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Kumaraswamy-Odd Rayleigh-G Family of Distributions with Applications

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Abstract

We propose a new generator of continuous distributions with at least four positive parameters called the Kumaraswamy-Odd Rayleigh-G family. Some special cases were presented. The plots of the Kumaraswamy Odd Rayleigh Log-Logistic (*KORLL*) distribution indicate that the distribution can take many shapes depending on the parameter values. The negative skewness and kurtosis indicates that the distribution has lighter tails than the normal distribution. The Monte Carlo simulation results indicate that the estimated biases decrease when the sample size increases. Furthermore, the root mean squared error estimates decay towards zero as the sample size increases. This part shows the consistency of the maximum likelihood estimators. From the considered analytical measures, the new *KORLL* provides the best fit to the analysed five real data sets indicating that this new model outclasses its competitors.

Keywords

Odd Rayleigh-G Family, Kumuraswamy Odd Rayleigh-G Family, Kumuraswamy Odd Rayleigh-Loglogistic Distribution, Monte Carlo Simulation, Skewness

1. Introduction

Researchers use different approaches to induct additional parameters to a continuous class of distributions, ostensibly because in many applications, these classical probability distributions do not fit real life data. In other words, all of these approaches extend the classical baseline probability distributions by introducing additional parameter(s) to the baselines, thereby making the extended baselines much more flexible to fit wide range of data from practical situations. With this approach, several generalized families of distributions have been proposed and applied to real life data in areas such as engineering, life sciences, environmental sciences, finance and medical sciences.

Recently, there are a lot of attempts in the statistics literature to generalize distributions. This generalization is mainly on a methodology proposed by many researchers, as in [1]. The most frequently used is the T-X approach by [2]. Some of the generalized families of distributions based on this approach in the literature include Weibull G family by [3], Lomax Generator of distributions by [4], Odd Generalized Exponential family by [5], Odd Lindley-G family by [6], Gompertz-G family by [7], Zubair-G family by [8], Odd Frechet G family by [9], Power Lindley G family by [10], Topp Leone Exponentiated-G Family by [11], Odd Chen-G family by [12], Burr X Exponential G family by [13], Inverse Lomax-G family by [14].

The objective of this paper is to propose a new family of distribution called the Kumaraswamy Odd Rayleigh-G family of distributions which has the capacity of providing more robust compound probability distribution when used in modelling real life data set. This new family adds three additional parameters to the baseline distribution.

The rest of this article is structured as follows: In Section 2, we defined the Kumaraswamy Odd-Rayleigh-G Family. In Section 3, we derive some models based on the *KORG* family. In Section 4 we present the estimation method used in estimating the parameters of non-linear models. We conduct a Monte Carlo simulation study using a Kumaraswamy Odd-Rayleigh Log-Logistic (*KORLL*D) model in Section 5. In Section 6, we apply the new model of *KORG* family to five real life datasets and compare their performance with some existing distributions. Lastly, Section7 concludes the paper.

2. Kumaraswamy Odd Rayleigh G (KOR-G) Family

Attempts have been made to define new families of probability distributions which enhance the flexibility in practical data modeling of well known baseline distributions. In the spirit of the T-X approach by [3], this paper defines the cumulative distribution as

$$F(x)=\int_a^{f(H(x))} z(t)\,\mathrm{d}t \tag{1}$$

where $f(H(x))$ is the function of the baseline cdf $H(x)$ of any continous random variable X. The function $f(H(x))$ must satisfy the following conditions

(a) $f(H(x)) \in (a,b)$;

(b) $f(H(x))$ is non-decreasing and monotonically differentiable;

(c) $f(H(x))$ tends to a as x tends to $-\infty$;

(d) $f(H(x))$ tends to b as x tends to ∞.

Let T be a random variable which is continuous with probability density function (pdf) $z(t)$ defined on the close interval [a,b].

In 2011, [15] introduced the Kumaraswamy-G family of distribution. The probability density function (pdf) and the cummulative distribution function (cdf) are given by:

$$f_{KG}(x)=2\lambda g(x;\zeta)G(x;\zeta)^{\alpha-1}\left[1-G(x;\zeta)^{\alpha}\right]^{\lambda-1},\quad x>0,\alpha,\lambda,\zeta>0 \tag{2}$$

$$F_{KG}(x)=1-\left[1-G(x;\zeta)^{\alpha}\right]^{\lambda},\quad x>0,\alpha,\lambda,\zeta>0 \tag{3}$$

where $G(x;\zeta)$ and $g(x;\zeta)$ are the cdf and pdf of the baseline distribution with parameter vector ζ.

The Odd Rayleigh-G family has pdf and cdf given by

$$f_{OR}(x)=\frac{g(x;\beta)G(x;\beta)}{\delta^2\left[\bar{G}(x;\beta)\right]^3}\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\},\quad x>0,\delta,\beta>0 \tag{4}$$

$$F_{OR}(x)=1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\},\quad x>0,\delta,\beta>0 \tag{5}$$

where $\bar{G}(x,\beta)=1-G(x,\beta)$.

Lemma I

The cdf of the proposed *KORG* family of distributions is given by

$$F_{KORG}(x)=1-\left\{1-\left[1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\}\right]^{\alpha}\right\}^{\lambda},\quad x>0,\alpha,\lambda>0 \tag{6}$$

where $x>0$, $\alpha,\lambda,\delta>0$ the vector β is the parameter of the baseline distribution $G(x,\beta)$ and $\bar{G}(x,\beta)=1-G(x,\beta)$.

Proof

From Equation (1),

$$F_{KORG}(x;\alpha,\lambda,\delta,\beta)=\int_0^{F_{OR}(x,\delta,\beta)}\alpha\lambda f_{OR}(t;\delta,\beta)\left[F_{OR}(t;\delta,\beta)\right]^{\alpha-1}\left[1-\left\{F_{OR}(t,\delta,\beta)\right\}^{\alpha}\right]^{\lambda-1}\mathrm{d}t \tag{7}$$

let

$$y=1-\left\{F_{OR}(t,\delta,\beta)\right\}^{\alpha}$$

$$\mathrm{d}y=-\alpha\left\{F_{OR}(t,\delta,\beta)\right\}^{\alpha-1}f_{OR}(t,\delta,\beta)\mathrm{d}t$$

if $t\to 0$, the $y\to 1$ and if $t\to x$, $y\to 1-\left\{F_{OR}(x,\delta,\beta)\right\}^{\alpha}$. So,

$$F_{KORG}(x;\alpha,\lambda,\delta,\beta)=-\int_1^{1-\left\{F_{OR}(x,\delta,\beta)\right\}^{\alpha}}\lambda y^{\lambda-1}\mathrm{d}y$$

$$F_{KORG}(x;\alpha,\lambda,\delta,\beta)=-y^{\lambda}I_1^{1-\left\{F_{OR}(x;\delta,\beta)\right\}^{\alpha}} \tag{8}$$

and

$$F_{KORG}(x;\alpha,\lambda,\delta,\beta)=1-\left[1-\left\{F_{OR}(x;\delta,\beta)\right\}^{\alpha}\right]^{\lambda} \tag{9}$$

and this can be written as

$$F_{KORG}(x;\alpha,\lambda,\delta,\beta)=1-\left[+1-\left\{1-\exp\left(-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right)\right\}^{\alpha}\right]^{\lambda} \tag{10}$$

whence the proof. if $x\to 0$, $F_{KORG}(x;\alpha,\lambda,\delta,\beta)\to 0$ and if

$$x \to \infty, \quad F_{KORG}(x;\alpha,\lambda,\delta,\beta) = 1 - \left[1 - 1^{\alpha}\right] = 1$$

From Equation (7), the pdf of the *KORG* family can be written as

$$f_{KORG}(x;\alpha,\lambda,\delta,\beta) = \alpha\lambda f_{OR}(x;\delta,\beta)\left[F_{OR}(x;\delta,\beta)\right]^{\alpha-1}\left[1-\left\{F_{OR}(x;\delta,\beta)\right\}^{\alpha}\right]^{\lambda-1} \quad (11)$$

And substituting Equations (4) and (5) in to 11 yields

$$\begin{aligned} f_{KORG}(x) &= \alpha\lambda \frac{g(x;\beta)G(x;\beta)}{\delta^2\left[\bar{G}(x;\beta)\right]^3}\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\} \\ &\times\left[1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\}\right]^{\alpha-1} \\ &\times\left[1-\left\{1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\}\right\}^{\alpha}\right]^{\lambda-1}, \quad x>0,\alpha,\lambda>0 \end{aligned} \quad (12)$$

Similarly, differentiating Equation (6) with respect to *x* will also yield Equation (12). Figure 1(a) illustrates the density function $f_{KORG}(x;\alpha,\lambda,\delta,\beta)$ with different parameter values. It is obvious from this graph that $f_{KORG}(x;\alpha,\lambda,\delta,\beta) > 0$ $\forall$ values of *x*. And to evaluate this integral

$$\int_0^{\infty} f_{KORG}(x;\alpha,\lambda,\delta,\beta)\,\mathrm{d}x$$

Let $y = 1-\left\{1-\exp\left(-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right)\right\}^{\alpha}$ Then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\alpha g(x;\beta)G(x;\beta)}{\delta^2\left[\bar{G}(x;\beta)\right]^3}\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\} \\ &\times\left[1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)^2\right\}\right]^{\alpha-1} \end{aligned}$$

and if $x = 0$, then $y = \left[1-\left(1-\exp(0)\right)^{\alpha}\right] = 1$

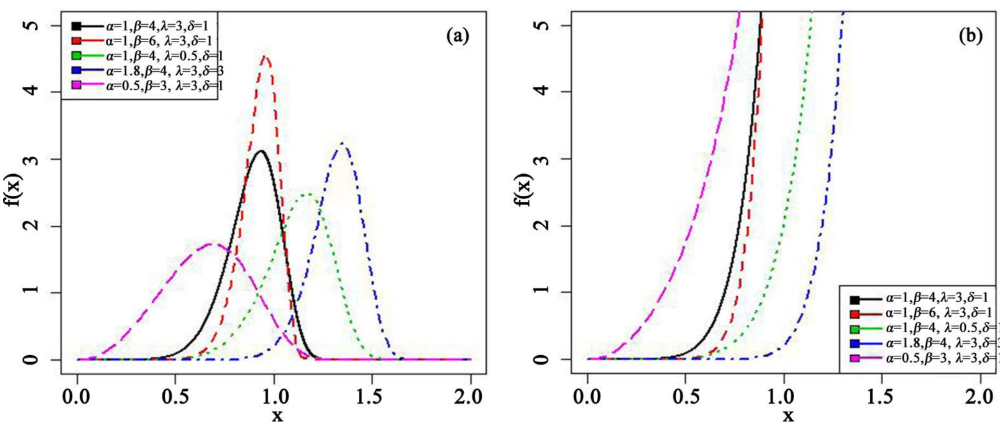

Figure 1. Density and hazard rate plots of *KORLL* distribution with varying parameter values.

if $x=\infty$, then $y=\left[1-\left(1-\exp(-\infty)\right)^{\alpha}\right]=0$
therefore

$$\int_0^{\infty} f_{KORG}\left(x;\alpha,\lambda,\delta,\beta\right)\mathrm{d}x=-\int_1^0 \lambda y^{\lambda-1}\mathrm{d}y=-y^{\lambda}I_1^0=1$$

$$\therefore \int_0^{\infty} f_{KORG}\left(x;\alpha,\lambda,\delta,\beta\right)\mathrm{d}x=1$$

which showed that $f_{KORG}\left(x;\alpha,\lambda,\delta,\beta\right)$ is a pdf for the continous random variable *X*. The Hazard function $\left(hf_{KORG}\right)$ and survival function $\left(sf_{KORG}\right)$ of the *KORG* family can be given as

$$hf_{KORG}(x)=\frac{\alpha\lambda g(x;\beta)G(x;\beta)}{\delta^2\left[\bar{G}(x;\beta)\right]^3\left[\left[1-\left\{1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right\}\right\}^{\alpha}\right]^{\lambda}\right]}$$
$$\times\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right\}\left[1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right\}\right]^{\alpha-1} \quad (13)$$
$$\times\left[1-\left\{1-\exp\left\{-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right\}\right\}^{\alpha}\right]^{\lambda-1},\quad x>0,\alpha,\lambda>0$$

and

$$sf_{KORG}=\left[1-\left\{1-\exp\left(-\frac{1}{2\delta^2}\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)^2\right)\right\}^{\alpha}\right]^{\lambda} \quad (14)$$

Quantile Function of *KORG*

The quantile function of *KROG* model can be given as

$$Q(u)=G^{-1}\left\{\frac{1}{1+\left[-2\delta^2\log\left(1-\left[1-(1-u)^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right)\right]^{-\frac{1}{2}}}\right\} \quad (15)$$

where G^{-1} is the quantile function of the baseline distribution.

3. Sub-Models of *KORG* Family

In this section, we considered two submodels of *KORG* family: Kumaraswamy Odd Rayleigh Log-Logistic (*KORLL*) and Kumaraswamy Odd Rayleigh Inverse Rayleigh (KORIR) distributions.

3.1. The *KORLL* Model

The cdf and pdf of log-logistic (*LL*) distribution are given as

$$G(x;\tau)=\frac{x^{\tau}}{1+x^{\tau}}$$

and

$$g(x;\tau)=\frac{\tau x^{\tau}}{\left(1+x^{\tau}\right)^{2}},\quad \tau>0,x>0$$

The quantile function of the *LL* distribution is given by

$$Q_{LL}(u)=\left(u^{-1}-1\right)^{-\frac{1}{\tau}}$$

where u is uniformly distributed in the interval (0, 1). Then, the *KORLL* distributon has the cdf given by:

$$F_{KORLL}(x;\alpha,\lambda,\tau,\delta)=1-\left\{1-\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha}\right\}^{\lambda} \tag{16}$$

The corresponding pdf of Equation (16) is given below:

$$f_{KORLL}(x;\alpha,\lambda,\tau,\delta)$$
$$=\frac{\alpha\lambda\tau x^{2\tau-1}}{\delta^{2}}\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha-1}\left\{1-\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha}\right\}^{\lambda-1} \tag{17}$$

The hazard function (*hf*), and survival function (*sf*) are presented below:

$$\begin{aligned}hf_{KORLL}(x)=&\frac{\alpha\lambda\tau x^{2\tau-1}}{\delta^{2}}\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha-1}\\&\times\left\{1-\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha}\right\}^{-1}\end{aligned} \tag{18}$$

$$sf_{KORLL}(x)=\left\{1-\left[1-\exp\left\{\frac{-x^{2\tau}}{2\delta^{2}}\right\}\right]^{\alpha}\right\}^{\lambda} \tag{19}$$

3.1.1. Quantile Function of *KORLL*

Lemma II

Let the random variable u be uniformly distributed on $(0,1)$. Define the random variable y as

$$y=\frac{1}{1+\left[-2\delta^{2}\log\left(1-\left[1-(1-u)^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right)\right]^{-\frac{1}{2}}} \tag{20}$$

then the random variable x defined as

$$x=\left(y^{-1}-1\right)^{-\frac{1}{\tau}} \tag{21}$$

has a kumaraswamy odd Rayleigh-Log-Logistic distribution *i.e.* $x\sim KORLL(\alpha,\lambda,\delta,\tau)$. And when $\alpha=\lambda=1$, x is distributed as $ORLL(\delta,\tau)$.

Figure 1 illustrates the various shapes of the density and hazard functions of the *KORLL* distribution at various parameter values. The density can be symmetric, skewed, and unimodal depending on the parameter values chosen. The hazard function can take many shapes depending on parameter values. This includes J-shaped and non-decreasing.

Table 1 presents the skewness and kurtosis of both the baseline log-logistic distribution and the *KORLL* distribution, computed from the quantile function in Equation (21) using Equation (22) and (23) respectively. For the choosen parameter values the skewness of the log-logistic ranged from −1.4352 to −0.0686, whereas that of the *KORLL* ranged from −0.0696 to 0.3479. Interms of skewness, it's clear that *KORLL* model is much more flexible than the log-logistic distribution. Similarly the kurtosis for the baseline and extended baseline distribution ranged from −2.9641 to −0.1024 and −0.1646 to 31.0576 for the choosen parameter values, respectively. This further suggest the flexibility of the *KORLL* over log-logistic distribution.

Table 1. Skewness and Kurtosis using different parameter values.

Parameters	*KORLL*		Log-Logistics	
	Skewness	Kurtosis	Skewness	Kurtosis
$\alpha = 0.5, \tau = 0.7, \lambda = 0.9, \delta = 0.4$	0.2554	1.0148	−1.4352	−2.9641
$\alpha = 1, \tau = 0.7, \lambda = 0.9, \delta = 0.4$	0.1572	0.4164	−1.4352	−2.9641
$\alpha = 0.5, \tau = 1.5, \lambda = 0.9, \delta = 0.4$	0.0308	0.0961	−0.3506	−0.5362
$\alpha = 0.5, \tau = 1.5, \lambda = 0.5, \delta = 0.4$	0.0036	0.5462	−1.4352	−2.9641
$\alpha = 7, \tau = 8, \lambda = 10, \delta = 7$	0.2113	0.9667	−0.0686	−0.2015
$\alpha = 5, \tau = 15, \lambda = 10, \delta = 8$	−0.0696	−0.1646	−0.0686	−0.2015
$\alpha = 7, \tau = 7, \lambda = 5, \delta = 4$	0.0479	31.0576	−0.0777	−0.2407
$\alpha = 5, \tau = 7, \lambda = 10, \delta = 8$	−0.0643	28.6411	−0.0777	−0.2407
$\alpha = 0.5, \tau = 0.9, \lambda = 0.5, \delta = 0.7$	0.1266	0.3387	−0.5444	−2.0176
$\alpha = 1, \tau = 0.4, \lambda = 0.9, \delta = 0.7$	0.3479	0.9374	−0.8794	−0.1024

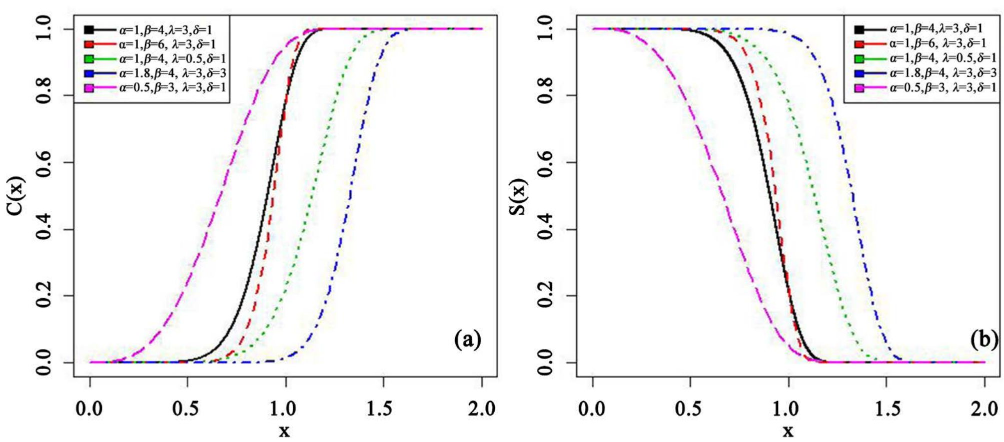

Figure 2. cdf and sf plots of *KORLL* distribution with varying parameter values.

3.1.2. Skewness and Kurtosis

The skewness and kurtosis of the *KORLL* distribution can easily be computed from the quantile function using the relation: the Bowley's skewness (by [16]) is based on the quantile defined as

$$S=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \tag{22}$$

And the Moor's Kurtosis by [17] is based on octiles given by

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \tag{23}$$

3.2. The KORIR Model

The cdf and pdf of the baseline Inverse Rayleigh distribution are given as

$$G(x;\beta)=\exp\left\{\frac{-\beta}{x^2}\right\}$$

and

$$g(x;\beta)=\frac{2\beta}{x^3}\exp\left\{\frac{-\beta}{x^2}\right\},\quad x>0,\beta>0$$

β is scale parameter. The qf is given by

$$Q(u)=\left\{\frac{-\beta}{\log(u)}\right\}^{\frac{1}{2}}$$

when u is uniformly distributed. The cdf and pdf of KORIR distribution is given as

$$F_{KORIR}(x;\alpha,\lambda,\tau,\delta)=1-\left\{1-\left[1-\exp\left\{-\frac{1}{2\delta^2}\left[\exp\left\{\frac{-\beta}{x^2}\right\}-1\right]^{-2}\right\}\right]^{\alpha}\right\}^{\lambda} \tag{24}$$

$$\begin{aligned}f_{KORIR}(x;\alpha,\lambda,\tau,\delta)&=\frac{2\alpha\lambda\beta\exp\left\{\frac{\beta}{x^2}\right\}\left(\exp\left\{\frac{-\beta}{x^2}\right\}-1\right)^{-1}\left(1-\exp\left\{-\frac{\beta}{x^2}\right\}\right)^{-2}}{\delta^2x^3}\\&\times\exp\left\{-\frac{1}{2\delta^2}\left[\exp\left\{\frac{\beta}{x^2}\right\}-1\right]^{-2}\right\}\left[1-\exp\left\{-\frac{1}{2\delta^2}\left[\exp\left\{\frac{\beta}{x^2}\right\}-1\right]^{-2}\right\}\right]^{\alpha-1}\\&\times\left\{1-\left[1-\exp\left\{-\frac{1}{2\delta^2}\left[\exp\left\{\frac{\beta}{x^2}\right\}-1\right]^{-2}\right\}\right]^{\alpha}\right\}^{\lambda-1},\quad x>0\end{aligned} \tag{25}$$

Quantile Function of KORIR

Lemma III

Let the random variable u be uniformly distributed on $(0,1)$. Define the random variable y as in Equation (20), then the random variable x_1 defined as

$$x_1 = \left\{ -\frac{\beta}{\log(y)} \right\}^{\frac{1}{2}} \tag{26}$$

has a kumaraswamy odd Rayleigh-Inverse Rayleigh distribution *i.e.* $x \sim KORIR(\alpha, \lambda, \delta, \beta)$. And when $\alpha = \lambda = 1$, x is distributed as $ROLL(\delta, \beta)$.

4. Estimation

The parameters of the $KORG$ family are estimated in this section using the method of maximum likelihood. Given a random sample of $x_1, x_2, x_3, x_4, \cdots, x_n$ of size n with parameters α, λ, δ and β from $KORG$ family of distribution, the pdf of $KORG$ can written as

$$f_{KORG}(x) = \alpha\lambda \frac{g(x;\beta) G(x;\beta)}{\delta^2 \left[\bar{G}(x;\beta)\right]^3} \exp\left\{ -\frac{Z_\beta^2}{2\delta^2} \right\} \left[1 - \exp\left\{ -\frac{Z_\beta^2}{2\delta^2} \right\} \right]^{\alpha-1} \times \left[1 - \left\{ 1 - \exp\left\{ -\frac{Z_\beta^2}{2\delta^2} \right\} \right\}^{\alpha} \right]^{\lambda-1} \tag{27}$$

where $Z_\beta = \left(\frac{G(x,\beta)}{\bar{G}(x,\beta)} \right)$.

Let $\vartheta = (\alpha, \lambda, \delta, \beta)^{\mathrm{T}}$ be the ($p \times 1$) parameter vector, then the log-likelihood function based on Equation (25) is given by

$$\begin{aligned} l(\vartheta) = {} & n \log\left(\frac{\alpha\lambda}{\delta^2} \right) + \sum_{i=1}^{n} \log\left(g(x;\beta) \right) + \sum_{i=1}^{n} \log\left(G(x;\beta) \right) - 3\sum_{i=1}^{n} \log\left(\bar{G}(x;\beta) \right) \\ & - \sum_{i=1}^{n} \frac{Z_\beta^2}{2\delta^2} (\alpha - 1) \sum_{i=1}^{n} \log\left(1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right) \\ & + (\lambda - 1) \sum_{i=1}^{n} \log\left(1 - \left\{ 1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right\}^{\alpha} \right) \end{aligned} \tag{28}$$

Partially differentiating the likelihood function yields the components of the score function $\boldsymbol{U}_{(\vartheta)} = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \delta}, \frac{\partial l}{\partial \beta} \right)^{\mathrm{T}}$ as follows

$$\begin{aligned} \frac{\partial l}{\partial \alpha} = {} & \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right) \\ & + (\lambda - 1) \sum_{i=1}^{n} \frac{Z_\beta Z_\beta' \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \left\{ 1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right\}^{\alpha-1}}{\delta^2 \left[1 - \left\{ 1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right\}^{\alpha} \right]} \end{aligned} \tag{29}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(1 - \left\{ 1 - \exp\left\{ \frac{-Z_\beta^2}{2\delta^2} \right\} \right\}^{\alpha} \right) \tag{30}$$

$$\frac{\partial l}{\partial \delta}=\frac{-2n}{\delta}+\sum_{i=1}^{n}\frac{Z_\beta^2}{\delta^3}+(\alpha-1)\sum_{i=1}^{n}\frac{Z_\beta^2\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}}{\delta^3\left(1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right)}$$
$$+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha Z_\beta^2\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\left\{1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right\}^{\alpha-1}}{\delta^3\left[1-\left\{1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right\}^{\alpha}\right]} \tag{31}$$

$$\frac{\partial l}{\partial \beta}=\sum_{i=1}^{n}\frac{g_m'(x;\beta)}{g(x;\beta)}+\sum_{i=1}^{n}\frac{G_m'(x;\beta)}{G(x;\beta)}-3\sum_{i=1}^{n}\frac{\bar{G}_m'(x;\beta)}{\bar{G}(x;\beta)}-\sum_{i=1}^{n}\frac{Z_\beta Z_\beta'}{\delta^2}$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{Z_\beta Z_\beta'\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}}{\delta^2\left[1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right]} \tag{32}$$
$$+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha Z_\beta Z_\beta'\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\left\{1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right\}^{\alpha-1}}{\delta^2\left[1-\left\{1-\exp\left\{\frac{-Z_\beta^2}{2\delta^2}\right\}\right\}^{\alpha}\right]}$$

where $Z_\beta'=\frac{\mathrm{d}Z_\beta}{\mathrm{d}\beta}$, $g_m'(x;\beta)=\frac{\mathrm{d}g(x;\beta)}{\mathrm{d}\beta}$, $G_m'(x;\beta)=\frac{\mathrm{d}G(x;\beta)}{\mathrm{d}\beta}$, and $\bar{G}_m'(x;\beta)=\frac{\mathrm{d}\bar{G}(x;\beta)}{\mathrm{d}\beta}$.

The estimators of the parameters can be obtained by setting Equations (29)-(32) to zero and solving numerically using Newton Rapson or any other iterative methods.

5. Monte Carlo Simulation

A Monte Carlo Simulation is conducted and the results of the bias and root mean squared error of the various estimated parameter values are presented in Table 2. The efficacy for the simulation study is to observe the performance of the maximum likelihood estimates and to see whether the simulated values of the model parameters approach the true parameter values or not. The Monte carlo simulation is described as follows:

(a) For known parameter values *i.e.* $\vartheta=(\alpha,\tau,\lambda,\delta)^{\mathrm{T}}$, samples of different sizes from the *KORLL* distribution were generated ($\alpha=0.5$, $\tau=0.7$, $\lambda=0.9$, and $\delta=0.4$) using the quantile function defined in Equation (21).

(b) Using the maximum likelihood method, we compute the MLE of $\hat{\alpha}_i$, $\hat{\tau}_i$, $\hat{\lambda}_i$, and $\hat{\delta}_i$ for the i^{th} replicate.

(c) Steps (a) and (b) are replicated N = 1000 times.

Table 2. A simulation results for the *KORLL* distribution.

n	Properties	$\alpha = 0.5$	$\tau = 0.7$	$\lambda = 0.9$	$\delta = 0.4$
50	Bias	0.0324	0.5088	0.8369	0.2625
	RMSE	0.6847	0.8095	1.9637	0.9301
	Est.	0.5324	1.2088	1.7369	0.6625
200	Bias	−0.0781	0.2628	0.7567	0.2155
	RMSE	0.1979	0.4535	1.7441	0.7009
	Est.	0.4219	0.9628	1.6567	0.6155
300	Bias	−0.0716	0.2095	0.6768	0.2526
	RMSE	0.1607	0.3817	1.6282	0.5791
	Est.	0.4284	0.9095	1.5768	0.6526
500	Bias	−0.0587	0.1547	0.5437	0.1942
	RMSE	0.1281	0.2979	1.2762	0.4519
	Est.	0.4413	0.8547	1.4437	0.5942
700	Bias	−0.0484	0.1189	0.4211	0.1435
	RMSE	0.1104	0.2468	1.0641	0.3582
	Est.	0.4516	0.8189	1.3211	0.5435
1000	Bias	−0.0348	0.0801	0.269	−0.0755
	RMSE	0.0944	0.2045	0.8391	0.4558
	Est.	0.4665	0.7801	1.169	0.3558

(d) The bias and RMSE for each sample size n are computed as

$$\hat{\vartheta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\vartheta}_i, \quad Bias\left(\hat{\vartheta}\right) = \hat{\vartheta} - \vartheta,$$

$$var\left(\hat{\vartheta}\right) = \sum_{i=1}^{N}\frac{\left(\hat{\vartheta}_i - \hat{\vartheta}\right)^2}{N} \tag{33}$$

$$RMSE\left(\hat{\vartheta}\right) = \left\{var\left(\hat{\vartheta}\right) + \left(Bias\left(\hat{\vartheta}\right)\right)^2\right\}^{\frac{1}{2}}$$

where $\hat{\vartheta}_i = \left(\hat{\alpha}, \hat{\delta}, \hat{\lambda}, \hat{\tau}\right)$ are the mle for each iteration $\left(n = 50, 200, 300, 500, 700, 1000\right)$. The simulation results in Table 2 have shown that based on the parameter values chosen, the estimated Biases decrease as the sample size n inreases. In addition, the estimated root mean squared errors decay towards zero as the sample size increases. These two observations illustrate the consistency of the maximum likelihood estimates.

6. Application

Here, we illustrate the applicability of the *KORLL* distribution to five data sets. Data set I represent survival times of 121 patients with breast cancer as reported by [18]. Data set II represents the Marine water as reported by [19]. Data set III represents 101 data points that reflect the stress-rupture life of kevlar 49/epoxy strands which were subjected to continuous persistent pressure at the 90 percent stress point until everything had collapsed as in [20]. Data set IV represents the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile as reported by [21]. Data set V is due to [21] which is a life times data relating to times (in months from 1st January, 2013 to 31st July, 2018) of 105 patients who were diagnosed with hypertension and received at least one treatment related to hypertension in the hospital where death is the event of interest.

We used a maxLik package by [22] in R by [23]. The analytical measures in comparing the model fit are the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Smaller values of the AIC statistic indicate better model fittings. The competing models are as follows:

(i) The Marshall Olkin Extended Log-Logistics (MOELL) as in [24] wth cdf

$$F_{MOELL}(x;\alpha,\tau,\delta)=1-\frac{\alpha^{\tau}\delta}{x^{\tau}+a^{\tau}\delta},\quad x>0$$

(ii) The Kumuraswamy Log-Logistic (KUMLL) as in [25] with cdf

$$F_{KUMLL}(x;\alpha,\tau,\delta,\gamma)=1-\left\{1-\left[1-\frac{1}{1+(x/\alpha)^{\tau}}\right]^{\delta}\right\}^{\gamma},\quad x>0$$

(iii) The Zografos-Balakrishnan Log-Logistic (ZBLL) as in [26] with cdf

$$F_{ZBLL}(x;\alpha,\tau,\lambda)=\frac{\gamma\left(\tau,\log\left[1+(x/\alpha)^{\lambda}\right]\right)}{\Gamma(\tau)},\quad x>0$$

$$\alpha,\lambda,\delta,\gamma>0$$

Based on the considered analytical measures, we have noted that the proposed *KORLL* model provides the best fit to the five analyzed real life data sets presented in **Tables 3-7**. This proposed model outperforms the other four competing

Table 3. MLEs of the Parameters with SEs (paranthesis), BIC, −ll, and AIC values for data set I.

Model	Estimates					BIC	−ll	AIC
	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\gamma}$			
KROLL	0.8201	0.6894	0.3432	6.4677	-	1177.12	578.9682	1165.936
	(0.5051)	(0.0881)	(0.3096)	(0.3043)				
KUMLL	5.9172	12.6227	-	0.5267	14.4993	1192.862	586.8401	1181.68
	(0.8224)	(1.1446)	-	(0.0443)	(3.5236)			
MOELL	13.8653	1.8536	-	5.6094	-	1181.2	587.5998	1181.2
	(2.4619)	(0.1901)	-	(2.8307)	-			
ZBLL	16.8813	1.6039	1.5947	-	-	1206.298	595.9553	1197.911
	(4.3020)	(0.1383)	(0.2628)					
LL	-	0.4335	-	-	-	1514.859	755.0317	1512.063
	-	(0.0307)						

Table 4. MLEs of the Parameters with SEs (paranthesis), BIC, −ll, and AIC values for Data set II.

Model	Estimates					BIC	−ll	AIC
	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\gamma}$			
KROLL	0.8887	0.3138	0.3037	3.4610	-	588.4807	287.0733	582.1466
	(0.2031)	(0.0389)	(0.1894)	(0.1641)				
KUMLL	7.6431	4.4116	-	0.2906	4.5788	595.8207	290.7433	589.4865
	(1.9310)	(2.4439)	-	(0.0673)	(3.8666)			
MOELL	15.7221	0.8058	-	14.1988	-	591.2842	290.2668	586.5335
	(5.1599)	(0.0747)	-	(4.3053)	-			
ZBLL	0.1492	1.1900	9.3955	-	-	598.5214	293.8854	593.7708
	(0.3282)	(0.2247)	(3.7651)					
LL	-	0.2528	-	-	-	666.7035	331.56	665.1199
	-	(0.0331)						

Table 5. MLEs of the Parameters with SEs (paranthesis), BIC, −ll, and AIC values for Data set III.

Model	Estimates					BIC	−ll	AIC
	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\gamma}$			
KROLL	0.8308	0.4859	0.1945	3.0197	-	1154.099	567.8195	1143.639
	(0.0034)	(0.0028)	(0.01779)	(0.0419)				
KUMLL	7.6605	16.3647	-	0.3284	7.7114	1167.367	574.4534	1156.907
	(0.3681)	(1.7609)	-	(0.0251)	(0.0123)			
MOELL	11.1388	1.2706	-	8.9333	-	1169.462	577.8084	1161.617
	(2.9676)	(0.1028)	-	(3.748)	-			
ZBLL	20.0089	1.0373	1.6079	-	-	1183.91	585.0321	1176.064
	(10.9152)	(0.0954)	(0.3745)					
LL	-	0.3761	-	-	-	1393.779	694.5821	1391.164
	-	(0.0293)						

Table 6. MLEs of the Parameters with SEs (paranthesis), BIC, −ll, and AIC values for Data set IV.

Model	Estimates					BIC	−ll	AIC
	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\gamma}$			
KROLL	0.8104	0.6072	0.2525	5.3920	-	563.4723	273.8725	555.7449
	(0.0006)	(0.0004)	(0.0354)	(0.0001)				
KUMLL	10.5183	13.4904	-	0.3931	3.0626	574.2929	279.2828	566.5657
	(2.2634)	(13.5114)	-	(0.0952)	(1.5268)			
MOELL	11.0024	1.6251	-	15.7128	-	569.8063	279.0054	564.0108

Continued

	(6.2578)	(0.1691)	-	(6.2920)	-			
ZBLL	18.6091	1.2735	1.7605	-	-	581.7961	285.0003	576.0006
	(3.5334)	(0.1401)	(0.2278)					
LL	-	0.3826	-	-	-	700.4312	348.2497	698.4994
	-	(0.0418)						

Table 7. MLEs of the Parameters with SEs (paranthesis), BIC, −ll, and AIC values for Data set V.

Model	Estimates					BIC	−ll	AIC
	$\hat{\alpha}$	$\hat{\tau}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\gamma}$			
KROLL	3.1172	0.4968	9.0421	5.8242	-	936.4692	458.9267	925.8534
	(0.6112)	(0.0738)	(3.7577)	(1.8566)				
KUMLL	8.7308	18.7688	-	0.7217	13.1942	961.8756	471.6299	951.2599
	(0.8082)	(1.4343)	-	(0.0473)	(0.8803)			
MOELL	28.1447	3.1661	-	3.1572	-	947.8358	470.9179	947.8358
	(1.5932)	(0.2767)	-	(0.9238)	-			
ZBLL	24.8004	2.4252	1.5230	-	-	981.1929	483.6155	973.2311
	(2.0922)	(0.2055)	(0.1536)					
LL	-	0.4251	-	-	-	1345.432	670.3889	1342.778
	-	(0.0321)						

extensions of the log-logistic distributions presented.

7. Conclusion

In this paper, a new family of distributions called the Kumaraswamy Odd Rayleigh G family which introduced three additional parameters to the baseline distribution is proposed and studied. This new family gives more flexibility and proved best fit, to a wide range of data from practical situations. The Monte Carlo simulation results indicated that the simulated values of the parameters of the sub-model of this family approached the true values as the sample size increases. Also, the root mean squared error estimates decay towards zero as the sample size becomes large. These facts suggest the consistency of the estimates. Based on the considered analytical measures, we concluded that the proposed family represented in this study by the Kumaraswamy Odd Rayleigh Log-Logistic distribution provided the best fit to the 5 analysed real life data sets, some of which are the survival times of 121 patients with Breast cancer and death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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