Scientific Research Publishing

Open Journal of Statistics, 2022, 12, 827-838
https://www.scirp.org/journal/ojs
ISSN Online: 2161-7198
ISSN Print: 2161-718X

# Management of a Complex Portfolio of Assets with Stochastic Drifts and Volatilities

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**How to cite this paper:** Yaméogo, W., Ouédraogo, K. and Barro, D. (2022) Management of a Complex Portfolio of Assets with Stochastic Drifts and Volatilities. *Open Journal of Statistics*, **12**, 827-838.
https://doi.org/10.4236/ojs.2022.126047

**Received:** December 2, 2022
**Accepted:** December 27, 2022
**Published:** December 30, 2022



## Abstract

In financial analysis risk quantification is essential for efficient portfolio management in a stochastic framework. In this paper we study the value at risk, the expected shortfall, marginal expected shortfall and value at risk, incremental value at risk and expected shortfall, the marginal and discrete marginal contributions of a portfolio. Each asset in the portfolio is characterized by a trend, a volatility and a price following a three-dimensional diffusion process. The interest rate of each asset evolves according to the Hull and White model. Furthermore, we propose the optimization of this portfolio according to the value at risk model.

## Keywords

Value at Risk, Expected Shortfall, Stochastic Process, Interest Rate

## 1. Introduction

In the financial world, portfolio management is essential. For that stochastic processes are used to describe the random evolution of financial assets. In their standard model, Black and Scholes [1] considered a model with constant volatility and drift. To solve one of the shortcomings of this model, other authors have assumed that the volatility and drift are stochastic. Stochastic volatility is used in quantitative finance to value derivatives, such as options. The modeling of interest rates has been the subject of work by several researchers. We can mention the Vasicek model, the Cox-Ingersoll-Ross (CIR) model, the Hull and White model (see [2]). The stochastic interest rate model of Hull and White $r$ is

$$\mathrm{d}r = \left(k(t) - ar(t)\right)\mathrm{d}t + \sigma \mathrm{d}B. \tag{1}$$

where $B$ is the standard Brownian motion, $\sigma$ the annualized instantaneous deviation of changes in the rate.

Applications of stochastic optimal control to management and finance problems were developed in the 1970s, notably with Merton's pioneering article on portfolio allocation. Portfolio management using mean-variance criteria was initially formulated by the American economist Harry Markowitz (one of the 1990 Nobel Prize winners in economics) in a static one-period framework. An efficient portfolio is one that presents the optimal risk/return trade-off for an investor. Optimization problems in finance using risk measures or the dynamic programming principle and the Hamiltonjacobibellman theory are studied in the works of Schied [3], Gundel [4], Barrieu and El Karoui [5], Yang and al [6] and Mbigili and al [7]. Marginal risks are important criterion in portfolio management [8] [9].

The major contribution of this paper is the study of the management of a large complex portfolio of assets with stochastc drifts and stochastic volatilities by these marginal, incremental risks and by the optimization of this portfolio by minimizing its potential loss for a fixed return. The price and volatility of each asset are correlated and the assets move independently. The rest of our work will be organized as follows. Section 2 is devoted to the preliminaries of the study, where properties on stochastic processes are presented, and Euler's theorem on homogeneous functions. In Section 3 we present the main result of our study. Finally in Section 4 we give model of VaR and optimization of portfolio.

## 2. Materials and Methods

In this section, we give important definitions and properties on stochastic processes, Euler theorem on the homogeneous functions. We refer the readers to [10]-[17] for stochastic processes applied to finance and to [18] for homogeneous functions. These results turn out to be necessary for our study.

### 2.1. An Overview of Stochastic Processes

A stochastic process is a one or multidimensional variable that depends on hazard and time. There are infinite examples of such processes. To stick to the field of finance, let us quote the price of a share, an interest rate, a stock market index or a set of variables including several rates, indices, exchange rates. Mathematically it is defined by the following.

Let $(\Omega, A, P)$ be a probability space, $(E, \varepsilon)$ a measurable space and $T$ a set, (for example $T = \mathbb{N}, \mathbb{R}, \mathbb{R}^d$). A stochastic process is a application $X : (\Omega, T) \to E$ which to the couple $(w, t)$ we associate $X(w, t)$ again denoted $X_t(w)$ such that, for any fixed $t \in T$, $(X_t)$ is a random variable on $(\Omega, A, P)$. In particular Brownian motion is a stochastic process which plays a key role in financial modelling.

The following proposition characterizes a standard Brownian motion.

**Proposition 1.** i) The standard Brownian motion $W(t)$ is distributed according to a Gaussian law $N(0, t)$ with $t \in \mathbb{R}$;

ii) Its increase between $t$ and $t+\Delta t$, $\Delta W = W(t+\Delta t) - W(t)$ is distributed according to a Gaussian law $N(0,\Delta t)$.

iii) W is a martingale: the conditional expectation
$E(W(t)|W(s)) = W(s)$.

An arithmetic Brownian motion $X(t)$ with parameters $\mu$ and $\sigma$ is expressed as a function of a Wiener process $W(t)$ as follows

$$W(t) = \frac{X(t) - \mu t}{\sigma}.$$

Calculation rules on Brownian motions.

The differential calculus on Wiener processes obeys the following rules:

i) $(\mathrm{d}W)^2 = \mathrm{d}t = V(\mathrm{d}W)$.

ii) $\mathrm{d}W \cdot \mathrm{d}t = 0$.

iii) $\mathrm{d}W_1(t) \cdot \mathrm{d}W_2(t) = \rho_{12}\mathrm{d}t = \mathrm{cov}(\mathrm{d}W_1, \mathrm{d}W_2)$ if $W_1$ and $W_2$ are two Wiener processes correlated and $\rho_{12}$ is the instantaneous correlation coefficient between $W_1$ and $W_2$.

iv) $\mathrm{d}W(t_1) \cdot \mathrm{d}W(t_2) = 0$ if $t_1 \neq t_2$.

Heuristic" computational rules applicable to the non-standard Brownian with parameters $\mu$ and $\sigma$:

v) $\mathrm{d}X^2 = \sigma^2 \mathrm{d}t$;

vi) $\mathrm{d}X\mathrm{d}t = 0$;

vii) $\mathrm{d}X_1(t)\mathrm{d}X_2(t) = \mathrm{cov}(\mathrm{d}X_1, \mathrm{d}X_2) = \sigma_{12}\mathrm{d}t = \rho_{12}\sigma_1\sigma_2\mathrm{d}t$

where $\rho_{12}$ is the instantaneous correlation coefficient between $X_1$ and $X_2$.

### 2.2. Homogeneous Function and Euler's Theorem

Euler's theorem associated to the name of Leonhard Euler, is a multivariate analysis result useful in thermodynamics, economics, and finance.

**Theorem 2.** [19] Let C be a cone of $\mathbb{R}^n$ and let k be a real number.

A multivariate function $f : C \to \mathbb{R}^m$ differentiable at any point is positively homogeneous of degree $k$ if and only if the following relation, satisfy: for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = kf(x). \tag{2}$$

Let $E$ and $F$ be two K-normalized vector spaces ($K = \mathbb{R}$ ou $\mathbb{C}$) and $C$ a cone of $E$ and $k$ a element of $K$. A differentiable function $f : C \to F$ is positively homogeneous of degree $k$ if and only if: for all $x \in C$, $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = kf(x)$.

The three following sections provides the main results of our study and $v'$ is the transpose of $v$.

## 3. Main Results

### 3.1. Hypothesis and Model

We consider a portfolio $P$ made up of a set of stocks. Each stock $i$ is defined as followed. $S_i$ is the stock price, $\sigma_i$ is the volatility of the stock which influences the dynamics of $S_i$ and the option price, $r_i$ is the interest rate. The stochastic interest rate model $r_i$ is that proposed by Hull and White ([2]). For all *i* $\sigma_{ir}$ is the annualized instantaneous deviation of changes in the rate supposed constant and know like a. In the same vein, the parameter $\theta$ is a risk premium that is added to the interest rate. The stochastic volatility $\sigma_i$ of the stock follows a process involving a recall force towards a value $f_i(S_i(t))$ where $f_i$ is a decreasing function of the price $S_i$. Morever it is assumed that the volatility and the interest rate are uncorrelated. The relative change in the price of $S_i$ and the volatility are correlated with $\rho_1$. $\varepsilon$ is the instantaneous standard deviation of the volatility $\sigma_i$, assumed constant. The interest rate, the volatility and the price $S_i$ follow the three-dimensional diffusion process below

$$\begin{cases} \mathrm{d}r_i = \left(k_i(t) - a_i r_i(t)\right)\mathrm{d}t + \sigma_{ir}\mathrm{d}B_1 \\ \dfrac{\mathrm{d}S_i}{S_i} = \left(r_i(t) + \theta\right)\mathrm{d}t + \rho\sigma_i(t)\mathrm{d}B_1 + \sigma_i(t)\sqrt{1-\rho^2}\mathrm{d}B_2 \\ \mathrm{d}\sigma_i = b\left(f(S_i) - \sigma_i(t)\right)\mathrm{d}t + \dfrac{\rho_1\varepsilon}{\sqrt{1-\rho^2}}\mathrm{d}B_2 + \varepsilon\sqrt{\dfrac{1-\rho^2-\rho_1^2}{1-\rho^2}}\mathrm{d}B_3 \end{cases} \tag{3}$$

The standard Brownian motions $B_1$, $B_2$ and $B_3$ are independent.

In the following subsection we calculate the VaR, the ES, the incremental VaR and ES and the contributions of each asset to the portfolio risk.

## 3.2. Computation of the Portfolio's VaR and ES

In actuarial and financial sciences, the Value at Risk (VaR) quantifies numerically the size of the loss for which there is a low probability of being exceeded. It is characterized by the confidence level, the time horizon chosen and the distribution of profit or loss. The VaR is a risk measure mainly used to measure the large portfolio market risk. For a confidence level $\alpha \in ]0;1[$ Value-at-Risk is the lower $\alpha$-quantile, given by

$$VaR(X;\alpha) = \inf\left\{x, F_X(x) \geq \alpha\right\} = F_X^{-1}(\alpha); \tag{4}$$

where $F_X^{-1}$ is the right continuous inverse of $F_X$.

Loyara *et al.* [19] showed that the VaR is also intrinsically linked to this function, making it possible to bridge the copula function with the VaR. While Yaméogo and al. [20] pointed out that VaR makes it possible to manage a large portfolio divided into several sectors each comprising two sub-sectors. More generally in multivariate study, for a random vector $X$ satisfying the regularity conditions, one defines the multidimensional $VaR$ at probability level $\alpha$ by:

$$VaR_\alpha(X) = \mathbb{E}\left[X \mid X \in \partial L(\alpha)\right]; \tag{5}$$

where $\partial L(\alpha)$ is the boundary of the $\alpha$-level set of $F_t$, the univariate component of the vector.

As for the Expected Shortfall it corresponds to the average size of losses above VaR. In the case of continuous variables, the definition of the Expected Shortfall coincides with that of Conditional Value at risk. This measure is very sensitive to the tail of the distribution and therefore it is more conservative than the VaR.

$$ES(T,p) = E\left[L \mid L \geq VaR(T,p)\right] = \frac{1}{1-p}\int_{VaR(T,p)}^{+\infty} x f_L(x)\,\mathrm{d}x; \tag{6}$$

where $f_L(x)$ is the density function of the loss and $F_L(x)$ will denote its distribution function.

The following proposition gives results one these risk measures.

**Proposition 3.** For all realization $x = (x_1, \cdots, x_n)'$ of the random vector X, the VaR and ES associated with of the portfolio P are given respectively by:

$$VaR_P(x_1, \cdots, x_n) = \alpha_p \sqrt{\sum_{i=1}^{n} x_i^2 \zeta_i} - \mu' x \tag{7}$$

and

$$ES_P(x_1, \cdots, x_n) = \xi_p \sqrt{\sum_{i=1}^{n} x_i^2 \zeta_i} - \mu' x, \quad \xi_p = \frac{1}{1-p}\int_p^1 \alpha_u \mathrm{d}u, \tag{8}$$

where $\alpha_p$ is the p-quantile of the standardized distributed Gaussian and $\mu$ is the mean of *X*.

*Proof.* The return of the portfolio is $R_P = \sum_{i=1}^{n} x_i R_i$ where $R_i = \frac{\mathrm{d}S_i}{S_i}$ is the return of each asset. The mathematical expectation being linear, the expected return of the portfolio $\mu_p = E(R_p) = E\left(\sum_{i=1}^{n} x_i R_i\right) = \sum_{i=1}^{n} x_i E(R_i) = \sum_{i=1}^{n} x_i \mu_i$ where $E(R_i) = \mu_i$. Under the assumption of independence, the variance matrix-covariance is diagonal of elements $V(R_i)$, $1 \leq i \leq n$. So, it comes that

$$E(R_i) = E\left(\frac{\mathrm{d}S_i}{S_i}\right) = E\left(r_i(t) + \theta\right)\mathrm{d}t = \mu_i \mathrm{d}t.$$

Furthermore by using the computations rules on the Brownian motion of the section 2.1 one has

$$V(R_i) = V\left(r_i(t) + \theta\right) = V\left(r_i(t)\right)$$

and

$$\mathrm{d}r_i = \left(k_i(t) - a_i r_i(t)\right)\mathrm{d}t + \sigma_{ir}\mathrm{d}W_1.$$

By integration, one obtain

$$r_i(t) = \mathrm{e}^{-a_i t} r_i(0) + \int_0^t \mathrm{e}^{a_i(s-t)} k_i(s)\mathrm{d}s + \sigma_{ir}\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i u}\mathrm{d}W(u) \tag{9}$$

where $r_i(t)$ is a Gaussian process since a sum of normal variables. So, we have

$$V\left(r_i(t)\right) = \sigma_{ir}^2\left(1 - \mathrm{e}^{-2a_i t}\right) \Big/ 2a_i \tag{10}$$

and

$$E\left(R_i\right)=E\left(\frac{\mathrm{d}S_i}{S_i}\right)=E\left(r_i\left(t\right)+\theta\right)\mathrm{d}t=\mu_i\mathrm{d}t=\left(\mathrm{e}^{-a_it}r_i\left(0\right)+\int_0^t\mathrm{e}^{a_i(s-t)}k_i\left(s\right)\mathrm{d}s+\theta\right)\mathrm{d}t. \quad (11)$$

Therefore $R_i$ is Gaussian distributed, which implies that

$$VaR\left(x_1,x_2,\cdots,x_n\right)=\alpha_p\sigma_p+\mu_P; \quad (12)$$

where $\sigma_p=\sqrt{x'Mx}$, $M$ is the variance-covariance matrix given by

$$M=\begin{pmatrix} \zeta_1 & 0 & & 0 & . & 0 \\ . & \zeta_2 & . & . & . & . \\ & . & . & 0 & . & 0 \\ 0 & . & 0 & . & \cdots & \\ . & . & . & . & \zeta_{n-1} & 0 \\ 0 & . & 0 & 0 & 0 & \zeta_n \end{pmatrix} \quad (13)$$

where $\zeta_i=\dfrac{\sigma_{r_i}^2\left(1-\mathrm{e}^{-2a_it}\right)}{2a_i}$.

From these result, one have

$$VaR_P\left(x_1,\cdots,x_n\right)=\alpha_p\sqrt{\sum_{i=1}^n x_i^2\zeta_i}-\mu'x$$

which gives

$$VaR_P\left(x_1,\cdots,x_n\right)=\alpha_p\sqrt{\sum_{i=1}^n x_i^2\zeta_i}-\sum_{i=1}^n x_i\left(\mathrm{e}^{-at}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right).$$

Furthermore; it comes that

$$VaR_P\left(x_1,\cdots,x_n\right)=\alpha_p\sqrt{\sum_{i=1}^n x_i^2\zeta_i}-\mu'x \quad \text{that is}$$

$$VaR_P\left(x_1,\cdots,x_n\right)=\alpha_p\sqrt{\sum_{i=1}^n x_i^2\zeta_i}-\sum_{i=1}^n x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right) \quad (14)$$

In the case where the returns are Gaussian distributed

$$VaR_P\left(x_1,\cdots,x_n\right)=\alpha_p\sqrt{x'Mx}-\mu'x$$

where $x=\left(x_1,\cdots,x_n\right)'$ and $\mu=\left(\mu_1,\cdots,\mu_n\right)'$. So, by setting $\xi_p=\dfrac{1}{1-p}\int_p^1\alpha_u\mathrm{d}u$, one have

$$ES_P\left(x_1,\cdots,x_n\right)=\xi_p\sqrt{x'Mx}-\mu'x \quad \text{which gives}$$

$$ES_P\left(x_1,\cdots,x_n\right)=\xi_p\sqrt{\sum_{i=1}^n x_i^2\zeta_i}-\sum_{i=1}^n x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right). \quad (15)$$

So, the result is proved as disserted.☐

### 3.3. Computation of Incremental Risks and Marginal Risks

To calculate the incremental value at risk, an investor needs to know the portfolio's standard deviation, the portfolio's rate of return, and the asset in question's rate of return and portfolio share.

Incremental value at risk (incremental VaR) is the amount of uncertainty added to or subtracted from a portfolio by purchasing or selling an investment. Investors use incremental value at risk to determine whether a particular investment should be undertaken, given its likely impact on potential portfolio losses. Incremental VaR tells you the precise amount of risk a position is adding or subtracting from the whole portfolio, while marginal VaR is just an estimation of the change in the total amount of risk. The idea of incremental value at risk was developed by Kevin Dowd in his 1999 book [21], "Beyond Value at Risk: The New Science of Risk Management." Incremental VaR is closely related to, but differs from, marginal VaR.

The following result gives expressions of VaR and ES as a function of their first partial derivatives.

**Proposition 4.** For all realization $x=(x_1,\cdots,x_n)$ of $X$ the VaR and the ES of the portfolio P are given respectively by:

$$VaR_P(x_1,\cdots,x_n)=\sum_{i=1}^{n}x_i\frac{\partial VaR_P(x_1,\cdots,x_n)}{\partial x_i} \tag{16}$$

and

$$ES_P(x_1,\cdots,x_n)=\sum_{i=1}^{n}x_i\frac{\partial ES_P(x_1,\cdots,x_n)}{\partial x_i}. \tag{17}$$

*Proof.* The risk measures $VaR$ and $ES$ satisfy respectively the relations

$$VaR(c\cdot x)=c_1\cdot VaR(x)$$

and

$$ES(c\cdot x)=c_1\cdot ES(x)$$

where $x\in\mathbb{R}^n$ and $c_1$ is a positive constant. This implies that they are homogeneous and differentiable functions. In particular the functions $VaR_p$ and $ES_p$ being differentiable and positively homogeneous functions of degree 1, we can apply Euler's theorem for $k=1$ to them and one has:

$$VaR_P(x_1,\cdots,x_n)=\sum_{i=1}^{n}x_i\frac{\partial VaR_P(x_1,\cdots,x_n)}{\partial x_i};$$

and also,

$$ES_P(x_1,\cdots,x_n)=\sum_{i=1}^{n}x_i\frac{\partial ES_P(x_1,\cdots,x_n)}{\partial x_i}.$$

The following result allows to quantify the marginal risks.

**Proposition 5.** The marginal risks associated to $VaR$ and $ES$ are given respectively by:

$$\frac{\partial VaR_P(x_1,\cdots,x_n)}{\partial x_i}=\alpha_p\frac{x_i}{\sqrt{\sum_{i=1}^{n}x_i^2\zeta_i}}-\left(\mathrm{e}^{-a_it}r(0)+I(a_i,s,k_i)+\theta\right) \tag{18}$$

and

$$\frac{\partial ES_P\left(x_1,\cdots,x_n\right)}{\partial x_i}=\xi_p\frac{x_i}{\sqrt{\sum_{i=1}^{n}x_i^2\zeta_i}}-\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right);\tag{19}$$

where $I\left(a_i,s,k_i\right)=\int_0^t\mathrm{e}^{a_i(s-t)}k_i\left(s\right)\mathrm{d}s$.

The contributions of each asset to risk are given by

$$x_i\frac{\partial VaR_P\left(x_1,\cdots,x_n\right)}{\partial x_i}=\alpha_p\frac{x_i^2}{\sqrt{\sum_{i=1}^{n}x_i^2\zeta_i}}-x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right)\tag{20}$$

and

$$x_i\frac{\partial ES_P\left(x_1,\cdots,x_n\right)}{\partial x_i}=\xi_p\frac{x_i^2}{\sqrt{\sum_{i=1}^{n}x_i^2\zeta_i}}-x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right)\tag{21}$$

The discrete marginal contributions of each asset to risk are given by:

$$\begin{aligned}&VaR_P\left(x_1,\cdots,x_i,\cdots,x_n\right)-VaR_P\left(x_1,\cdots,0,\cdots,x_n\right)\\&=\alpha_p\frac{x_i^2\zeta_i}{\sqrt{\sum_{l=1}^{n}x_l^2\zeta_l}+\sqrt{\sum_{j=1,i\neq j}^{n}x_j^2\zeta_j}}-x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right)\end{aligned}\tag{22}$$

and

$$\begin{aligned}&ES_P\left(x_1,\cdots,x_i,\cdots,x_n\right)-ES_P\left(x_1,\cdots,0,\cdots,x_n\right)\\&=\xi_p\frac{x_i^2\zeta_i}{\sqrt{\sum_{l=1}^{n}x_l^2\zeta_l}+\sqrt{\sum_{j=1,i\neq j}^{n}x_j^2\zeta_j}}-x_i\left(\mathrm{e}^{-a_it}r\left(0\right)+I\left(a_i,s,k_i\right)+\theta\right).\end{aligned}\tag{23}$$

*Proof.* The relations (16) and (17) are obtained by deriving $VaR_P\left(x_1,\cdots,x_i,\cdots,x_n\right)$ and $ES_P\left(x_1,\cdots,x_i,\cdots,x_n\right)$ from relations 12 and 13. Finally (18) = $x_i$·(16) and (19) = $x_i$·(17). □

## 4. Model of VaR and Optimization of Portfolio P

In this section we try to optimize the portfolio P under certain constraints.

### 4.1. Optimization of Portfolio P According to the VaR Model

In this subsection, it is a question of determining the composition of the portfolio which minimizes the VaR, for a given profitability. The corresponding optimization problem is as follows:

$$\begin{cases}Min\ VaR_p\left(x\right)\\ sc\ E\left(X\right)=c\ \text{ and }\sum_{i=1}^{n}x_i=1\end{cases}\tag{24}$$

where $E\left(X\right)$ is the expected return of the portfolio. The solution to this problem is given by the following proposition.

**Proposition 6.** *The composition of the efficient portfolio is given by:*

$$x^{opt} = \left(x_1^{opt}, x_2^{opt}, \cdots, x_n^{opt}\right) \quad \textit{where}$$

$$x_i^{opt} = \frac{1}{\zeta_i} \frac{\left[c - \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)\right]\mu_i - c\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right) + \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}\right)}{\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j} - \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)^2} \quad (25)$$

and $\mu_i = \mathrm{e}^{-a_i t} r(0) + \int_0^t \mathrm{e}^{a_i(s-t)} k_i(s)\mathrm{d}s + \theta$ for $i = 1, \cdots, n$.

*Proof.* Assume VaR and ES be the portfolio P returns.

$\begin{cases} Min\ VaR_p(x) \\ E(X) = c \text{ and } \sum_{i=1}^{n} x_i = 1 \end{cases}$ equivalently $\begin{cases} Min\ \delta_p(x) \\ E(X) = c \text{ and } \sum_{i=1}^{n} x_i = 1 \end{cases}$ which gives also $\begin{cases} Min\ x'Mx \\ sc\ E(X) = c \text{ and } \sum_{i=1}^{n} x_i = 1 \end{cases}$ By replacing $E(X)$ by its expression, one has

$$\begin{cases} Min\ x'Mx \\ sc \sum_{i=1}^{n} x_i \left(\mathrm{e}^{-a_i(t-s)}(\alpha_s - b) + b + \theta\right) = c \text{ and } \sum_{i=1}^{n} x_i = 1 \end{cases} \quad (26)$$

Let us use the method of Lagrange multipliers. The Lagrangian is given by: $L(x, \lambda_1, \lambda_2) = x'Mx - \lambda_1(x'\mu - c) - \lambda_2 x' \cdot I$ where $x'u = \sum_{i=1}^{n} x_i \left(\mathrm{e}^{-a_i(t-s)}(\alpha_s - b) + b + \theta\right)$ the vector of $n$ components $I = (1, 1, \cdots, 1)'$ and $\lambda_1$ and $\lambda_2$ are Lagrange multipliers.

By differentiating the Lagrangian with respect to $x$ we have

$$\frac{\partial L(x, \theta, \alpha)}{\partial x} = 2Mx' - \lambda_1\mu - \lambda_2.$$

$\frac{\partial L(x, \theta, \alpha)}{\partial x} = 0$ equivalently it follows that $2Mx' - \lambda_1\mu - \lambda_2 = 0$ So, one obtains easily $Mx' = \frac{1}{2}(\lambda_1\mu + \lambda_2)$.

Finally

$$x' = M^{-1}\left(\frac{1}{2}(\lambda_1\mu + \lambda_2)\right). \quad (27)$$

From the system $\begin{cases} \sum_{i=1}^{n} x_i = 1 \\ E(X) = c \end{cases}$, one obtain the Lagrange multipliers such as

$$\lambda_1 = \frac{2\left[c - \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)\right]}{\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j} - \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)^2} \quad (28)$$

for the first and the second is given by

$$\lambda_2 = \left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left[1-\frac{\left[c-\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)\right]}{\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}-\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)^2}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)\right];$$

$$\lambda_2 = \frac{-c\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)+\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}\right)}{\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}-\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)^2}.$$

Furthermore the extrema are given by

$$x_i = \frac{1}{\zeta_i}\frac{\left[c-\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)\right]\mu_i - c\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)+\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}\right)}{\sum_{j=1}^{n}\frac{\mu_j^2}{\zeta_j}-\left(\sum_{j=1}^{n}\frac{1}{\zeta_j}\right)^{-1}\left(\sum_{j=1}^{n}\frac{\mu_j}{\zeta_j}\right)^2}$$

for $i = 1,\cdots,n$.

### 4.2. Application of an Optimization Problem to the Portfolio

In this section we solve the optimization problem in the particular case where P contains 5 assets and for fixed values of some parameters. One has

$$k_i(t) = a_i g_t(0) + g_t'(0) + \frac{\sigma_{ir}}{2a_i}\left(1-\mathrm{e}^{-2a_i t}\right); \tag{29}$$

where $g_t(0)$ is the instantaneous forward rate for date *t* implied in the yield curve at 0 and $g_t'(0)$ is the slope of the instantaneous forward rate curve.

$$\mu_i = \mathrm{e}^{-a_i t} r(0) + \int_0^t \mathrm{e}^{a_i(s-t)} k_i(s)\,\mathrm{d}s + \theta$$

which gives

$$\mu_i = \mathrm{e}^{-at} r(0) + \int_0^t \mathrm{e}^{a(s-t)}\left[a_i g_s(0) + g_s'(0) + \frac{\sigma_{ir}}{2a_i}\left(1-\mathrm{e}^{-2a_i s}\right)\right]\mathrm{d}s + \theta.$$

Let's choose $\theta = 0.2$; $g_t(0) = 0.5$ and $g_t'(0) = 2$.

Table 1 gives parameters associated with the 5 stocks and shares (last column) of each asset to have an efficient portfolio.

In other words, the solution of the particular optimization problem is:

$$x^{opt} = \left(x_1^{opt}, x_2^{opt}, x_3^{opt}, x_4^{opt}, x_5^{opt}\right)$$

$$x^{opt} = (0.9987551; 0.9859565; -3.742564; 1.759097; 0.9987551). \tag{30}$$

A positive value of $x_i$ represents a long position: one have bought asset *i*. A negative value represents a short position: asset i has been borrowed.

**Table 1.** Parameters $a_i$, $\sigma_{ir}$ and shares $x_i^{opt}$ of asssets in efficient portfolio.

| Stock i | $a_i$ | $\sigma_{ir}$ | $x_i^{opt}$ |
|---|---|---|---|
| Stock 1 | 0.1 | 0.091 | 0.9987551 |
| Stock 2 | 0.22 | 0.0710 | 0.9859565 |
| Stock 3 | 0.01 | 0.015 | −3.742564 |
| Stock 4 | 0.05 | 0.09 | 1.759097 |
| Stock 5 | 0.04 | 0.03 | 0.9987551 |

## 5. Conclusion

This study allowed us to contribute to the management of a large complex portfolio of assets (defined by (3)) with stochastic drifts and volatilities. It gives interesting results. We have proven that when the interest rate of each asset evolves according to the Hull and White model, the profitability of the portfolio is Gaussian. In addition, the value at risk, the expected shortfall, marginal expected shortfall and value at risk, incremental value at risk and expected shortfall, the marginal and discrete marginal contributions of a portfolio do not depend on the correlation forces between assets prices and volatilities. Similarly, the optimal portfolio allocation does not depend on correlation forces between assets prices and volatilities.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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