

The Estimation of the Spot Volatility for Diffusion Process

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Abstract

In this paper, we propose a Gasser-Müller type spot volatility estimator (abbreviated as GM type estimator) for diffusion process, which is weighted by integrals, it is different from the kernel spot volatility estimator discussed by Kristensen (2010). Under more general conditions, the asymptotic unbiasedness and the asymptotic normality of the GM type estimator are derived. The simulation results show that the GM type spot volatility estimator has good estimation effect, and its mean square error tends to be less than that of the kernel spot volatility estimator discussed by Kristensen (2010), so it provides a selection method for estimating the spot volatility in high frequency data environment.

Keywords

Diffusion Process, Spot Volatility, GM Type Estimation, Asymptotic Property

1. Introduction

Volatility is the main component of describing the price movement towards financial market, so we need to use the observed actual data to estimate volatility. However, the traditional low-frequency data cannot meet the needs of high-frequency traders in today's market, and the common models in the traditional low-frequency field are no longer applicable to the high-frequency field. In order to avoid errors caused by setting any model, more and more scholars consider using non-parametric methods to estimate volatility in high-frequency data.

Consider the diffusion process $\{X_t\} = \{X_t : t \geq 0\}$, which satisfies that

$$dX_t = \mu_t dt + \sigma_t dW_t, \quad (1.1)$$

where $\{W_t\}$ is a standard Brownian motion, while $\{\mu_t\}$ and $\{\sigma_t\}$ are adapted stochastic processes. The process $\{\sigma_t\}$ is usually denoted the spot volatility process, while $\{\mu_t\}$ is the drift process. In the diffusion process, the volatility is an important measure to reflect the volatility of financial products' logarithmic price.

Under the condition that asset prices obey the diffusion process, the quadratic variation of the price process converges in probability to the integrated volatility. So quadratic variation, also known as realized volatility, is a consistent estimator of integrated volatility, and has attracted more and more attention. For example, Barndorff—Nielsen and Shephard [1] (2002) prove the central limit theorem of realized volatility. Christensen and Podolskij [2] (2005) give another estimator of integral volatility, range-based realized volatility, and prove the consistency and central limit theorem of this estimator under diffusion model. A nonparametric estimator is proposed by Li Guo and Bao [3] (2013) for the class of integrated self-weighted cross volatilities. Under assuming that processes X and Y are sampled with microstructure noise and in an asynchronous way, the asymptotic properties, which include consistency and asymptotic normality, are obtained. There is still a lot of further research on the integrated volatility estimator, see also, Zhang, Mykland and At-Sahalia [4] (2005), Hansen, Large and Lunde [5] (2008), Li, *et al.* [6] (2013), Jing and Liu and Kong [7] (2015) and Li, Xie and Zheng [8] (2016).

Merton [9] (1976) first proposed the jump-diffusion model. Later, Duffie, Pan and Singleton [10] (2000), Pan [11] (2002), Eraker, Johannes and Polson [12] (2003) found that asset price process contains improper mutation behavior, so jump part should be added in diffusion model. Previous studies have shown that integrated volatility is generally estimated by realized multiple power variation and realized threshold volatility when asset prices obey jump-diffusion model. Barndorff-Nielsen and Shephard [13] [14] (2004, 2006) proposed realized power variation and realized bipower variation, and proved that realized bipower variation is a robust consistent estimator of integrated volatility with jump process. By using the properties of Brown's motion path, when the price obeys the jump-diffusion model with finite jump, another estimator of integrated volatility, the realized threshold volatility, is proposed, and the central limit theorem is obtain by Mancini [15] [16] (2004, 2009). Christensen and Podolskij [17] (2012) put forward the realized multiple power variation of based-range. They also show the robustness of the estimator to jump behavior in probability limit and asymptotic distribution. The further research can be seen in Todorov [18] [19] (2009, 2010), At-Sahalia and Jacod [20] [21] (2009, 2011), Tauchen and Zhou [22] (2011).

The integrated volatility characterizes the total volatility over a certain period of time for the past, but investors pay more attention to the spot volatility at a certain time, so we need to use high-frequency data to estimate the spot volatility. As the price process obeys the diffusion process, the rolling sample estimator for spot volatility was first proposed by Foster and Nelson [23] (1996). This rolling

sample method is extended by Andreou and Ghysels [24] (2002) to construct the rolling sample variance to estimate the quadratic variation of asset returns, and a rolling sample estimator with many similar characteristics to the instantaneous volatility estimator is proposed. They further discussed the asymptotically effective bandwidth and weighted form of the quadratic variation estimator, and established the relationship between the instantaneous volatility estimator and the integrated volatility estimator.

Moreover, Fan and Wang [25] (2008), Shigeyoshi and Simona [26] (2011), Zu and Boswijk [27] (2014), Yu, *et al.* [28] (2014), Mancini, Mattiussi and Renò [29] (2015), Curato, Mancino and Recchioni [30] (2018) and Liu, *et al.* [31] (2018) *et al.* have done a lot of research on instantaneous volatility estimation. Fan and Wang [25] (2008) employs a bivariate diffusion to model the price and volatility of an asset and investigates kernel type estimators of spot volatility based on high-frequency return data. And they establish both pointwise and global asymptotic distributions for the estimators. Zu and Peter [27] (2014) constructs a spot volatility estimate for high frequency financial data with market microstructure noise. At the same time, the consistency of the estimator is proved and the asymptotic distribution of the estimator is derived. Curato, Mancino and Recchioni [30] (2018) defines a new nonparametric estimation of spot volatility based on the relationship between Laplace transformation of price process and fluctuation process of Brownian semi-martingale model, and proves the consistency and asymptotic normality of the estimation. The simulation results confirm the theoretical results of the proposed estimator in the presence of noise effects on the microstructure.

According to the idea of Nadaraya-Watson kernel regression estimation, Kristensen [32] (2010) proposes the NW type kernel estimation of spot volatility σ_t

$$\hat{\sigma}_\tau^2 = \sum_{i=1}^n K_h(t_i - \tau) (\Delta X_{t_i})^2, \quad \tau \in [0, T] \quad (1.2)$$

where T is the time interval, $t_i = i\delta, i = 0, 1, \dots, n$, $\delta = T/n$, $\Delta X_{t_i} = X_{t_i} - X_{t_{i-1}}$, $K_h(s - \tau) = K\left(\frac{s - \tau}{h}\right)/h$, $K: R \mapsto R$ is a kernel, $h > 0$ is the bandwidth. And the asymptotic property of the kernel estimator is proved in Kristensen [32] (2010).

In non-parameter kernel regression estimation, there are two popular kernel methods, Nadaraya-Watson estimator $\hat{\mu}_{n,NW}(t)$ and Gasser-Müller estimator $\hat{\mu}_{n,GM}(t)$. Gasser and Müller [33] (1979) gave

$$E(\hat{\mu}_{n,NW}(t)) = h^{-1} \int_0^1 K\left(\frac{t-s}{h}\right) \mu(s) ds + O\left(n^{-\gamma_\mu} + (nh)^{-\gamma_K} + n^{-(\delta_0-1)}\right), \quad (1.3)$$

and

$$E(\hat{\mu}_{n,GM}(t)) = h^{-1} \int_0^1 K\left(\frac{t-s}{h}\right) \mu(s) ds + O\left(n^{-\gamma_\mu}\right), \quad (1.4)$$

where $\gamma_\mu > 0, \gamma_K > 0$ and $\delta_0 > 1$. The two equations imply that Gasser-Müller

estimator $\hat{\mu}_{n,GM}(t)$ has a better remainder term for the bias than Nadaraya-Watson estimator $\hat{\mu}_{n,NW}(t)$.

In view of this advantage, this paper will study the Gasser-Müller integrated type spot volatility estimator (or the GM type spot volatility estimator)

$$\hat{\sigma}_{\tau,GM}^2 = \frac{1}{\delta} \sum_{i=1}^n (\Delta X_{t_i})^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds, \quad \tau \in [0, T]. \tag{1.5}$$

We will demonstrate the asymptotic property of the GM type instantaneous volatility estimator under more general conditions, including asymptotic unbiasedness and asymptotic normality.

The remainder of the paper is organized as follows. In Section 2, asymptotic property of the GM type instantaneous volatility estimator are proved. Simulation study is put in Section 3. Finally, our conclusion is given in Section 4.

2. Asymptotic Property of Estimation

Throughout the paper, we will use the following basic assumptions.

- 1) Let $\delta = T/n, t_i = i\delta, i = 0, 1, \dots, n$.
- 2) (a) The processes $\{\mu_t\}$ and $\{\sigma_t\}$ are jointly independent of $\{W_t\}$, absolutely integral and locally bounded; (b) There exist constants $\lambda > 0$ and $C > 0$ such that

$$E|\sigma_t - \sigma_s| \leq C|t - s|^\lambda. \tag{2.1}$$

- (c) Let $f(t) = \sigma_t^2$. Assume that $f(t)$ is m times differentiable where $m \geq 0$, and there exist constants $\alpha > 0$ and $C > 0$ such that

$$E|f^{(m)}(t) - f^{(m)}(\tau)| \leq C|t - \tau|^\alpha. \tag{2.2}$$

- 3) (a) The kernel $K(u)$ is bounded in R^1 and satisfies that $\int_{-\infty}^{+\infty} K(u) du = 1$, $\int_{-\infty}^{+\infty} |K(u)| du < \infty$, $\int_{-\infty}^{+\infty} u^j K(u) du = 0$ for $j = 1, 2, \dots, m$, and $\int_{-\infty}^{+\infty} |u|^r |K(u)| du < \infty$ for some $r \geq m + \alpha$ and $r \geq \lambda$; (b) $K(u)$ satisfies the β th order Lipschitz condition where $\beta > 0$.

- 4) As $n \rightarrow \infty$, $h_n \rightarrow 0$ and $nh_n^{1+1/\beta} \rightarrow \infty$.

Notice that when $f(t)$ is not differentiable, $m = 0$ in (c) of (2). In this case, the conditions (2.1) and (2.2) are identical.

Bandi and Nguyen [34] (2003) and Bandi and Phillips [35] (2003) assumed that σ_t and μ_t are at least twice continuously differentiable and satisfy Lipschitz condition. Kristensen [32] (2010) relaxed the restriction of differentiability and allowed the order of differentiability $m \geq 0$. Our assumption (2) is to refer to the assumption A.2 - A.4 in Kristensen [32] (2010).

In this section, we first establish the asymptotic unbiasedness of the kernel estimator (1.5), and then study its asymptotic normality.

Theorem 2.1. *Assume that the conditions (1)-(4) hold. Then*

$$E(\hat{\sigma}_{\tau,GM}^2 | \{\sigma_t\}) - \sigma_\tau^2 = O_p\left(n^{-1} + \frac{1}{n^\beta h_n^{1+\beta}} + h_n^{m+\alpha}\right), \tag{2.3}$$

furthermore

$$E\left(\hat{\sigma}_{\tau,GM}^2 - \sigma_{\tau}^2\right) = O\left(n^{-1} + \frac{1}{n^{\beta} h_n^{1+\beta}} + h_n^{m+\alpha}\right). \tag{2.4}$$

Proof We introduce some denotations. Let $\Delta\tilde{X}_{t_i} = \int_{t_{i-1}}^{t_i} \sigma_s dW_s$ and

$$\tilde{\sigma}_{\tau,GM}^2 = \frac{1}{\delta} \sum_{i=1}^n \left(\Delta\tilde{X}_{t_i}\right)^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds. \tag{2.5}$$

Obviously,

$$\begin{aligned} \hat{\sigma}_{\tau,GM}^2 - \tilde{\sigma}_{\tau,GM}^2 &= \frac{1}{\delta} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \mu_s ds + \int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \\ &\quad - \frac{1}{\delta} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \\ &= \frac{1}{\delta} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \mu_s ds\right)^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \\ &\quad + \frac{2}{\delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mu_s ds \int_{t_{i-1}}^{t_i} \sigma_s dW_s \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds. \end{aligned} \tag{2.6}$$

Due to boundedness of μ_t , it follows that

$$\begin{aligned} \frac{1}{\delta} \left| \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \mu_s ds\right)^2 \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \right| &\leq C\delta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |K_h(s-\tau)| ds \\ &= C\delta \int_0^T |K_h(s-\tau)| ds \\ &\leq C\delta \int_{-\infty}^{\infty} |K(u)| du \\ &\leq Cn^{-1}. \end{aligned} \tag{2.7}$$

Note that

$$\int_{t_{i-1}}^{t_i} \sigma_s dW_s \stackrel{law}{=} \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 ds\right)^{1/2} U_i, \tag{2.8}$$

where U_1, U_2, \dots, U_n are independently and identically standard normal $N(0,1)$ and independent of the process σ . Let

$$A_n = \frac{2}{\delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \int_{t_{i-1}}^{t_i} \mu_s ds \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 ds\right)^{1/2} U_i, \tag{2.9}$$

Since $E(A_n | \{\sigma_t\}) = 0$, from (2.6)-(2.8), we find that

$$E\left(\hat{\sigma}_{\tau,GM}^2 | \{\sigma_t\}\right) - E\left(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_t\}\right) = O(n^{-1}). \tag{2.10}$$

Therefore, it is sufficient to show that

$$E\left(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_t\}\right) - \sigma_{\tau}^2 = O_p\left(\frac{1}{n^{\beta} h_n^{1+\beta}} + h_n^{m+\alpha}\right). \tag{2.11}$$

By (2.8), we have that

$$E\left[\left(\Delta\tilde{X}_{t_i}\right)^2 | \{\sigma_t\}\right] = E\left[\left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 | \{\sigma_t\}\right] = \int_{t_{i-1}}^{t_i} \sigma_s^2 ds, \tag{2.12}$$

and

$$\begin{aligned} E\left(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_t\}\right) - \sigma_{\tau}^2 &= \frac{1}{\delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds E\left[\left(\Delta\tilde{X}_{t_i}\right)^2 | \{\sigma_t\}\right] - \sigma_{\tau}^2 \\ &= \frac{1}{\delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \int_{t_{i-1}}^{t_i} \sigma_v^2 dv - \sigma_{\tau}^2. \end{aligned} \tag{2.13}$$

By the mean value theorem of integrals, there exist $\tilde{t}_i \in [t_{i-1}, t_i] (i = 1, 2, \dots, n)$ such that

$$E(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_i\}) - \sigma_\tau^2 = \sum_{i=1}^n K_h(\tilde{t}_i - \tau) \int_{t_{i-1}}^{t_i} \sigma_v^2 dv - \sigma_\tau^2. \tag{2.14}$$

We know that $K(u)$ satisfies the β th Lipschitz condition and σ_i^2 is bounded in $[0, T]$. It follows that

$$\begin{aligned} & \left| \sum_{i=1}^n K_h(\tilde{t}_i - \tau) \int_{t_{i-1}}^{t_i} \sigma_v^2 dv - \int_0^T K_h(v - \tau) \sigma_v^2 dv \right| \\ &= \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \{K_h(\tilde{t}_i - \tau) - K_h(v - \tau)\} \sigma_v^2 dv \right| \\ &\leq C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{1}{h_n} \left| \frac{\tilde{t}_i - v}{h_n} \right|^\beta \sigma_v^2 dv \leq \frac{C\delta^\beta}{h_n^{1+\beta}} \int_0^T \sigma_v^2 dv \leq \frac{C\delta^\beta}{h_n^{1+\beta}}, \end{aligned} \tag{2.15}$$

hence

$$\begin{aligned} E(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_i\}) - \sigma_\tau^2 &= \int_0^T K_h(v - \tau) \sigma_v^2 dv - \sigma_\tau^2 + O\left(\frac{\delta^\beta}{h_n^{1+\beta}}\right) \\ &= \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} K(u) \sigma_{\tau+uh_n}^2 du - \sigma_\tau^2 + O\left(\frac{\delta^\beta}{h_n^{1+\beta}}\right) \\ &= \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} K(u) \sigma_{\tau+uh_n}^2 du - \sigma_\tau^2 \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} K(u) du + O\left(\frac{\delta^\beta}{h_n^{1+\beta}}\right). \end{aligned} \tag{2.16}$$

Recalling $f(t) = \sigma_t^2$ in the assumption (2) and $f(t)$ is m times differentiable and its m th derivative $f^{(m)}(t)$ satisfies the α th order Lipschitz condition in $L_1([0, T])$ where $m \geq 0$ and $\alpha > 0$. By Taylor formula, we have that

$$\begin{aligned} & \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} K(u) \sigma_{\tau+uh_n}^2 du - \sigma_\tau^2 \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} K(u) du \\ &= \sum_{j=1}^{m-1} \frac{f^{(j)}(\tau) h_n^j}{j!} \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} u^j K(u) du + \frac{h_n^m}{m!} \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} f^{(m)}(\tau + \theta u h_n) u^m K(u) du, \end{aligned} \tag{2.17}$$

where $|\theta| \leq 1$. Note that $\int_{-\infty}^{+\infty} u^j K(u) du = 0, j = 1, 2, \dots, m$ and $\int_{-\infty}^{\infty} |u|^r |K(u)| du < \infty$ where $r \geq m + \alpha$, we have

$$\int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} u^j K(u) du = -\int_{-\frac{\tau}{h_n}}^{\tau} u^j K(u) du - \int_{\frac{T-\tau}{h_n}}^{\infty} u^j K(u) du = O(h_n^{r-j}), \tag{2.18}$$

while

$$\begin{aligned} & \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} f^{(m)}(\tau + \theta u h_n) u^m K(u) du \\ &= \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} \left[f^{(m)}(\tau + \theta u h_n) - f^{(m)}(\tau) \right] u^m K(u) du \\ &\quad - f^{(m)}(\tau) \int_{-\frac{h_n}{\tau}}^{\tau} u^m K(u) du - f^{(m)}(\tau) \int_{\frac{T-\tau}{h_n}}^{\infty} u^m K(u) du \\ &= \int_{-\frac{h_n}{\tau}}^{\frac{T-\tau}{h_n}} \left[f^{(m)}(\tau + \theta u h_n) - f^{(m)}(\tau) \right] u^m K(u) du + O(h_n^{r-m}), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
 & E \left| \int_{\frac{\tau}{h_n}}^{\frac{T-\tau}{h_n}} \left[f^{(m)}(\tau + \theta u h_n) - f^{(m)}(\tau) \right] u^m K(u) du \right| \\
 & \leq \int_{\frac{\tau}{h_n}}^{\frac{T-\tau}{h_n}} E \left| f^{(m)}(\tau + \theta u h_n) - f^{(m)}(\tau) \right| |u|^m |K(u)| du \tag{2.20} \\
 & \leq C \int_{\frac{\tau}{h_n}}^{\frac{T-\tau}{h_n}} |\theta u h_n|^\alpha |u|^m |K(u)| du = O(h_n^\alpha).
 \end{aligned}$$

Combining (2.17)-(2.20), we find that

$$\int_{\frac{\tau}{h_n}}^{\frac{T-\tau}{h_n}} K(u) \sigma_{\tau+uh_n}^2 du - \sigma_\tau^2 \int_{\frac{\tau}{h_n}}^{\frac{T-\tau}{h_n}} K(u) du = O_p(h_n^{m+\alpha}). \tag{2.21}$$

We obtain the desire result (2.11) from (2.16) and (2.21). Complete the proof.

Theorem 2.2. Assume that assumptions (1)-(3) hold, and $\beta > 1/2$ in (b) of (3). If the bandwidth h_n satisfies

$$nh_n^{(1/2+\beta)/(\beta-1/2)} \rightarrow \infty, \tag{2.22}$$

$$nh_n^{2(m+\alpha)+1} \rightarrow 0, \tag{2.23}$$

then

$$\frac{\sqrt{\delta^{-1} h_n} (\hat{\sigma}_{\tau,GM}^2 - \sigma_\tau^2)}{\sqrt{2\sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz}} \xrightarrow{d} N(0,1). \tag{2.24}$$

Here we give some explanations for the conditions.

Remark 2.1. Obviously, $h_n \rightarrow 0$ from (2.23). Note that $(1/2 + \beta)/(\beta - 1/2) > 1 + 1/\beta$ as $\beta > 1/2$, so that $nh_n^{1+1/\beta} \rightarrow \infty$ from (2.23). It implies that the assumption (4) naturally holds under (2.22) and (2.23). Therefore, the assumption (4) doesn't appear in the assumption of Theorem 2.2.

Remark 2.2. Normally, the used kernel function $K(u)$ is derivative bounded. Thus, $K(u)$ usually satisfies $\beta \geq 1$ order Lipschitz condition. In this case, $1 < (1/2 + \beta)/(\beta - 1/2) \leq 3$. Thus it can be seen that (2.22) is a reasonable condition.

Remark 2.3. The condition (2.23) is used in Kristensen [32] (2010, Theorem 3 and Theorem 4). It requires that σ_i^2 has smoothness. If $m = \alpha = 1$, then (2.23) is equivalent to $nh_n^5 \rightarrow 0$, while if $m = 2$ and $\alpha = 1$ (refer to Bandi and Nguyen [34] (2003) and Bandi and Phillips [35] (2003)), then (2.23) is equivalent to $nh_n^7 \rightarrow 0$.

To prove the theorem, we introduce the following lemmas.

Lemma 2.1. (Rosenthal [36], 1970) Let $\{\xi_i : i \geq 1\}$ be a sequence of independent random variables with $E(\xi_i) = 0$ and $E(|\xi_i|^p) < \infty$ for some $p \geq 2$. Then there exists constant C no depending on n such that

$$E \left| \sum_{i=1}^n \xi_i \right|^p \leq C \left\{ \sum_{i=1}^n E|\xi_i|^p + \left(\sum_{i=1}^n E\xi_i^2 \right)^{p/2} \right\}. \tag{2.25}$$

Lemma 2.2. Let the kernel $K(u)$ satisfies $\int_{-\infty}^{+\infty} |K(u)| du < \infty$, $\int_{-\infty}^{+\infty} |u|^r |K(u)| du < \infty$ where $r > 0$, and there exists constants $\beta > 0$ and $C > 0$ such that

$$|K(u) - K(v)| \leq C|v - u|^\beta \text{ for any } u, v \in \mathbb{R}. \tag{2.26}$$

If $h \rightarrow 0$ and $nh^{1+\beta} \rightarrow \infty$, then for any given $\tau \in (0, T)$,

$$\delta \sum_{i=1}^n |K_h(s_i - \tau)| = \int_{-\infty}^{\infty} |K(u)| du + O\left(h^r + (nh^{1+\beta})^{-1}\right), \tag{2.27}$$

where $t_{i-1} \leq s_i \leq t_i$.

Proof By the mean value theorem, there exist $\hat{t}_i \in [t_{i-1}, t_i]$ such that

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |K_h(s - \tau)| ds = \delta_n \sum_{i=1}^n |K_h(\hat{t}_i - \tau)|. \tag{2.28}$$

Hence,

$$\begin{aligned} \delta \sum_{i=1}^n |K_h(s_i - \tau)| &= \delta \sum_{i=1}^n |K_h(s_i - \tau)| - \delta \sum_{i=1}^n |K_h(\hat{t}_i - \tau)| + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |K_h(s - \tau)| ds \\ &\leq \delta \sum_{i=1}^n \left| |K_h(s_i - \tau)| - |K_h(\hat{t}_i - \tau)| \right| + \int_0^T |K_h(s - \tau)| ds \\ &\leq \delta \sum_{i=1}^n |K_h(s_i - \tau) - K_h(\hat{t}_i - \tau)| + \int_{-\infty}^{\infty} |K(u)| du + O(h^r) \\ &= \int_{-\infty}^{\infty} |K(u)| du + O\left(h^r + (nh^{1+\beta})^{-\beta}\right). \end{aligned} \tag{2.29}$$

Complete the proof.

Proof of Theorem 2.2. Let

$$a_i(\mu, \sigma) = \int_{t_{i-1}}^{t_i} K_h(s - \tau) ds \int_{t_{i-1}}^{t_i} \mu_s ds \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 ds \right)^{1/2}. \tag{2.30}$$

Recalling A_n defined in (2.9), we have that $A_n = \frac{2}{\delta} \sum_{i=1}^n a_i(\mu, \sigma) U_i$. Hence, by Lemma 2.1,

$$\begin{aligned} E(A_n^4 | \{\mu_i, \sigma_i\}) &\leq \frac{C}{\delta^4} \left\{ \sum_{i=1}^n a_i^4(\mu, \sigma) + \left(\sum_{i=1}^n a_i^2(\mu, \sigma) \right)^2 \right\} \\ &\leq \frac{C}{\delta^4} \left\{ \frac{\delta^9}{h_n^3} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |K_h(s - \tau)| ds + \left(\frac{\delta^4}{h_n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |K_h(s - \tau)| ds \right)^2 \right\} \\ &\leq \frac{C\delta^4}{h_n^2} \leq \frac{C}{n^4 h_n^2}. \end{aligned} \tag{2.31}$$

It implies that $E(A_n^4) \leq Cn^{-4}h_n^{-2}$. From this and (2.6)-(2.9), it follows that, for any given $\varepsilon > 0$

$$\begin{aligned} &P\left(\sqrt{\delta^{-1}h_n} \left| \hat{\sigma}_{\tau,GM}^2 - \tilde{\sigma}_{\tau,GM}^2 \right| > \varepsilon\right) \\ &\leq P\left(\sqrt{\delta^{-1}h_n} |A_n| > \varepsilon/2\right) \\ &\leq C\left(\delta^{-1}h_n\right)^2 E(A_n^4) \\ &\leq Cn^{-2} \rightarrow 0, \end{aligned} \tag{2.32}$$

that is

$$\sqrt{\delta^{-1}h_n}(\hat{\sigma}_{\tau,GM}^2 - \tilde{\sigma}_{\tau,GM}^2) = o_p(1). \tag{2.33}$$

Therefore, the remaining task is to show that

$$\frac{\sqrt{\delta^{-1}h_n}(\tilde{\sigma}_{\tau,GM}^2 - \sigma_\tau^2)}{\sqrt{2\sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz}} \xrightarrow{d} N(0,1). \tag{2.34}$$

Let

$$y_{n,i}(\tau) = \frac{1}{\delta} \int_{t_{i-1}}^{t_i} \sigma_v^2 dv \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds, \tag{2.35}$$

then

$$E(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_\tau\}) = \frac{1}{\delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_v^2 dv \int_{t_{i-1}}^{t_i} K_h(s-\tau) ds = \sum_{i=1}^n y_{n,i}(\tau), \tag{2.36}$$

$$Var(\tilde{\sigma}_{\tau,GM}^2 | \{\sigma_\tau\}) = \frac{2}{\delta^2} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_v^2 dv \right)^2 \left(\int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \right)^2 = 2 \sum_{i=1}^n y_{n,i}^2(\tau), \tag{2.37}$$

while

$$\begin{aligned} & \sqrt{\delta^{-1}h_n} \frac{\tilde{\sigma}_{\tau,GM}^2 - \sigma_\tau^2}{\sqrt{2\sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz}} \\ &= \frac{\tilde{\sigma}_{\tau,GM}^2 - \sum_{i=1}^n y_{n,i}(\tau)}{\sqrt{2\sum_{i=1}^n y_{n,i}^2(\tau)}} \sqrt{\frac{\delta^{-1}h_n \sum_{i=1}^n y_{n,i}^2(\tau)}{\sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz}} + \sqrt{\delta^{-1}h_n} \frac{\sum_{i=1}^n y_{n,i}(\tau) - \sigma_\tau^2}{\sqrt{2\sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz}} \\ &=: I_{1,n}(\tau) \cdot I_{2,n}(\tau) + I_{3,n}(\tau). \end{aligned} \tag{2.38}$$

First, consider $I_{2,n}$. Denote that

$$J_{1,n}(\tau) = \frac{h_n}{\delta^3} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_v^2 dv \right)^2 \left(\int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \right)^2 - h_n \int_0^T \sigma_s^4 K_h^2(s-\tau) ds, \tag{2.39}$$

$$J_{2,n}(\tau) = h_n \int_0^T \sigma_s^4 K_h^2(s-\tau) ds - \sigma_\tau^4 h_n \int_0^T K_h^2(s-\tau) ds, \tag{2.40}$$

$$J_{3,n}(\tau) = \sigma_\tau^4 h_n \int_0^T K_h^2(s-\tau) ds + \sigma_\tau^4 h_n \int_{-\infty}^0 K_h^2(s-\tau) ds. \tag{2.41}$$

Let $z = (s-\tau)/h_n$, we have

$$\begin{aligned} & \delta^{-1}h_n \sum_{i=1}^n y_{n,i}^2(\tau) - \sigma_\tau^4 \int_{-\infty}^{\infty} K^2(z) dz \\ &= \frac{h_n}{\delta^3} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_v^2 dv \right)^2 \left(\int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \right)^2 - \sigma_\tau^4 h_n \int_{-\infty}^{\infty} K_h^2(s-\tau) ds \\ &= J_{1,n}(\tau) + J_{2,n}(\tau) + J_{3,n}(\tau). \end{aligned} \tag{2.42}$$

By the mean value theorem, there exists $v_i, s_i, \tilde{s}_i \in [t_{i-1}, t_i]$ ($i = 1, 2, \dots, n$), such that

$$\begin{aligned} J_{1,n}(\tau) &= \frac{h_n}{\delta^3} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_v^2 dv \right)^2 \left(\int_{t_{i-1}}^{t_i} K_h(s-\tau) ds \right)^2 - h_n \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^4 K_h^2(s-\tau) ds \\ &= \delta h_n \sum_{i=1}^n \sigma_{v_i}^4 K_h^2(s_i-\tau) - \delta h_n \sum_{i=1}^n \sigma_{\tilde{s}_i}^4 K_h^2(\tilde{s}_i-\tau) \\ &= \delta h_n \sum_{i=1}^n (\sigma_{v_i}^4 - \sigma_{s_i}^4) K_h^2(s_i-\tau) + \delta h_n \sum_{i=1}^n \sigma_{s_i}^4 \{K_h^2(s_i-\tau) - K_h^2(\tilde{s}_i-\tau)\} \\ &=: Q_{1,n}(\tau) + Q_{2,n}(\tau). \end{aligned} \tag{2.43}$$

Using Lemma 2.2, we get that

$$\begin{aligned}
 E|Q_{1,n}(\tau)| &\leq C\delta h_n \sum_{i=1}^n E|\sigma_{v_i}^2 - \sigma_{\tilde{s}_i}^2| K_h^2(s_i - \tau) \\
 &\leq C\delta^2 \sum_{i=1}^n \delta |K_h(s_i - \tau)| \leq C\delta^\lambda,
 \end{aligned}
 \tag{2.44}$$

and

$$\begin{aligned}
 |Q_{2,n}(\tau)| &\leq C\delta h_n \sum_{i=1}^n |K_h(s_i - \tau) - K_h(\tilde{s}_i - \tau)| \cdot |K_h(s_i - \tau) + K_h(\tilde{s}_i - \tau)| \\
 &\leq \frac{C\delta^{1+\beta}}{h_n^\beta} \sum_{i=1}^n |K_h(s_i - \tau) + K_h(\tilde{s}_i - \tau)| \leq C\left(\frac{\delta}{h_n}\right)^\beta,
 \end{aligned}
 \tag{2.45}$$

thus

$$J_{1,n}(\tau) = O_p\left(n^{-\lambda} + (nh_n)^{-\beta}\right).
 \tag{2.46}$$

Using the integral transform $u = (s - \tau)/h_n$ for $J_{2,n}(\tau)$, and by (2) and (3), we know that

$$\begin{aligned}
 E|J_{2,n}(\tau)| &= E\left|\int_{-\tau/h_n}^{(T-\tau)/h_n} (\sigma_{uh_n+\tau}^4 - \sigma_\tau^4) K^2(u) du\right| \\
 &\leq C\int_{-\tau/h_n}^{(T-\tau)/h_n} E|\sigma_{uh_n+\tau}^2 - \sigma_\tau^2| K^2(u) du \leq Ch_n^\lambda \int_{-\infty}^\infty |u|^\lambda |K(u)| du \leq Ch_n^\lambda.
 \end{aligned}
 \tag{2.47}$$

Note that $\int_{-\infty}^{+\infty} |u|^\alpha |K(u)| du < \infty$, we find that

$$\begin{aligned}
 |J_{3,n}(\tau)| &\leq C\sigma_\tau^4 \int_{(T-\tau)/h_n}^\infty |K(u)| du + C\sigma_\tau^4 \int_{-\infty}^{-\tau/h_n} |K(u)| du \\
 &\leq \frac{C\sigma_\tau^4 h_n^\alpha}{(T-\tau)^\alpha} \int_{(T-\tau)/h_n}^\infty |u|^\alpha |K(u)| du + \frac{C\sigma_\tau^4 h_n^\alpha}{-\tau^\alpha} \int_{-\infty}^{-\tau/h_n} |u|^\alpha |K(u)| du \leq Ch_n^\alpha.
 \end{aligned}
 \tag{2.48}$$

Combining (2.42) and (2.46)-(2.48), we have

$$\delta^{-1} h_n \sum_{i=1}^n y_{n,i}^2(\tau) - \sigma_\tau^4 \int_{-\infty}^\infty K^2(z) dz = o_p(1).
 \tag{2.49}$$

It follows that

$$I_{2,n}(\tau) \xrightarrow{p} 1.
 \tag{2.50}$$

Next, consider $I_{1,n}$. We defined $c_{n,i}(\tau) = y_{n,i}(\tau) / \sqrt{\sum_{i=1}^n y_{n,i}^2(\tau)}$, thus

$$I_{1,n}(\tau) \stackrel{law}{=} \sum_{i=1}^n c_{n,i}(\tau) \bar{U}_i^2,
 \tag{2.51}$$

where $\bar{U}_i^2 = (U_i^2 - 1) / \sqrt{2}$ are independent and identically distributed random variables with mean zero and variance one. According to (2.49), we obtain

$$\begin{aligned}
 c_{n,i}(\tau) &= \frac{y_{n,i}(\tau)}{\sqrt{\frac{T}{nh_n} \sigma_\tau^4 \int_{-\infty}^\infty K^2(z) dz + o_p\left(\frac{1}{nh_n}\right)}} = \frac{\frac{1}{\delta h_n} \int_{t_{i-1}}^{t_i} \sigma_v^2 dv \int_{t_{i-1}}^{t_i} K\left(\frac{s-\tau}{h_n}\right) ds}{\sqrt{\frac{T}{nh_n} \sigma_\tau^4 \int_{-\infty}^\infty K^2(z) dz + o_p\left(\frac{1}{nh_n}\right)}} \\
 &\leq \frac{\frac{C}{nh_n}}{\sqrt{\frac{T}{nh_n} \sigma_\tau^4 \int_{-\infty}^\infty K^2(z) dz + o_p\left(\frac{1}{nh_n}\right)}} \leq C\left(\frac{1}{nh_n}\right)^{1/2},
 \end{aligned}
 \tag{2.52}$$

and $\sum_{i=1}^n c_{n,i}^2(\tau) = 1$, by Barndorff-Nielsen and Shephard [14] (2006, Cor. 3.1), we obtain

$$I_{1,n}(\tau) | \{\sigma_t\} \xrightarrow{d} N(0,1). \tag{2.53}$$

Finally, consider $I_{3,n}$. By (2.3), we know

$$\sum_{i=1}^n y_{n,i}(\tau) - \sigma_\tau^2 = O_p \left(n^{-1} + \frac{1}{n^\beta h_n^{1+\beta}} + h_n^{m+\alpha} \right), \tag{2.54}$$

then

$$\begin{aligned} I_{3,n}(\tau) &= O_p \left(n^{-1} + \frac{1}{n^\beta h_n^{1+\beta}} + h_n^{m+\alpha} \right) (nh_n)^{1/2} \\ &= O_p \left(n^{-1/2} h_n^{1/2} + \frac{1}{n^{\beta-1/2} h_n^{1/2+\beta}} + n^{1/2} h_n^{m+\alpha+1/2} \right) \\ &= O_p \left(n^{-1/2} h_n^{1/2} + \frac{1}{(nh_n^{(1/2+\beta)/(\beta-1/2)})^{\beta-1/2}} + (nh_n^{2m+2\alpha+1})^{1/2} \right). \end{aligned} \tag{2.55}$$

By (2.22) and (2.23), we obtain

$$I_{3,n}(\tau) = o_p(1). \tag{2.56}$$

Therefore, it follows (2.34) from (2.50), (2.53) and (2.56). Complete the proof.

3. Simulation

Here, we will show the performance of the Gasser-Müller integrated type spot volatility estimator $\hat{\sigma}_{\tau,GM}^2$ (see 1.5) proposed in this paper and compare it with the kernel estimator $\hat{\sigma}_\tau^2$ proposed by Kristensen [32] (2010). We consider the following stochastic volatility model as referred in Kanaya and Kristensen [37] (2016),

$$\begin{aligned} dX_t &= \sigma_t dW_{1,t}, \\ d\sigma_t^2 &= \beta(\alpha - \sigma_t^2) dt + \kappa \sigma_t^2 dW_{2,t}, \end{aligned}$$

where $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions, parameters are respectively $\beta = 0.510$, $\alpha = 0.476$ and $\kappa^2 = 0.0518$ (refer to Kanaya and Kristensen [37] (2016), page 888). In order to simulate the above model, we use Euler discretization scheme (see Kloeden and Platten [7], 1999) to obtain the following model.

$$\Delta X_{i\delta} = \sigma_{(i-1)\delta} \sqrt{\delta} \varepsilon_{1,i}, \tag{3.1}$$

$$\Delta \sigma_{i\delta}^2 = \beta(\alpha - \sigma_{(i-1)\delta}^2) \delta + \kappa \sigma_{(i-1)\delta}^2 \sqrt{\delta} \varepsilon_{2,i}. \tag{3.2}$$

where $\{\varepsilon_{1,i}\}$ and $\{\varepsilon_{2,i}\}$ are i.i.d $N(0,1)$, $\delta > 0$ is the length of the discretization step.

In the actual operation of the simulation, we assume that the drift term $\mu_t = 0$, and the observed time is $T = 2$ (48 hours). The specific steps are as follows:

- 1) Get the true value $\{\sigma_t\}$. According to model (3.2), the truth sequence

$\{\sigma_t\}$ of σ_t is obtained, because the discretized model is approximated to the original continuity model and the simulation interval is as small as possible, so the time interval $\delta = \Delta/100$ is selected in model (3.2), where $\Delta^{-1} = 3 \times 60 \times 24 \times 2$.

2) Produce samples. $\Delta X_{i\delta}$ is sampled at 20 seconds, 1 minute, 5 minutes, 10 minutes, 15 minutes and 30 minutes sampling frequency. Model (3.1) is used to generate simulated sample $\{\Delta X_{i\delta}, i = 1, 2, \dots, n\}$. The numbers of samples corresponding to various sampling frequencies are

$$\delta^{-1} = 3 \times 60 \times 24 \times 2, 60 \times 24 \times 2, 12 \times 24 \times 2, 6 \times 24 \times 2, 4 \times 24 \times 2, 2 \times 24 \times 2,$$

respectively. At the same time, the sample is repeated 400 times and 400 sets of sample data are obtained for each sampling case. Define

$\tilde{\sigma}_{t_i,j}^2, i = 1, 2, \dots, n; j = 1, 2, \dots, 400$ as the estimated value at time t_i using the group j sample data.

3) Choose kernel function and bandwidth. In the process of calculating estimators $\hat{\sigma}_{\tau,GM}^2$ and $\hat{\sigma}_{\tau}$, we choose the Gaussian kernel. Since the choice of the kernel function has a great influence on estimates on the boundary (see Silverman [38] (1986)), in order to avoid the boundary effects, we only considered estimated values of the $[1/2, 3/2]$. Because the selection of bandwidth has a great influence on the internal estimation performance, we apply the cross-validated mentioned in Kristensen [32] (2010) to obtain the optimal bandwidth, namely, the optimum bandwidth is the minimum bandwidth for

$$\widehat{ISE}(h_n) = \sum_{i=1}^n I\{T_l \leq t_{i-1} \leq T_u\} \left[\frac{(\Delta X_{t_i})^2}{\delta} - \hat{\sigma}_{-i,t_i}^2 \right], \tag{3.3}$$

where $\hat{\sigma}_{-i,t_i}^2$ is based on the samples removed from the i th sample to calculate the spot volatility estimator of the time $t_i = i\delta$, T_l and T_u are severally the upper and lower limits of the time span we consider, $T_l = 1/2, T_u = 3/2$. Using the simulated sample $\{\Delta X_{i\delta}\}$ and (3.3), we acquire the optimal bandwidth of the kernel spot volatility estimator and the GM type spot volatility estimator under the different time intervals.

We want to measure the accuracy of the estimator (*i.e.* estimation effect) by the bias, the variance and the integrated mean square error of this estimator. The calculation formulas for the bias, the variance and the integrated mean square error of the estimators are respectively

$$\text{Bias}^2 = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{400} \sum_{j=1}^{400} \tilde{\sigma}_{t_i,j}^2 - \sigma_{t_i}^2 \right)^2,$$

$$\text{Var} = \frac{1}{n} \sum_{i=1}^n \frac{1}{400} \sum_{j=1}^{400} \left(\tilde{\sigma}_{t_i,j}^2 - \frac{1}{400} \sum_{j=1}^{400} \tilde{\sigma}_{t_i,j}^2 \right)^2,$$

and

$$\text{IMSE} = \frac{1}{n} \sum_{i=1}^n \frac{1}{400} \sum_{j=1}^{400} (\tilde{\sigma}_{t_i,j}^2 - \sigma_{t_i}^2)^2.$$

Obviously, $\text{IMSE} = \text{Var} + \text{Bias}^2$.

Table 1. Performance in interior.

Sampling frequency	Type	Bias ²	Var	IMSE
20 seconds	$\hat{\sigma}_{\tau,GM}^2$	0.46374	7.65629	8.12003
	$\hat{\sigma}_{\tau}$	0.46388	7.65646	8.12033
1 minute	$\hat{\sigma}_{\tau,GM}^2$	1.59960	15.46301	17.06261
	$\hat{\sigma}_{\tau}$	1.60059	15.46395	17.06454
5 minutes	$\hat{\sigma}_{\tau,GM}^2$	2.75254	28.00331	30.75585
	$\hat{\sigma}_{\tau}$	2.76279	28.00738	30.77016
10 minutes	$\hat{\sigma}_{\tau,GM}^2$	4.16189	38.34402	42.50591
	$\hat{\sigma}_{\tau}$	4.16406	38.38035	42.54441
15 minutes	$\hat{\sigma}_{\tau,GM}^2$	5.04384	47.84588	52.88972
	$\hat{\sigma}_{\tau}$	5.02933	47.90483	52.93416
30 minutes	$\hat{\sigma}_{\tau,GM}^2$	7.52807	89.87464	97.40271
	$\hat{\sigma}_{\tau}$	7.33064	90.20958	97.54365

[Note: Integrated sq.bias ($\times 10^{-4}$), variance ($\times 10^{-4}$), and IMSE ($\times 10^{-4}$) of estimators].

The bias, the variance and the integrated mean square error of the GM instantaneous volatility estimator and the kernel instantaneous volatility estimator under the different time intervals are reported in **Table 1**.

According to the simulation results shown in **Table 1**, we find that the higher the sampling frequency, the better the accuracy of both the GM type spot volatility estimator $\hat{\sigma}_{\tau,GM}^2$ and the kernel spot volatility estimator $\hat{\sigma}_{\tau}$, indicating that both estimators are suitable for high frequency data. At the same time, we see that the IMSE of $\hat{\sigma}_{\tau,GM}^2$ is slightly smaller than that of $\hat{\sigma}_{\tau}$. This shows that the GM type spot volatility estimator provides a selection method for estimating the spot volatility, especially in high frequency data environment, its estimation performance is better.

4. Conclusion

We propose a Gasser-Müller type spot volatility estimator for diffusion process, and prove its asymptotic unbiasedness and asymptotic normality under mild conditions. By simulation analysis, we find that the GM type spot volatility estimator is a better estimator. Therefore, it provides a selection method for estimating the spot volatility in high frequency data environment.

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Authors' Contributions

All authors carried out the mathematical studies, and read and approved the final manuscript.

Conflicts of Interest

The authors declare that they have no competing interests.

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