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Analysis of Variance for Three-Way Unbalanced Mixed Effects Interactive Model

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Abstract

In the study, a method of solving ANOVA problems based on an unbalanced three-way mixed effects model with interaction for data when factors A and B are fixed, and factor C is random was presented, and the required EMS was derived. Under each of the appropriate null hypotheses, it was observed that none of the derived EMS was unbiased for the other. Unbiased estimators of the mean squares were determined to test hypotheses. With the unbiased estimators, appropriate F-statistics as well as their corresponding pseudo-degrees of freedom were obtained. The theoretical results presented in the paper were illustrated using a numerical example.

Keywords

Expected Mean Square (EMS), Pseudo-Degree of Freedom, Unbiased Estimate, Hypotheses, Multi-Factor Experiments

1. Introduction

The role of multi-factor experiments in agriculture, engineering and other fields cannot be overemphasized. Through a multi-factor experiment, it is possible to test the interaction effect of two or more factors. Sometimes, a multi-factor experiment conducted to compare factor levels and factor level combinations results in unbalanced data. Often, in the case of the analysis of variance (ANOVA) for unbalanced data, an exact F-test does not exist. As a remedy to this problem, authors have recommended some methods of testing effects in various multi-factor ANOVA problems. Consequently, [1] proposed an exact permutation test for fixed effects ANOVA based on balanced and unbalanced data. [2] derived expected mean squares for the unbalanced two-way random effects model with integer degrees of freedom. The F-test statistics for testing main effects as well as the interaction effects based on the two-way mixed effects model were derived by [3].

From the foregoing, it is obvious that a reasonable number of studies have been carried out on the unbalanced two-way fixed effects, random effects and mixed effects models. However, not much attention has been given to the unbalanced three-way analysis of variance problems, especially such problems requiring mixed factor effects. There are basically six cases of the unbalanced three-way mixed effects crossed classification models. This paper deals with hypothesis testing problems arising from one of the six cases, in which two (*A* and *B*) of the three factors are fixed and the other factor (*C*) is random. The remaining parts of this paper are organized in the following manner. Section 2 has to do with the model specification and the necessary notations. In Section 3, theoretical results pertaining to expected mean squares, F-statistics and the corresponding pseudo-degrees of freedom are derived. A numerical example and the conclusion of this paper are presented in Sections 4 and 5 respectively.

2. Model Specification and Restriction

The three-way unbalanced mixed effects cross-classification model with interaction terms, in which factors A and B are fixed while factor C is random is given by [4] and [5] as

$$X_{ijkl} = \mu + A_i + B_j + C_k + (AB)_{ij} + (AC)_{ik} + (BC)_{jk} + (ABC)_{ijk} + e_{ijkl}, \quad \begin{cases} i = 1, 2, \cdots, a \\ j = 1, 2, \cdots, b \\ k = 1, 2, \cdots, c \\ l = 1, 2, \cdots, n_{ijk} \end{cases} \tag{1}$$

where: X_{ijkl}, denotes the 1st observation at the *i*th level of factor *A*, the *j*th level of factor *B*, and the *k*th level of factor *C*, μ denotes the overall mean, A_i denotes the effect of the *i*th level of factor *A*. B_j denotes the effect of the *j*th level of factor *B*, C_k denotes the effect of the *k*th level of factor *C*, $(AB)_{ij}$, $(AC)_{ik}$, $(BC)_{jk}$ denotes the effects of the two-factor interactions $A \times B$, $A \times C$, $B \times C$, respectively, $(ABC)_{ijk}$ denotes the effect of the three-factor interaction $A \times B \times C$, e_{ijkl} denotes the customary error term.

The Model (1) is called unbalanced three-way mixed effects cross-classification model with interaction if the following assumptions by [4].

From Equation (1), if factor A and B are fixed while factor C is random.

The model follows the following assumptions as:

i) The effects $A_i's$ and $B_j's$ are assumed to be fixed subject to the constraint $\sum_{i}^{a} A_i = \sum_{j}^{b} B_j = \sum_{ij}^{ab} (AB)_{ij} = 0$;

ii) $C_k's$ are assumed to be randomly and normally distributed with mean zero and variance σ_θ^2. *i.e.* $C_k \sim N(0, \sigma_C^2)$, $(AC)_{ik} \sim N(0, \sigma_{AC}^2)$, $(BC)_{jk} \sim N(0, \sigma_{BC}^2)$ and $(ABC)_{ijk} \sim N(0, \sigma_{ABC}^2)$.

iii) $C_k's$ are uncorrelated with one another $C_k's$ and $e_{ijk}'s$ that is

$$E\left(C_k C_k'\right) = 0, k \neq k'$$

and

$$E\left(C_k e_{ijk}\right) = 0 \quad \text{for all} \quad (i, j, k)\text{'s;}$$

iv) Error terms are normally distributed with mean zero and variance σ_e^2, they are mutually independent *i.e.* $e_{ijkl} \sim N\left(0, \sigma_e^2\right)$.

Under the assumptions above, we consider the following notations so as to derive the requisite expected mean squares.

Now, let

$$\begin{aligned} N &= \sum_i^a N_i = \sum_j^b N_j = \sum_k^c N_k = \sum_i^a \sum_j^b n_{ij.} = \sum_i^a \sum_k^c n_{i.k} = \sum_j^b \sum_k^c n_{.jk} \\ &= \sum_i^a \sum_j^b \sum_k^c n_{ijk} = \sum_i^a \sum_j^b \sum_k^c n_{i..} = \sum_i^a n_{ijk} = \sum_j^b n_{ijk} = \sum_k^c n_{ijk} \end{aligned} \tag{2}$$

3. Main Results

The mean squares due to the three main effects and four interaction terms for Model (1) are

$$MS_A = \frac{\left[\sum_i^a N_i \left(\bar{X}_{i...} - \bar{X}_{....}\right)^2\right]}{a-1}, \tag{3}$$

$$MS_B = \frac{\left[\sum_j^b N_j \left(\bar{X}_{.j..} - \bar{X}_{....}\right)^2\right]}{b-1}, \tag{4}$$

$$MS_C = \frac{\left[\sum_k^c N_k \left(\bar{X}_{..k.} - \bar{X}_{....}\right)^2\right]}{c-1}, \tag{5}$$

$$MS_{AB} = \frac{\sum_{ij}^{ab} n_{ij.} \left(\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....}\right)^2}{(a-1)(b-1)}, \tag{6}$$

$$MS_{AC} = \frac{\sum_{ik}^{ac} n_{i.k} \left(\bar{X}_{i.k.} - \bar{X}_{i...} - \bar{X}_{..k.} + \bar{X}_{....}\right)^2}{(a-1)(c-1)}, \tag{7}$$

$$MS_{BC} = \frac{\sum_{jk}^{bc} n_{.jk} \left(\bar{X}_{.jk.} - \bar{X}_{.j..} - \bar{X}_{..k.} + \bar{X}_{....}\right)^2}{(b-1)(c-1)}, \tag{8}$$

$$MS_{ABC} = \frac{\sum_{ijk}^{abc} n_{ijk} \left(\bar{X}_{ijk.} - \bar{X}_{ij..} - \bar{X}_{i.k.} - \bar{X}_{.jk.} + \bar{X}_{i...} + \bar{X}_{.j..} + \bar{X}_{..k.} - \bar{X}_{....}\right)^2}{(a-1)(b-1)(c-1)}, \tag{9}$$

and

$$MS_E = \frac{\sum_i^a \sum_j^b \sum_k^c \sum_l^{n_{ijk}} \left(X_{ijkl} - \bar{X}_{ijk.} \right)^2}{N - abc}, \tag{10}$$

where: MS_A is the Mean Square for factor A, MS_B is the Mean Square for factor B, MS_C is the Mean Square for factor C, MS_{AB} is the Mean Square for the interaction factor A and B, MS_{AC} is the Mean Square for the interaction factor A and C, MS_{BC} is the Mean Square for interaction of factor B and C, MS_{ABC} is the Mean Square for interaction of factor A, B and C, and MS_E is the Mean Square for error term.

Using Brute Force Method, the expected mean squares of Equation (1) when factor A and B is fixed while factor C is random the expected mean square are shown in theorem 1.

Theorem 1: Given the model in (1), the expected mean square due to factor A is

$$E\left[MS_A\right] = \frac{\sum_i^a N_i A_i^2}{a-1} + K_1 \sigma_{AC}^2 + K_2 \sigma_{ABC}^2 + \sigma_e^2 .$$

where $K_1 = \dfrac{\sum_i^a N_i^{-1} \sum_k^c n_{ijk}^2 - N^{-1} \sum_{ik}^{ac} n_{ijk}^2}{a-1}$, $K_2 = \dfrac{N^{-1} \sum_{ijk}^{abc} n_{ijk}^2}{a-1}$ and σ_e^2 is the error variance.

Proof:

Using Equation (1), we have

$$\bar{X}_{i...} = \mu + A_i + 0 + \bar{C}_. + 0 + \left(\overline{AC}\right)_{i.} + \left(\overline{BC}\right)_{..} + 0 + \bar{e}_{i...} \tag{11}$$

and

$$\bar{X}_{.....} = \mu + 0 + 0 + \bar{C}_. + 0 + \left(\overline{AC}\right)_{..} + \left(\overline{BC}\right)_{..} + \left(\overline{ABC}\right)_{...} + \bar{e}_{....} \tag{12}$$

Substituting (11), (12) into (3), and taking expectations, we have

$$\begin{aligned} E\left[MS_A\right] &= \frac{1}{a-1} E\left[\sum_i^a N_i \left[A_i + \left(\overline{AC}\right)_{i.} - \left(\overline{AC}\right)_{..} - \left(\overline{ABC}\right)_{...} + \left(\bar{e}_{i...} - \bar{e}_{....}\right) \right]^2 \right] \\ &= \frac{1}{a-1} E\left[\sum_i^a N_i \left[A_i + \left[\left(\overline{AC}\right)_{i.} - \left(\overline{AC}\right)_{..} \right] - \left(\overline{ABC}\right)_{...} + \left(\bar{e}_{i...} - \bar{e}_{....}\right) \right]^2 \right] \\ &= \frac{1}{a-1} E\left[\sum_i^a N_i \left[A_i^2 + \left[\left(\overline{AC}\right)_{i.} - \left(\overline{AC}\right)_{..} \right]^2 - \left(\overline{ABC}\right)_{...}^2 + \left(\bar{e}_{i...} - \bar{e}_{....}\right)^2 \right] \right] \end{aligned}$$

where

$$\left(\overline{AC}\right)_{i.} = \frac{N\left(\overline{AC}\right)_{..}}{\sum_i^a N_i}, \quad E\left(\overline{AC}\right)_{..}^2 = \frac{\sum_{ik}^{ac} n_{ijk}^2 \sigma_{AC}^2}{N^2} \quad \text{and} \quad E\left(\overline{ABC}\right)_{...}^2 = \frac{\sum_{ijk}^{abc} n_{ijk}^2 \sigma_{ABC}^2}{N^2}.$$

$$=\frac{1}{a-1}E\left[\sum_{i}^{a}N_i\left[A_i^2+\left(\overline{AC}\right)_{i.}^2-2\left(\overline{AC}\right)_{i.}\left(\overline{AC}\right)_{..}+\left(\overline{AC}\right)_{..}^2+\left(\overline{ABC}\right)_{...}^2+\left(\overline{e}_{i...}-\overline{e}_{....}\right)^2\right]\right]$$

$$=\frac{1}{a-1}E\left[\sum_{i}^{a}N_iA_i^2+\sum_{i}^{a}N_i\left(\overline{AC}\right)_{i.}^2-2\sum_{i}^{a}N_i\left(\overline{AC}\right)_{i.}\left(\overline{AC}\right)_{..}+N\left(\overline{AC}\right)_{..}^2\right.$$
$$\left.+N\left(\overline{ABC}\right)_{...}^2+\sum_{i}^{a}N_i\left(\overline{e}_{i...}-\overline{e}_{....}\right)^2\right]$$

$$=\frac{1}{a-1}E\left[\sum_{i}^{a}N_iA_i^2+\sum_{i}^{a}N_i\left(\overline{AC}\right)_{i.}^2-N\left(\overline{AC}\right)_{..}^2+N\left(\overline{ABC}\right)_{...}^2+\sum_{i}^{a}N_i\left(\overline{e}_{i...}^2-2\overline{e}_{i...}\overline{e}_{....}+\overline{e}_{....}^2\right)\right]$$

$$=\frac{1}{a-1}\left[\sum_{i}^{a}N_iE\left(A_i^2\right)+\sum_{i}^{a}N_iE\left(\overline{AC}\right)_{i.}^2-NE\left(\overline{AC}\right)_{..}^2+NE\left(\overline{ABC}\right)_{...}^2+\sum_{i}^{a}N_iE\left(\overline{e}_{i...}^2\right)\right.$$
$$\left.-2\sum_{i}^{a}N_iE\left(\overline{e}_{i...}\right)E\left(\overline{e}_{....}\right)+NE\left(\overline{e}_{....}^2\right)\right]$$

$$=\frac{1}{a-1}\left[\sum_{i}^{a}N_iE\left(A_i^2\right)+\sum_{i}^{a}N_i^{-1}\sum_{k}^{c}n_{ijk}^2\sigma_{AC}^2-N^{-1}\sum_{ik}^{ac}n_{ijk}^2\sigma_{AC}^2+N^{-1}\sum_{ijk}^{abc}n_{ijk}^2\sigma_{ABC}^2\right.$$
$$\left.+\sum_{i}^{a}N_i\frac{\sigma_e^2}{N_i}-2N\frac{\sigma_e^2}{N}+N\frac{\sigma_e^2}{N}\right]$$

$$=\frac{1}{a-1}\left[\sum_{i}^{a}N_iE\left(A_i^2\right)+\left(\sum_{i}^{a}N_i^{-1}\sum_{k}^{c}n_{ijk}^2-N^{-1}\sum_{ik}^{ac}n_{ijk}^2\right)\sigma_{AC}^2+N^{-1}\sum_{ijk}^{abc}n_{ijk}^2\sigma_{ABC}^2+(a-1)\sigma_e^2\right]$$

$$E\left[MS_A\right]=\frac{\sum_{i}^{a}N_iE\left(A_i^2\right)}{a-1}+\frac{1}{a-1}\left(\sum_{i}^{a}N_i^{-1}\sum_{k}^{c}n_{ijk}^2-N^{-1}\sum_{ik}^{ac}n_{ijk}^2\right)\sigma_{AC}^2$$
$$+\frac{1}{a-1}N^{-1}\sum_{ijk}^{abc}n_{ijk}^2\sigma_{ABC}^2+\frac{1}{a-1}(a-1)\sigma_e^2$$

$$E\left[MS_A\right]=\frac{\sum_{i}^{a}N_iA_i^2}{a-1}+K_1\sigma_{AC}^2+K_2\sigma_{ABC}^2+\sigma_e^2 .$$

where $\overline{e}_{i...}=\frac{N\overline{e}_{....}}{\sum_{i}^{a}N_i}$. This completes the proof.

Similarly, if MS_B and MS_C denote the mean squares due to factor B and factor C respectively. Then

$$E\left[MS_B\right]=\frac{\sum_{j}^{b}N_jE\left(B_j\right)^2}{b-1}+K_3\sigma_{BC}^2+K_4\sigma_{ABC}^2+\sigma_e^2 \tag{13}$$

$$E\left[MS_C\right]=K_\theta\sigma_C^2+K_5\sigma_{AC}^2+K_6\sigma_{BC}^2+K_7\sigma_{ABC}^2+\sigma_e^2 \tag{14}$$

$$E\left[MS_{AB}\right]=\frac{\sum_{i}^{a}\sum_{j}^{b}n_{ij.}E\left(AB\right)_{ij}^2}{(a-1)(b-1)}+K_8\sigma_{ABC}^2+\sigma_e^2 \tag{15}$$

$$E\left[MS_{AC}\right]=K_9\sigma_{AC}^2+K_{10}\sigma_{ABC}^2+\sigma_e^2 \tag{16}$$

$$E\left[MS_{BC}\right]=K_{11}\sigma_{BC}^2+K_{12}\sigma_{ABC}^2+\sigma_e^2 \tag{17}$$

$$E\left[MS_{ABC}\right]=K_{13}\sigma_{ABC}^2+\sigma_e^2 \tag{18}$$

and

$$E\left[MS_e\right]=\sigma_e^2 \tag{19}$$

where:

$$K_1=\frac{\sum_i^a N_i^{-1}\sum_k^c n_{ijk}^2-N^{-1}\sum_{ik}^{ac} n_{ijk}^2}{a-1}, K_2=\frac{N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{a-1},$$

$$K_3=\frac{\sum_j^b N_j^{-1}\sum_k^c n_{ijk}^2-N^{-1}\sum_{jk}^{bc} n_{ijk}^2}{b-1}, K_4=\frac{N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{b-1}, K_5=\frac{N^{-1}\sum_{ik}^{ac} n_{ijk}^2}{c-1},$$

$$K_6=\frac{N^{-1}\sum_{jk}^{bc} n_{ijk}^2}{c-1}, K_7=\frac{N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{c-1}, K_8=\frac{N^{-1}\sum_k^c n_{ijk}^2+N^{-1}3\sum_{ijk}^{ab} n_{ijk}^2}{(a-1)(b-1)},$$

$$K_9=\frac{N-\sum_i^b N_i^{-1}\sum_k^c n_{ijk}^2+N^{-1}\sum_{ik}^{ac} n_{ijk}^2}{(a-1)(c-1)}, K_{10}=\frac{N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{(a-1)(c-1)}, \tag{20}$$

$$K_{11}=\frac{N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{(b-1)(c-1)}, K_{12}=\frac{N-N\sum_k^c n_{ijk}^2+N^{-1}\sum_{jk}^{ac} n_{ijk}^2}{(b-1)(c-1)},$$

$$K_{13}=\frac{N^{-1}\sum_j^b n_{ijk}^2+N^{-1}\sum_{ijk}^{abc} n_{ijk}^2}{(a-1)(b-1)(c-1)}$$

A major step in the derivation of the F-statistics is to find the unbiased estimates of the mean squares due to the main factors and the interactions. Therefore, the unbiased estimates are presented as follows.

Theorem 2: Given the model in (1), if factors A and B are fixed while factor C is random, then $MS_\phi=\phi_1 MS_{AC}+\phi_2 MS_{ABC}+(1-\phi_1-\phi_2)MS_E$ is an unbiased estimate of

$$MS_A=k_1\sigma_{AC}^2+k_3\sigma_{ABC}^2+\sigma_e^2 .$$

where, MS_ϕ is the unbiased estimate of the mean square for factor A.

Proof:

If we assume that MS_{AC} and MS_{ABC} are independent, we take expectations, to have

$$E(MS_\phi)=\phi_1 E(MS_{AC})+\phi_2 E(MS_{ABC})+(1-\phi_1-\phi_2)E(MS_E)$$

$$E(MS_\phi)=\phi_1\left(K_9\sigma_{AC}^2+K_{10}\sigma_{ABC}^2+\sigma_e^2\right)+\phi_2\left(k_{13}\sigma_{ABC}^2+\sigma_e^2\right)+(1-\phi_1-\phi_2)\sigma_e^2$$

$$E(MS_\phi)=\phi_1 K_9\sigma_{AC}^2+\phi_1 K_{10}\sigma_{ABC}^2+\phi_1\sigma_e^2+\phi_2 K_{13}\sigma_{ABC}^2+\phi_2\sigma_e^2+\sigma_e^2-\phi_1\sigma_e^2-\phi_2\sigma_e^2$$

But

$$\phi_1=\frac{K_1}{K_9},\quad \phi_2=\frac{K_2}{K_{13}}-\frac{K_1 K_{10}}{K_9 K_{13}} \tag{21}$$

$$E\left(MS_{\phi}\right)=\frac{K_1}{K_9}K_9\sigma_{AC}^2+\frac{K_1}{K_9}K_{10}\sigma_{ABC}^2+\frac{K_2}{K_{13}}-\frac{K_1K_{10}}{K_9K_{13}}\left(K_{13}\sigma_{ABC}^2\right)+\sigma_e^2$$
$$=K_1\sigma_{AC}^2+\left(\frac{K_1}{K_9}K_{10}+\frac{K_2}{K_{13}}K_{13}-\frac{K_1K_{10}}{K_9K_{13}}k_{13}\right)\sigma_{ABC}^2+\sigma_e^2$$

$E\left(MS_{\phi}\right)=k_1\sigma_{AC}^2+k_2\sigma_{ABC}^2+\sigma_e^2$ as required.

Similarly, it can be shown that $MS_{\lambda}=\lambda_1 MS_{BC}+\lambda_2 MS_{ABC}+\left(1-\lambda_1-\lambda_2\right)MS_E$ is an unbiased estimate.

$$MS_B=k_3\sigma_{AC}^2+k_4\sigma_{ABC}^2+\sigma_e^2,$$

$MX_{\gamma}=\gamma_1 MS_{AC}+\gamma_2 MS_{BC}+\gamma_3 MS_{ABC}+\left(1-\gamma_1-\gamma_2-\gamma_3\right)MS_E$ is an unbiased estimate.

$$MS_C=k_5\sigma_{AC}^2+k_6\sigma_{BC}^2+k_7\sigma_{ABC}^2+\sigma_e^2,$$

$MS_{\pi}=\pi_2 MS_{ABC}+\left(1-\pi_1\right)MS_E$ is an unbiased estimate.

$$MS_{ABC}=k_8\sigma_{ABC}^2+\sigma_e^2,$$

$MS_{\rho}=\rho_1 MS_{AC}+\rho_2 MS_{ABC}+\left(1-\rho_1-\rho_2\right)MS_E$ is an unbiased estimate.

$$MS_{AC}=k_9\sigma_{AC}^2+k_{10}\sigma_{ABC}^2+\sigma_e^2,$$

and

$MS_{\tau}=\tau_1 MS_{BC}+\tau_2 MS_{ABC}+\left(1-\tau_1-\tau_2\right)MS_E$ is an unbiased estimate.

$$MS_{BC}=k_{11}\sigma_{BC}^2+k_{12}\sigma_{ABC}^2+\sigma_e^2.$$

The F-statistics for the main effects and interactions effect are given below:

$$F_A=\frac{MS_A}{MS_{\phi}}, \tag{22}$$

$$F_B=\frac{MS_B}{MS_{\lambda}}, \tag{23}$$

$$F_C=\frac{MS_C}{MS_{\gamma}}, \tag{24}$$

$$F_{AB}=\frac{MS_{AB}}{MS_{\pi}}, \tag{25}$$

$$F_{AC}=\frac{MS_{AC}}{MS_{\rho}}, \tag{26}$$

$$F_{BC}=\frac{MS_{BC}}{MS_{\tau}}, \tag{27}$$

and

$$F_{ABC}=\frac{MS_{ABC}}{MS_e}, \tag{28}$$

where: F_A is the F-statistic for factor A, F_B is the F-statistic for factor B, F_C is the F-statistic for factor C, F_{AB} is the F-statistics for the interaction factors A and B, F_{AC} is the F-statistics for the interaction factors A and C, F_{BC} is the F-statistics for the interaction factors B and C, and F_{ABC} is the F-statistics for the interaction factors A, B and C.

Having presented the necessary F-statistic, we also have to determine the pseudo-degree of freedom corresponding to this statistic. Using the [6], the Welch Satterthwaite equation is used to determine the pseudo-degrees of freedom in this paper.

Theorem 3: Given the model in (1) and Welch Satterthwaite Equation, let f_ϕ be the pseudo-degree of freedom for factor *A*. Then

$$f_\phi = \left[\left(\widehat{MS_\phi}\right)^2\right]\left[\frac{\phi_1^2\left(MS_{AC}\right)^2}{f_{AC}} + \frac{\phi_2^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\phi_1-\phi_2\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1}, \quad (29)$$

Proof:

Recall that

$$MS_\phi = \phi_1 MS_{AC} + \phi_2 MS_{ABC} + \left(1-\phi_1-\phi_2\right)MS_e, \quad (30)$$

If we assumed that MS_{AC} and MS_{ABC} are independent, we obtain

$$\operatorname{var}\left(MS_\phi\right) = \phi_1^2 \operatorname{var}\left(MS_{AC}\right) + \phi_2^2 \operatorname{var}\left(MS_{ABC}\right) + \left(1-\phi_1-\phi_2\right)^2 \operatorname{var}\left(MS_e\right), \quad (31)$$

Recall that $f_e = N - abc$

Since

$$MS_e = \frac{\sum_{ijkl}^{abcn_{ijk}}\left(X_{ijkl} - \bar{X}_{ijk.}\right)}{N-abc} = \frac{SS_e}{f_e} \text{ and } \frac{SS_e}{\sigma_e^2} \sim \chi^2_{f_e},$$

we have

$$\operatorname{var}\left(MS_e\right) = \frac{\operatorname{var}\left(SS_e\right)}{f_e^2}, \quad (32)$$

$$\begin{aligned}
&\operatorname{var}\left(MS_e\right) = \frac{2\left(\sigma_e^2\right)^2 f_e}{f_e^2} = \frac{2\left(\sigma_e^2\right)^2}{f_e}, \operatorname{var}\left(MS_\phi\right) = \frac{2\left(\sigma_\phi^2\right)^2}{f_\phi},\\
&\operatorname{var}\left(MS_{AC}\right) = \frac{2\left(\sigma_{AC}^2\right)^2}{f_{AC}}, \operatorname{var}\left(MS_{BC}\right) = \frac{2\left(\sigma_{BC}^2\right)^2}{f_{BC}},\\
&\operatorname{var}\left(MS_{ABC}\right) = \frac{2\left(\sigma_{ABC}^2\right)^2}{f_{ABC}}
\end{aligned} \quad (33)$$

Extending our idea of (33) into (31), leads to

$$\frac{2\left(\sigma_\phi^2\right)^2}{f_\phi} = \phi_1^2\left(\frac{2\left(\sigma_{AC}^2\right)^2}{f_{AC}}\right) + \phi_2^2\left(\frac{2\left(\sigma_{ABC}^2\right)^2}{f_{ABC}}\right) + \left(1-\phi_1-\phi_2\right)^2\left(\frac{\left(2\sigma_e^2\right)^2}{f_e}\right)$$

where:

$\sigma_e^2 = MS_e, \sigma_\phi^2 = MS_\phi, \sigma_{AC}^2 = MS_{AC}, \sigma_{BC}^2 = MS_{BC}$ and $\sigma_{ABC}^2 = MS_{ABC}$.

Consequently

$$\begin{aligned}
2\left(MS_\phi\right)^2 f_\phi^{-1} = {} & 2\phi_1^2\left(MS_{AC}\right)^2 f_{AC}^{-1} + 2\phi_2^2\left(MS_{ABC}\right)^2 f_{ABC}^{-1}\\
& + 2\left(1-\phi_1-\phi_2\right)^2\left(MS_e\right)^2 f_e^{-1}
\end{aligned}$$

$$f_\phi = \left[\left(MS_\phi\right)^2\right]\left[\frac{\phi_1^2\left(MS_{AC}\right)^2}{f_{AC}} + \frac{\phi_2^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\phi_1-\phi_2\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{34}$$

The degree of freedom associated with F_A is

$$f_{f_A, f_\phi}^{0.05}$$

Similarly,

$$f_\lambda = \left[\left(MS_\lambda\right)^2\right]\left[\frac{\lambda_1^2\left(MS_{BC}\right)^2}{f_{BC}} + \frac{\lambda_2^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\lambda_1-\lambda_2\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{35}$$

The degree of freedom associated with F_B is

$$f_{f_B, f_\lambda}^{0.05}$$

$$f_\gamma = \left[\left(MS_\gamma\right)^2\right] \cdot \left[\frac{\gamma_1^2\left(MS_{AC}\right)^2}{f_{AC}} + \frac{\gamma_2^2\left(MS_{BC}\right)^2}{f_{BC}} + \frac{\gamma_3^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\gamma_1-\gamma_2-\gamma_3\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{36}$$

The degree of freedom associated with F_C is

$$f_{f_C, f_\gamma}^{0.05}$$

$$f_\pi = \left[\left(MS_\pi\right)^2\right]\left[\frac{\pi_1^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\pi_1\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{37}$$

The degree of freedom associated with interaction factor A and B is

$$f_{f_{AB}, f_\pi}^{0.05}$$

$$f_\rho = \left[\left(MS_\rho\right)^2\right]\left[\frac{\rho_1^2\left(MS_{AC}\right)^2}{f_{AC}} + \frac{\rho_2^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\rho_1-\rho_2\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{38}$$

The degree of freedom associated with interaction factor A and C is

$$f_{f_{AC}, f_\rho}^{0.05}$$

$$f_\tau = \left[\left(MS_\tau\right)^2\right]\left[\frac{\tau_1^2\left(MS_{BC}\right)^2}{f_{BC}} + \frac{\gamma_2^2\left(MS_{ABC}\right)^2}{f_{ABC}} + \frac{\left(1-\tau_1-\tau_2\right)^2\left(MS_e\right)^2}{f_e}\right]^{-1} \tag{39}$$

The degree of freedom associated with interaction factor B and C is

$$f_{f_{BC}, f_\tau}^{0.05}$$

The degree of freedom associated with the interaction factors $A \times B \times C$ is

$$f_{f_{ABC}, f_e}^{\alpha}$$

This does not involve obtaining any expression, where $f_\lambda, f_\gamma, f_\pi, f_\rho$ and f_τ represent the pseudo degrees of freedom for factors B and C, the interactions $A \times B$, $A \times C$ and $B \times C$. While f_{ABC} and f_e are the numerator and denominator degrees of freedom respectively (**Table 1**).

Table 1. Table for the unbiased estimates of the mean squares under the null hypothesis.

FACTORS	HYPOTHESIS	UNBIASED ESTIMATES OF THE MEAN SQUARES	APPROXIMATE F-STATISTIC
A	$H_{0A}: A_1 = \cdots = A_n = 0$	$MS_\phi = \phi_1 MS_{AC} + \phi_2 MS_{ABC} + (1-\phi_1-\phi_2) MS_E$	$\frac{MS_A}{MS_\phi}$
B	$H_{0B}: B_1 = \cdots = B_n = 0$	$MS_\lambda = \lambda_1 MS_{BC} + \lambda_2 MS_{ABC} + (1-\lambda_1-\lambda_2) MS_E$	$\frac{MS_B}{MS_\lambda}$
C	$H_{0C}: \sigma_C^2 = 0$	$MS_\gamma = \gamma_1 MS_{AC} + \gamma_2 MS_{BC} + \gamma_3 MS_{ABC} + (1-\gamma_1-\gamma_2-\gamma_3) MS_E$	$\frac{MS_B}{MS_\gamma}$
AB	$H_{0AB}: (AB)_1 = \cdots = (AB)_n = 0$	$MS_\pi = \pi_1 MS_{ABC} + (1-\pi_1) MS_E$	$\frac{MS_{AB}}{MS_\pi}$
AC	$H_{0AC}: \sigma_{AC}^2 = 0$	$MS_\rho = \rho_1 MS_{AC} + \rho_2 MS_{ABC} + (1-\rho_1-\rho_2) MS_E$	$\frac{MS_{AC}}{MS_\rho}$
BC	$H_{0BC}: \sigma_{BC}^2 = 0$	$MS_\tau = \tau_1 MS_{BC} + \tau_2 MS_{ABC} + (1-\tau_1-\tau_2) MS_E$	$\frac{MS_{BC}}{MS_\tau}$
ABC	$H_{0ABC}: \sigma_{ABC}^2 = 0$		$\frac{MS_{ABC}}{MS_E}$

4. Numerical Example

Consider a three-factorial experiment involving factor A (Solvents-water, ethanol, ether), factor B (Volumes of solute-25, 50 and 100 ml) and factor C (Time-40, 50, 60 and 70 mins) respectively. The solvents are of varying polarities. Arbitrary volumes of 25, 50 and 100 ml were chosen while the extraction was done at intervals of time 40, 50, 60 and 70 mins. The major aim is to determine the efficiency of different quantities of solvents on the extraction of soluble components of lemon grass per unit time.

In the experiment, the sample (lemon grass) was dried in the oven at 45˚C for 1440 mins. The dried sample was pulverized and 1 g of pulverized sample was used for each solvent in a typical extraction 1g of sample was dissolved in 25 ml of water for 40 mins. At the end of the time, the solute was filtered using a suitable filter paper (Whatman). The solution was then vaporized at 105˚C for 720 mins leaving the remaining extract which was weighted in an analytical balance. The process was repeated and replicated three times for 50, 60 and 70 mins. A similar procedure was done using different volumes of ethanol and ether as extracts at different durations of 40, 50, 60 and 70mins. Results of the extraction are shown in Table 2.

Using the information in Table 2 as well as the formulae for computing MS_A, MS_B, MS_C, MS_{AB}, MS_{AC}, MS_{BC}, MS_{ABC} and MS_E, we have

$$MS_A = 72.4819, MS_B = 50.5353, MS_C = 67.0578,$$
$$MS_{AB} = 53.4128, MS_{BC} = 25.4262, MS_{AC} = 86.2273,$$
$$MS_{ABC} = 193.7391 \text{ and } MS_E = 1.9473.$$

Again, the constants are calculated as follows:

$$k_1 = 7.7153, k_2 = 1.2531, k_3 = 6.8274, k_4 = 1.2531,$$
$$k_5 = 1.7325, k_6 = 2.2840, k_7 = 0.8354, k_8 = 8.6482,$$
$$k_9 = 11.2040, k_{10} = 0.4177, k_{11} = 0.4177,$$
$$k_{12} = 67.3453, k_{13} = 4.8323$$

The ANOVA table for the data is shown in **Table 3**.

Our hypothesis for factor A shall be $H_{0A} : A_1 = \cdots = A_n = 0$;

Similarly, our hypothesis for factor B shall be $H_{0B} : B_1 = \cdots = B_n = 0$;

Our hypothesis for factor C shall be $H_{0C} : \sigma_C^2 = 0$;

Our hypothesis for factor A and B shall be $H_{0AB} : (AB)_1 = \cdots = (AB)_n = 0$;

Our hypothesis for factor A and C shall be $H_{0AC} : \sigma_{AC}^2 = 0$;

Our hypothesis for factor B and C shall be $H_{0BC} : \sigma_{BC}^2 = 0$;

And Our hypothesis for factor A, B and C shall be $H_{0ABC} : \sigma_{ABC}^2 = 0$.

Table 2. The extract data.

Time (mins) (C)	Solvents (B)	Volumes (A)			$X_{..k.}$	N_k	$\bar{X}_{..k.}$
		25	50	100			
40	1	1.0	5.0	8.0	145	20	7.25
		2.0	2.0	9.0			
		3.0					
	2	2.0	13.0	12.0			
		4.0		15.0			
				16.0			
	3	4.0	3.0	15.0			
		7.0	5.0	14.0			
		5.0					
50	1	12.0	16.0	2.0	222	21	10.5714
		11.0	17.0				
			14.0				
	2	13.0	8.0	13.0			
		15.0	9.0	15.0			
			8.0	11.0			
	3	2.0	10.0	14.0			
		5.0		11.0			
		4.0		12.0			

Continued

		6.0	12.0	5.0			
	1	5.0		4.0			
		2.0					
60		5.0	12.0	6.0	123	19	6.4737
	2	7.0	11.0				
	3	2.0	13.0	3.0			
		4.0	15.0	2.0			
		5.0		4.0			
	1	13.0	3.0	1.0			
		14.0	5.0	2.0			
				4.0			
	2	9.0	13.0	6.0	156	21	7.4286
70		9.0		4.0			
		7.0		5.0			
	3	2.0	6.0	15.0			
			8.0	13.0			
			5.0	12.0			
	$X_{i\ldots}$	180	213	253			
	N_i	29	23	29			
	$\bar{X}_{i\ldots}$	6.2069	9.2069	8.7241			
	$X_{..j.}$	178	248	220			
	N_i	26	26	29			
	$\bar{X}_{..j.}$	6.8462	9.8462	7.5862			

Table 3. Complete ANOVA table for extraction solution.

S.V	d.f	SS	MS	Expected mean square	f-ratio
A	2	144.9638	72.4819	$\frac{\sum_i^a N_i A_i^2}{a-1} + 7.7153\sigma_{AC}^2 + 1.2531\sigma_{ABC}^2 + \sigma_e^2$	0.7373
B	2	101.0706	50.5353	$\frac{\sum_j^b N_j B_j^2}{b-1} + 6.8274\sigma_{BC}^2 + 1.2531\sigma_{ABC}^2 + \sigma_e^2$	0.9649
C	3	201.1734	67.0578	$26.963\sigma_C^2 + 1.7325\sigma_{AC}^2 + 2.2840\sigma_{BC}^2 + 0.8354\sigma_{ABC}^2 + \sigma_e^2$	1.4645
AB	4	213.6512	53.4128	$\frac{\sum_{ij}^{ab} n_{ij.} E(AB)_{ij}^2}{(a-1)(b-1)} + 8.6482\sigma_{ABC}^2 + \sigma_e^2$	0.2757
AC	6	152.5572	86.2273	$11.2040\sigma_{AC}^2 + 0.4177\sigma_{ABC}^2 + \sigma_e^2$	1
BC	6	517.3638	25.3261	$0.4177\sigma_{BC}^2 + 67.3453\sigma_{ABC}^2 + \sigma_e^2$	1
ABC	12	2324.8692	193.7391	$4.8323\sigma_{ABC}^2 + \sigma_e^2$	99.4911
Error	54	87.6285	1.9473	σ_e^2	
Total	80				

NOTE: UEMS= UNBIASED ESTIMATES OF THE MEAN SQUARES.

5. Conclusion

In this study, we have presented a hypothesis testing procedure based on an unbalanced three-way cross-classification mixed effects model with interaction when factors A and B are fixed while factor C is random. From the theoretical results obtained in this study, it was observed that exact F-tests do not exist for any of the hypotheses to be tested. As a consequence, approximate F-tests were considered. A numerical example was given to illustrate theoretical our results.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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