hat{\zeta}_{1,2}(\eta_{2,2})\hat{\phi}_{w,1}(\eta_{2,2})+\hat{\zeta}_{2,2}(\eta_{2,2})\hat{\phi}_{w,2}(\eta_{2,2})$$
$$+\hat{\varphi}_{h_1}(\eta_{2,2})-\hat{\zeta}_{1,2}(s)\hat{\phi}_{w,1}(s)-\hat{\zeta}_{2,2}(s)\hat{\phi}_{w,2}(s)-\hat{\varphi}_{h_1}(s)\Big) \tag{4.20}$$

Using relation (4.19), we notice that $\hat{\phi}_{w,2}(\eta_{2,2})=0$. By isolating $\hat{\phi}_{w,2}(s)$ in relation (4.20), the expected result is obtained.

4.2. The Laplace Transforms of the Functions $\phi_{d,i}(u)$, $i=1,2$

Theorem 4.2. *The Laplace transforms of the Gerber-Shiu functions* $\phi_{d,i}(u), i=1,2$ *verify the following relations*:

$$\hat{\phi}_{d,1}(s)+\left(\varepsilon_1\hat{\chi}_{2,1}(s)+\varepsilon_2\hat{\zeta}_{2,1}(s)\right)\hat{\phi}_{d,2}(s)$$
$$=\varepsilon_1\left(\hat{\chi}_{2,1}(\eta_{2,1})\hat{\phi}_{d,2}(\eta_{2,1})\right)+\varepsilon_2\left(\hat{\zeta}_{2,1}(\eta_{2,1})\hat{\phi}_{d,2}(\eta_{2,1})\right)+\frac{1}{s+\eta_{1,1}} \tag{4.21}$$

$$\hat{\phi}_{d,2}(s)+\left(\varepsilon_3\hat{\chi}_{1,2}(s)+\varepsilon_4\hat{\zeta}_{1,2}(s)\right)\hat{\phi}_{d,1}(s)$$
$$=\varepsilon_3\left(\hat{\chi}_{1,2}(\eta_{2,2})\hat{\phi}_{d,1}(\eta_{2,2})\right)+\varepsilon_4\left(\hat{\zeta}_{1,2}(\eta_{2,2})\hat{\phi}_{d,1}(\eta_{2,2})\right)+\frac{1}{s+\eta_{1,2}} \tag{4.22}$$

where:

- $\eta_{2,1}$ *and* $\eta_{2,2}$ *are defined in relation* (3.9);
- $\chi_{2,1}(x);\chi_{1,2}(x)$ *are defined in relation* (3.14);
- $\zeta_{2,1}(x);\zeta_{1,2}(x)$ *are defined in relation* (3.16);
- ε_1 *and* ε_2 ; ε_3 *are* ε_4 *are respectively defined by relations* (4.3) *and* (4.4);
- **Proof of relation (4.21).**

Using Equation (3.56), he follows:

$$\begin{aligned}\hat{m}^{\star}_{1,1}(s)&=\int_0^{\infty}\mathrm{e}^{-su}\int_0^{u}\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{1,1}(v)\mathrm{d}v\mathrm{d}u\\&=\int_0^{\infty}\int_0^{u}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{1,1}(v)\mathrm{d}v\mathrm{d}u\\&=\frac{1}{s+\eta_{1,1}}\int_0^{\infty}\mathrm{e}^{\eta_{1,1}v}\mathrm{e}^{-(s+\eta_{1,1})v}\Upsilon^{\star}_{1,1}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,1}}\int_0^{\infty}\mathrm{e}^{-sv}\Upsilon^{\star}_{1,1}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{1,1}(s)\end{aligned} \tag{4.23}$$

$$\int_0^{\infty}\mathrm{e}^{-su}T_{\eta_{2,1}}\Upsilon^{\star}_{1,1}(v)\mathrm{d}u=T_sT_{\eta_{2,1}}\Upsilon^{\star}_{1,1}(0) \tag{4.24}$$

$$\int_0^{\infty}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,1}u}\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})\mathrm{d}u=\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})\int_0^{\infty}\mathrm{e}^{-(s+\eta_{1,1})u}\mathrm{d}u=\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1}) \tag{4.25}$$

$$\begin{aligned}\hat{m}^{\star}_{2,1}(s) &= \int_0^{\infty} e^{-su}\int_0^{u} e^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{2,1}(v)\,dv\,du \\ &= \int_0^{\infty}\int_0^{u} e^{-su}e^{-\eta_{1,1}(u-v)}\Upsilon^{\star}_{2,1}(v)\,dv\,du \\ &= \frac{1}{s+\eta_{1,1}}\int_0^{\infty} e^{\eta_{1,1}v}e^{-(s+\eta_{1,1})v}\Upsilon^{\star}_{2,1}(v)\,dv \\ &= \frac{1}{s+\eta_{1,1}}\int_0^{\infty} e^{-sv}\Upsilon^{\star}_{2,1}(v)\,dv \\ &= \frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{2,1}(s)\end{aligned} \tag{4.26}$$

$$\int_0^{\infty} e^{-su}T_{\eta_{2,1}}\Upsilon^{\star}_{2,1}(v)\,du = T_sT_{\eta_{2,1}}\Upsilon^{\star}_{2,1}(0) \tag{4.27}$$

$$\int_0^{\infty} e^{-su}e^{-\eta_{1,1}v}\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})\,du = \hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})\int_0^{\infty} e^{-(s+\eta_{1,1})u}\,du = \frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1}) \tag{4.28}$$

By relations (4.23) to (4.28), we have:

$$\begin{aligned}\hat{\phi}_{d,1}(s) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{1,1}(s)+T_sT_{\eta_{2,1}}\Upsilon^{\star}_{1,1}(0)-\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})\right) \\ &+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{2,1}(s)+T_sT_{\eta_{2,1}}\Upsilon^{\star}_{2,1}(0)-\frac{1}{s+\eta_{1,1}}\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})\right)+\frac{1}{s+\eta_{1,1}}\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{d,1}(s) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(\eta_{1,1}+\eta_{2,1})}\left(\frac{\hat{\Upsilon}^{\star}_{1,1}(s)-\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})}{s+\eta_{1,1}}-\frac{\hat{\Upsilon}^{\star}_{1,1}(s)-\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})}{s-\eta_{2,1}}\right) \\ &+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{\eta_{1,1}+\eta_{2,1}}\left(\frac{\hat{\Upsilon}^{\star}_{2,1}(s)-\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})}{s+\eta_{1,1}}-\frac{\hat{\Upsilon}^{\star}_{2,1}(s)-\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})}{s-\eta_{2,1}}\right)+\frac{1}{s+\eta_{1,1}}\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{d,1}(s) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(s+\eta_{1,1})(s-\eta_{2,1})}\left(\hat{\Upsilon}^{\star}_{1,1}(\eta_{2,1})-\hat{\Upsilon}^{\star}_{1,1}(s)\right) \\ &+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{(s+\eta_{1,1})(s-\eta_{2,1})}\left(\hat{\Upsilon}^{\star}_{2,1}(\eta_{2,1})-\hat{\Upsilon}^{\star}_{2,1}(s)\right)+\frac{1}{s+\eta_{1,1}}\end{aligned} \tag{4.29}$$

$$\begin{aligned}\hat{\phi}_{d,1}(s) &= \frac{\lambda_1\eta_{1,1}\eta_{2,1}(1-\alpha)}{(\delta+\lambda_1)(s+\eta_{1,1})(s-\eta_{2,1})}\Big(\hat{\chi}_{1,1}(\eta_{2,1})\hat{\phi}_{d,1}(\eta_{2,1}) \\ &+\hat{\chi}_{2,1}(\eta_{2,1})\hat{\phi}_{d,2}(\eta_{2,1})-\hat{\chi}_{1,1}(s)\hat{\phi}_{d,1}(s)-\hat{\chi}_{2,1}(s)\hat{\phi}_{d,2}(s)\Big) \\ &+\frac{\alpha\beta\eta_{1,1}\eta_{2,1}}{(s+\eta_{1,1})(s-\eta_{2,1})}\Big(\hat{\zeta}_{1,1}(\eta_{2,1})\hat{\phi}_{d,1}(\eta_{2,1})+\hat{\zeta}_{2,1}(\eta_{2,1})\hat{\phi}_{d,2}(\eta_{2,1}) \\ &-\hat{\zeta}_{1,1}(s)\hat{\phi}_{d,1}(s)-\hat{\zeta}_{2,1}(s)\hat{\phi}_{d,2}(s)\Big)+\frac{1}{s+\eta_{1,1}}\end{aligned} \tag{4.30}$$

Using relation (4.29), we notice that $\hat{\phi}_{d,1}(\eta_{2,1})=0$. By isolating $\hat{\phi}_{d,1}(s)$ in relation (4.30), the expected result is obtained.

- **Proof of the relation (4.22)**

$$\begin{aligned}\hat{m}_{1,2}^{\star}(s)&=\int_0^{\infty}\mathrm{e}^{-su}\int_0^{u}\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}^{\star}(v)\mathrm{d}v\mathrm{d}u\\&=\int_0^{\infty}\int_0^{u}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{1,2}^{\star}(v)\mathrm{d}v\mathrm{d}u\\&=\frac{1}{s+\eta_{1,2}}\int_0^{\infty}\mathrm{e}^{\eta_{1,2}v}\mathrm{e}^{-(s+\eta_{1,2})v}\Upsilon_{1,2}^{\star}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,2}}\int_0^{\infty}\mathrm{e}^{-sv}\Upsilon_{1,2}^{\star}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}^{\star}(s)\end{aligned}\tag{4.31}$$

$$\int_0^{\infty}\mathrm{e}^{-su}T_{\eta_{2,2}}\Upsilon_{1,2}^{\star}(v)\mathrm{d}u=T_sT_{\eta_{2,2}}\Upsilon_{1,2}^{\star}(0)\tag{4.32}$$

$$\int_0^{\infty}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})\mathrm{d}u=\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})\int_0^{\infty}\mathrm{e}^{-(s+\eta_{1,2})u}\mathrm{d}u=\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})\tag{4.33}$$

$$\begin{aligned}\hat{m}_{2,2}^{\star}(s)&=\int_0^{\infty}\mathrm{e}^{-su}\int_0^{u}\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{2,2}^{\star}(v)\mathrm{d}v\mathrm{d}u\\&=\int_0^{\infty}\int_0^{u}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}(u-v)}\Upsilon_{2,2}^{\star}(v)\mathrm{d}v\mathrm{d}u\\&=\frac{1}{s+\eta_{1,2}}\int_0^{\infty}\mathrm{e}^{\eta_{1,2}v}\mathrm{e}^{-(s+\eta_{1,2})v}\Upsilon_{2,2}^{\star}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,2}}\int_0^{\infty}\mathrm{e}^{-sv}\Upsilon_{2,2}^{\star}(v)\mathrm{d}v\\&=\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}^{\star}(s)\end{aligned}\tag{4.34}$$

$$\int_0^{\infty}\mathrm{e}^{-su}T_{\eta_{2,2}}\Upsilon_{2,2}^{\star}(v)\mathrm{d}u=T_sT_{\eta_{2,2}}\Upsilon_{2,2}^{\star}(0)\tag{4.35}$$

$$\int_0^{\infty}\mathrm{e}^{-su}\mathrm{e}^{-\eta_{1,2}u}\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})\mathrm{d}u=\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})\int_0^{\infty}\mathrm{e}^{-(s+\eta_{1,2})u}\mathrm{d}u=\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})\tag{4.36}$$

By relations (4.31) to (4.36) he follows:

$$\begin{aligned}\hat{\phi}_{d,2}(s)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}^{\star}(s)+T_sT_{\eta_{2,2}}\Upsilon_{1,2}^{\star}(0)-\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})\right)\\&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}^{\star}(s)+T_sT_{\eta_{2,2}}\Upsilon_{2,2}^{\star}(0)-\frac{1}{s+\eta_{1,2}}\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})\right)+\frac{1}{s+\eta_{1,2}}\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{d,2}(s)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(\eta_{1,2}+\eta_{2,2})}\left(\frac{\hat{\Upsilon}_{1,2}^{\star}(s)-\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})}{s+\eta_{1,2}}-\frac{\hat{\Upsilon}_{1,2}^{\star}(s)-\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})}{s-\eta_{2,2}}\right)\\&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{\eta_{1,2}+\eta_{2,2}}\left(\frac{\hat{\Upsilon}_{2,2}^{\star}(s)-\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})}{s+\eta_{1,2}}-\frac{\hat{\Upsilon}_{2,2}^{\star}(s)-\hat{\Upsilon}_{2,2}^{\star}(\eta_{2,2})}{s-\eta_{2,2}}\right)+\frac{1}{s+\eta_{1,2}}\end{aligned}$$

$$\begin{aligned}\hat{\phi}_{d,2}(s)=&\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(s+\eta_{1,2})(s-\eta_{2,2})}\left(\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})-\hat{\Upsilon}_{1,2}^{\star}(s)\right)\\&+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{(s+\eta_{1,2})(s-\eta_{2,2})}\left(\hat{\Upsilon}_{1,2}^{\star}(\eta_{2,2})-\hat{\Upsilon}_{2,2}^{\star}(s)\right)+\frac{1}{s+\eta_{1,2}}\end{aligned}\tag{4.37}$$

$$\begin{aligned}\hat{\phi}_{d,2}(s) = &\frac{\lambda_2\eta_{1,2}\eta_{2,2}(1-\alpha)}{(\delta+\lambda_2)(s+\eta_{1,2})(s-\eta_{2,2})}\Big(\hat{\chi}_{1,2}(\eta_{2,2})\hat{\phi}_{d,1}(\eta_{2,2}) \\ &+\hat{\chi}_{2,2}(\eta_{2,2})\hat{\phi}_{d,2}(\eta_{2,2}) - \hat{\chi}_{1,2}(s)\hat{\phi}_{d,1}(s) - \hat{\chi}_{2,2}(s)\hat{\phi}_{d,2}(s)\Big) \\ &+\frac{\alpha\beta\eta_{1,2}\eta_{2,2}}{(s+\eta_{1,2})(s-\eta_{2,2})}\Big(\hat{\zeta}_{1,2}(\eta_{2,2})\hat{\phi}_{d,1}(\eta_{2,2}) + \hat{\zeta}_{2,2}(\eta_{2,2})\hat{\phi}_{d,2}(\eta_{2,2}) \\ &-\hat{\zeta}_{1,2}(s)\hat{\phi}_{d,1}(s) - \hat{\zeta}_{2,2}(s)\hat{\phi}_{d,2}(s)\Big) + \frac{1}{s+\eta_{1,2}}\end{aligned} \quad (4.38)$$

Using relation (4.37), we notice that $\hat{\phi}_{d,2}(\eta_{2,2}) = 0$. By isolating $\hat{\phi}_{d,2}(s)$ in relation (4.38), the expected result is obtained.

5. Conclusion

In this work, the integro-differential equations of Gerber-Shiu function and their Laplace transforms in a compound risk model perturbed by Brownian motion with variable premium and tail dependence via the Spearman copula are obtained. Determining the ultimate ruins probabilities associated with this risk model remains a future investigation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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