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# Sub-Differential Characterizations of Non-Smooth Lower Semi-Continuous Pseudo-Convex Functions on Real Banach Spaces

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## Abstract

In this paper, we characterize lower semi-continuous pseudo-convex functions $f: X \to \mathbb{R} \cup \{+\infty\}$ on convex subset of real Banach spaces $K \subset X$ with respect to the pseudo-monotonicity of its Clarke-Rockafellar Sub-differential. We extend the results on the characterizations of non-smooth convex functions $f: X \to \mathbb{R} \cup \{+\infty\}$ on convex subset of real Banach spaces $K \subset X$ with respect to the monotonicity of its sub-differentials to the lower semi-continuous pseudo-convex functions on real Banach spaces.

## Keywords

Real Banach Spaces, Pseudo-Convex Functions, Pseudo-Monotone Maps, Sub-Differentials, Lower Semi-Continuous Functions and Approximate Mean Value Inequality

## 1. Introduction

Convex optimization, which studies the problem of minimizing convex functions over convex sets, plays important roles in many branches of applied mathematics. The foremost reason is that; it is very suitable to extremum problems. For instance, some necessary conditions for the existence of a minimum also become sufficient in the in terms of convexity. And convex optimization can be a smooth or a non-smooth convex optimization. Since the concept of convexity does not satisfy some mathematical models, various generalizations of convexity such as quasi-convexity and pseudo-convexity, which retain some important properties of convexity and equally provide a better representation of reality, were introduced in the literature to fill these gaps.

While the quasi-convexity property of a function guarantees the convexity of their sublevel sets, the pseudo-convexity property implies that the critical points are minimizers [1]. One of the features of convexity of functions is the relationship it has with the monotonicity of some maps. For example, a differentiable function is said to be convex if and only if its gradient is a monotone map. In non-smooth analysis, the generalized convexity of functions can be equally characterized in terms of the generalized monotonicity of their related operators [2].

The concepts of pseudo-convexity, traced to [3], within his research on analytical functions and independently introduced into the field of optimization by [4], have many applications in mathematical programming and economic problems [5] [6] [7]. And pseudo-monotonicity, introduced by [8] as a generalization of monotone operators, has been used to describe a property of consumer's demand correspondence [9]. Although the simplest class of pseudo-monotone operators consists of gradients of pseudo-convex functions, there are some monotone operators that are not sub-differentials [9]. And generalized monotonicity of maps is frequently used in complementarity problems, equilibrium problems and variational inequalities [10].

## 2. Preliminaries

Let $X$ be a real Banach space with norm $\|.\|$, $X^*$ be its topological dual and $\langle x^*, x\rangle$ be the duality pairing between $x \in X$ and $x^* \in X^*$. We denote the closed segment $[x, y] = \{\lambda x + (1-\lambda) y : \lambda \in [0,1]\}$ for $x, y \in X$, and define $(x, y]$, $[x, y)$ and $(x, y)$ similarly.

**Definition 2.1** [11] Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function, the effective domain is defined by

$$\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}.$$

**Definition 2.2** [12] A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous at $x \in X$ if and only if: $\forall \lambda \in \mathbb{R}$, such that $\lambda < f(x)$, $\exists V \subset U(x)$: $\lambda < f(y)$ $\forall y \in V$.

**Definition 2.3** [7] [13] A lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasi-convex, if for any $x, y \in X$ and $z \in [x, y]$ we have

$$f(z) \le \max\{f(x), f(y)\}. \tag{1}$$

**Definition 2.4** [7] [13] [14] A lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be strictly quasi-convex, if the inequality (1) is strict when $x \ne y$.

**Definition 2.5** [2] Let $T : X \to X^*$ be a multivalued operator with domain

$D(T)=\{x\in X:T(x)\neq\varnothing\}$. $T$ is said to be quasi-monotone if for any $x,y\in X$, $x^*\in T$ and $y^*\in T(y)$, we have

$$\langle x^*,y-x\rangle>0\Rightarrow\langle y^*,y-x\rangle\geq 0.$$

**Definition 2.6** [2] Let $T:X\to X^*$ be a multivalued operator with domain $D(T)=\{x\in X:T(x)\neq\varnothing\}$. $T$ is said to be pseudo-monotone if for any $x,y\in X$, $x^*\in T$ and $y^*\in T(y)$, we have

$$\langle x^*,y-x\rangle\geq 0\Rightarrow\langle y^*,y-x\rangle\geq 0. \tag{2}$$

**Definition 2.7** [7] Let $T:X\to X^*$ be a multivalued operator with domain $D(T)=\{x\in X:T(x)\neq\varnothing\}$. $T$ is said to be strictly pseudo-monotone if for any different two points $x,y\in X$, $x^*\in T$ and $y^*\in T(y)$, we have

$$\langle x^*,y-x\rangle\geq 0\Rightarrow\langle y^*,y-x\rangle>0. \tag{3}$$

**Definition 2.8** [15] An operator $\partial$ that associates to any lower semi-continuous function $f:X\to\mathbb{R}\bigcup\{+\infty\}$ and a point $x\in X$ a subset $\partial f(x)$ of $X^*$ is a sub-differential if it satisfies the following properties:

1) $\partial f(x)=\{x^*\in X^*:\langle x^*,y-x\rangle+f(x)\leq f(y),\forall y\in X\}$, whenever $f$ is convex;

2) $0\in\partial f(x)$, whenever $x\in\operatorname{dom}f$ is a local minimum of $f$;

3) $\partial(f+g)(x)\subset\partial f(x)+\partial g(x)$, whenever $g$ is a real a real-valued convex continuous function which is $\partial$-differentiable at $x$.

Where $g$-differentiable at $x$ means that both $\partial g(x)$ and $\partial(-g)(x)$ are non-empty. We say that $f$ is $\partial$-differentiable at $x$ when $\partial f(x)$ is non-empty while $\partial f(x)$ are called the sub-gradients of $f$ at $x$.

**Definition 2.9** [15] The Clarke-Rockafellar generalized directional derivative of $f$ at $x_0\in\operatorname{dom}(f)$ in the direction $d\in X$ is given by

$$f^{\uparrow}(x_0,d)=\sup_{\varepsilon>0}\limsup_{\substack{x\to_f x_0\\ \lambda\searrow 0}}\inf_{d'\in B_\varepsilon(d)}\frac{f(x+\lambda d')-f(x)}{\lambda}, \tag{4}$$

where $B_\varepsilon(d)=\{d'\in X:\|d'-d\|<\varepsilon\}$, $\lambda\searrow 0$ indicates the fact that $\lambda>0$ and $\lambda\to 0$, and $x\to f^{x_0}$ means that both $x\to x_0$ and $f(x)\to f(x_0)$;

While,

**Definition 2.10** [15] The Clarke-Rockafellar sub-differential of $f$ at $x_0$ is defined by

$$\partial f(x_0)=\{x^*\in X^*:(x^*,d)\leq f^{\uparrow}(x_0,d),\forall d\in X\}; \tag{5}$$

if $x_0\in X\setminus\operatorname{dom}(f)$, then

$$\partial f(x_0)=\varnothing, \text{[7]}.$$

**Definition 2.11** [2] [7] A lower semi-continuous function $f:X\to\mathbb{R}\bigcup\{+\infty\}$ is said to be quasi-convex (with respect to Clarke-Rockerfeller Sub-differentials) if for any $x,y\in X$,

$$\exists x^*\in\partial f(x):\langle x^*,y-x\rangle>0\Rightarrow\forall z\in[x,y],\ f(z)\leq f(y). \tag{6}$$

**Definition 2.12** [7] [16] A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be pseudo-convex (with respect to Clarke-Rockerfeller Subdifferentials) if for any $x, y \in X$:

$$\exists x^* \in \partial f(x): \langle x^*, y-x \rangle \geq 0 \Rightarrow f(x) \leq f(y). \tag{7}$$

**Definition 2.13** [2] [7] [14] A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be strictly pseudo-convex (with respect to Clarke-Rockerfeller Subdifferentials) if for any two different points $x, y \in X$:

$$\exists x^* \in \partial f(x): \langle x^*, y-x \rangle \geq 0 \Rightarrow f(x) < f(y), \text{ when } x \neq y. \tag{8}$$

**Definition 2.14** [7] A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be radially continuous if for all $x, y \in X$, $f$ is continuous on $[x, y]$.

**Definition 2.15** [7] A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be radially non-constant if for all $x, y \in X$, with $x \neq y$, $f \not\equiv \text{constant}$ on $[x, y]$.

**Definition 2.16** A sub-differential operator $\partial f: X \to X^*$ is said to said to be quasi-monotone if for any $x, y \in X$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$, we have

$$\langle x^*, y-x \rangle > 0 \Rightarrow \langle y^*, y-x \rangle \geq 0. \tag{9}$$

**Definition 2.17** A sub-differential operator $\partial f: X \to X^*$ is said to said to be quasi-monotone if for any $x, y \in X$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$, we have

$$\langle x^*, y-x \rangle \geq 0 \Rightarrow \langle y^*, y-x \rangle \geq 0. \tag{10}$$

**Theorem 2.1.** (Approximate mean value inequality). Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a Clarke-Rockafellar sub-differentiable lower semi-continuous (l.s.c.) function on a Banach space *X*. Let $a, b \in X$ with $a \in \operatorname{dom} f$ and $a \neq b$. Let $\rho \in \mathbb{R}$ be such that $\rho \leq f(b)$. Then, there exist $c \in [a, b)$ and $x_n \to f^C$ and $x_n^* \in \partial f(x_n)$ such that

1) $\liminf_{n \to +\infty} \langle x_n^*, c - x_n \rangle \geq 0$;
2) $\liminf_{n \to +\infty} \langle x_n^*, b - a \rangle \geq \rho - f(a)$.

**Proof.** [15].

**Lemma 2.2.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a Clarke-Rockafeller sub-differentiable lower semi-continuous (l.s.c.) function on a Banach space *X*. Let $a, b \in X$ with $f(a) < f(b)$. Then, there exist $c \in [a, b)$, and two sequences $c_n \to c$, and $c_n^* \in \partial f(c_n)$ with

$$\langle c_n^*, x - c_n \rangle > 0 \text{ for every } x = c + \lambda(b-a) \text{ with } \lambda > 0.$$

**Proof.** By **Theorem 2.1**, there exists an $x_0 \in [a, b)$ and a sequence $x_n \to f^C$ and $x_n^* \in \partial f(x_n)$ verifying

$$\liminf_{n \to +\infty} \langle x_n^*, c - x_n \rangle \geq 0 \text{ and } \liminf_{n \to +\infty} \langle x_n^*, b - a \rangle > 0. \tag{11}$$

Putting $x = c + \lambda(b-a)$ with $\lambda > 0$ it holds

$$\langle x_n^*, x - x_n \rangle = \langle x_n^*, c - x_n \rangle + \lambda \langle x_n^*, b - a \rangle > 0 \tag{12}$$

for *n* very large. ■

We consider the relationship between pseudo-convexity and quasi-convexity.

**Theorem 2.3.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller subdifferentiable function on a Banach space *X*. Then, *f* is quasi-convex if and only if $\partial f$ is quasi-monotone.

**Proof.** We show that if *f* is not quasi-convex, then $\partial f$ is not quasi-monotone.

Suppose that there exist some $x, y, z$ in *X* with $z \in [x, y]$ and $f(z) > \max\{f(x), f(y)\}$. According to Lemma 2.2 applied with $a = x$ and $b = z$, there exists a sequence $y_n \in \operatorname{dom}\partial f$ and $y_n^* \in \partial f(y_n)$ such that

$$y_n \to \bar{y} \in [x, z], \quad \bar{y} \neq z \quad \text{and} \quad \langle y_n^*, y - y_n \rangle > 0. \tag{13}$$

Let $0 < \lambda \leq 1$ be such that $z = \bar{y} + \lambda(y - \bar{y})$ and set $z_n = y_n + \lambda(y - y_n)$, so that $z_n \to z$. Since *f* is lower semi-continuous, we may pick $n \in \mathbb{N}$ very large with $f(z_n) > f(y)$. Apply Lemma 2.2 again with $a = y$ and $b = z_n$ to find sequences $x_k \in \operatorname{dom}\partial f$, $x_k^* \in \partial f(x_k)$ such that

$$x_k \to \bar{x} \in [y, z_n], \quad \bar{x} \neq z_n \quad \text{and} \quad \langle x_k^*, y_n - x_k \rangle > 0. \tag{14}$$

In particular, $\bar{x} \neq y_n$ and

$$\langle y_n^*, \bar{x} - y_n \rangle = \frac{\|\bar{x} - y_n\|}{\|y - y_n\|} \langle y_n^*, y - y_n \rangle > 0; \tag{15}$$

hence, $\langle y_n^*, x_k - y_n \rangle > 0$ for *k* sufficiently large. But $\langle y_n^*, y_n - x_k \rangle > 0$, showing that $\partial f$ is not quasi-monotone.

Conversely, we suppose that *f* is quasi-convex and show that $\partial f$ is quasi-monotone. Let $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ with $\langle x^*, y - x \rangle > 0$. We need to verify that $f^{\uparrow}(y, x - y) \leq 0$. We fix $\varepsilon > 0$ and $\omega \in (0, \varepsilon)$ such that $\langle x^*, v - x \rangle > 0$ for all $v \in B_\omega(y)$.

We fix $v \in B_\omega(y)$. Since $f^{\uparrow}(y, x - y) > 0$ we can find $\varepsilon' \in (0, \varepsilon - \omega)$, $u \in B_{\varepsilon'}(x)$ and $t \in (0,1)$ such that $f(u + t(v - u)) > f(u)$. From the quasi-convexity of *f* we deduce that $f(u) < f(v)$, whence,

$$f(v + \lambda(u - v)) \leq f(v) \quad \text{for all} \quad \lambda \in (0,1),$$

so that

$$\inf_{\mu \in B_\varepsilon(x-y)} \frac{f(v + \lambda\mu) - f(v)}{\lambda} \leq \frac{f(v + \lambda(u - v)) - f(v)}{\lambda} \leq 0 \quad \text{for all} \quad \lambda \in (0,1).$$

Combining the inequalities and for any $\varepsilon > 0$ there exists $\omega > 0$ such that

$$\sup_{\substack{v \in B_\omega(y) \\ \lambda \in (0,1)}} \left[ \inf_{\mu \in B_\varepsilon(x-y)} \frac{f(v + \lambda\mu) - f(v)}{\lambda} \right] \leq 0,$$

which shows that $f^{\uparrow}(y, x - y) \leq 0$. ■

## 3. Sub-Differential Characterization of Pseudo-Convex Functions

**Theorem 3.1.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) function on a Banach space $X$ such that $f$ Clarke-Rockafeller suddifferentiable. Consider the following assertions:

(i) $f$ is pseudoconvex.

(ii) $f$ is quasiconvex and ( $0 \in \partial f(x) \Rightarrow x$ is a global minimum of $f$).

Then, (i) implies (ii). And (ii) implies (i) if $f$ is radially continuous.

**Proof.** (i) $\Rightarrow$ (ii). We want to prove that $f$ is quasiconvex. Suppose to the contrary that for some $x, y \in X$, $z \in (x, y)$ we have $f(z) > \max\{f(x), f(y)\}$. Since $f$ is lower semicontinuous, we can find some $\varepsilon > 0$ such that $f(z') > \max\{f(x), f(y)\}$, for all $z' \in B_\varepsilon(z)$. Since $z$ cannot be a local nor global minimizers, there exist some $v \in B_\varepsilon(z)$ such that $f(v) < f(z)$. From **Lemma 2.2**, there exist $u_n \to u \in [v, z)$ and $u_n^* \in \partial f(u_n^*)$ such that

$$\langle u_n^*, z - u_n \rangle > 0.$$

But since $z \in (x, y)$, either of the following must hold

$$\langle u_n^*, x - u_n \rangle > 0 \text{ or } \langle u_n^*, y - u_n \rangle > 0.$$

Therefore,

$$f(u_n) \le \max\{f(x), f(y)\}.$$

which is a contradiction.

(ii) $\Rightarrow$ (i). Let $x \in \operatorname{dom}\partial f$, $y \in X$, and $x^* \in \partial f(x)$ such that $\langle x^*, y - x \rangle \ge 0$. If $0 \in \partial f(x)$, then $x$ is a global minimum of $f$ and $f(x) \le f(y)$ in particular. Otherwise, $[0 \notin \partial f(x)]$, there exist $d \in X$ such that $\langle x^*, d \rangle > 0$. We define a sequence $\{y_n\}$ by

$$y_n = y + \left(\frac{1}{2n\|d\|}\right) d.$$

For every $n \in \mathbb{N}$, the point $y_n$ satisfies

$$y_n \in B_{1/n}(y),$$

$$\langle x^*, y_n - x \rangle = \langle x^*, y_n - y \rangle + \langle x^*, y - x \rangle \ge \left(\frac{1}{2n\|d\|}\right) \langle x^*, d \rangle > 0.$$

Using (7), we obtain that, for every $n$, $f(y_n) \ge f(x)$ and by radial continuity of $f$, $f(y) \ge f(x)$. ■

**Theorem 3.2.** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller sub-differentiable function. Consider the following assertions:

(i) $f$ is pseudo-convex.

(ii) $\partial f$ is pseudo-monotone

Then, (i) implies (ii). And (ii) implies (i) if $f$ is radially continuous.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $x^* \in \partial f(x)$ such that $\langle x^*, y - x \rangle \ge 0$. By **Theorem 3.1**, $f$ is quasi-convex. By **Theorem 2.3**, we conclude that $\partial f$ is quasi-monotone. Hence, $\langle y^*, y - x \rangle \ge 0$, for all $y^* \in \partial f(y)$. Suppose to the contrary that for some $y^* \in \partial f(y)$, we have $\langle y^*, y-x\rangle = 0$. From (7), we obtain $f(x) \geq f(y)$.

However, since $f^{\uparrow}(x, y-x) > 0$, there exist $\varepsilon > 0$, such that for some $x_n \to x$, $\lambda_n \searrow 0$ and for all $y' \in B_\varepsilon(y)$, we have $f(x_n + t_n(y'-x_n)) > f(x_n)$. By the quasiconvexity of *f*, it implies that $f(y') > f(x_n)$ for every $y' \in B_\varepsilon(y)$. In particular, $f(y) > f(x)$ because *f* is lower semicontinuous. Thus, $f(y') \geq f(y)$. This shows that *y* is a local minimum and also a global minimum, which is a contradiction since we can have that $f(y) > f(x_n)$.

(ii) $\Rightarrow$ (i). Using Theorem, we prove that $f$ is pseudoconvex. Since $\partial f$ is pseudomonotone, $\partial f$ is quasimonotone. By Theorem 3.1, $f$ is quasiconvex. On the other hand, if *x* is not a minimizer of *f*, there exists $y \in X$ such that $f(y) < f(x)$. Using **Lemma 2.2,** we find $u \in \text{dom}\partial f$ and $u^* \in \partial f(u)$ such that $\langle u^*, x-u\rangle > 0$ and by the pseudo-monotonicity of $\partial f$, $\langle x^*, x-u\rangle > 0$ for every $x^* \in \partial f(x)$. Hence, 0 does not $\partial f(x)$. Consequently, *f* satisfies condition $0 \in \partial f(x)$, which implies that *x* is a global minimum of *f*, which completes the proof. ■

**Theorem 3.3.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller subdifferentiable function on a Banach space *X*. Consider the following assertions:

(i) $f$ is strictly pseudoconvex.

(ii) $f$ is strictly quasiconvex and ($0 \in \partial f(x) \Rightarrow x$ is a global minimum of *f*),

Then, (i) implies (ii). And (ii) implies (i) if $f$ is radially continuous.

**Proof.** (i) $\Rightarrow$ (ii). We want to prove that $f$ is strictly quasiconvex. Let *f* be a strictly pseudo-convex function, then by **Theorem 3.1,** the function $f$ is quasiconvex and satisfies the optimality condition

$$0 \in \partial f(x) \Rightarrow \ (x \text{ is a global minimum of } f).$$

Since $f$ is quasiconvex, then according to [13], it suffices to prove that $f$ is radially non-constant. Assume by contradiction that there exists a closed segment $[x, y]$ with $x \neq y$ where with $f$ is constant. Let $z \in (x, y)$ and apply the strict pseudo-convexityproperty to *x* and *z*, then

$$f(z) = f(x) \Rightarrow \left(\forall z^* \in \partial f(z) : \langle z^*, x-z\rangle < 0\right).$$

Using the same argument for *z* and *y* we obtain

$$f(z) = f(y) \Rightarrow \left(\forall z^* \in \partial f(z) : \langle z^*, y-z\rangle < 0\right).$$

Since $\partial f(z)$ is nonempty, it follows that for all $z^* \in \partial f(z)$, $\langle z^*, x-y\rangle < 0$ and $\langle z^*, x-y\rangle > 0$), which is a contradiction.

(ii) $\Rightarrow$ (i). Assume that *f* satisfies condition ii) and $f$ is radially continuous. Then by Theorem 3.1, $f$ is pseudoconvex. We prove that $f$ is pseudo-convex. Suppose by contradiction that there exist $x \neq y$ in *X* and $x^* \in \partial f(x)$ such that

$$\langle x^*, y-x\rangle \geq 0 \quad \text{and} \quad f(x) \geq f(y).$$

Then, it follows by pseudo-convexity property that

$$\forall z \in [x, y], \quad f(z) = f(x).$$

Since $f$ is quasi-convex, then we have

$$\forall z \in [x, y], \quad f(z) \geq f(x) \geq f(y).$$

So $f$ is not radially non-constant on $X$ (since $f$ is constant on $[x, y]$) which contradicts the fact $f$ is strictly quasi-convex.

**Theorem 3.4.** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) function such that $f$ is radially Clarke-Rockafeller differentiable. Consider the following assertions:

(i) $f$ is strictly pseudo-convex.

(ii) $\partial f$ is strictly pseudomonotone

Then, (i) implies (ii). And (ii) implies (i) if $f$ is radially continuous.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $f$ is strictly pseudoconvex. We want to prove that $\partial f$ is strictly pseudomonotone. Suppose to the contrary that there exist two distinct points $x, y \in X$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ such that

$$\langle x^*, y - x \rangle \geq 0 \text{ and } \langle y^*, y - x \rangle \leq 0.$$

Since $f$ is strictly pseudoconvex, we have that

$$f(x) < f(y) \text{ and } f(y) < f(x).$$

Which is a contradiction. Therefore, $\partial f$ is strictly pseudomonotone.

(ii) $\Rightarrow$ (i). Suppose that $f$ satisfies condition (ii) and $f$ is radially continuous. We want to prove that $f$ is strictly pseudoconvex. Suppose to the contrary that there exist two distinct points $x, y \in X$, and $x^* \in \partial f(x)$ such that

$$\langle x^*, y - x \rangle \geq 0 \text{ and } f(x) \geq f(y).$$

Then,

$$\langle x^*, z - x \rangle \geq 0 \text{ for all } z \in [x, y]. \tag{16}$$

By theorem 3.2, $f$ is quasiconvex. Consequently, $f$ must be constant on $[x, y]$. Contrarily, from (15) and the strict monotonicity of $\partial f(x)$, we have

$$\langle x^*, z - x \rangle > 0, \forall z \in (x, y) \text{ and } \forall z^* \in \partial f(z). \tag{17}$$

Pick $z_0 \in (x, y)$ such that $\partial f(z_0) \neq \varnothing$ (such a $z_0$ exists since $f$ is a radially Clarke-Rockafeller subdifferentiable function). Choose any $z_0^* \in \partial f(z_0)$. Then, $\langle z_0^*, z_0 - x \rangle > 0$. Therefore, $\langle z_0^*, y - z_0 \rangle > 0$. Consequently, there exist $\varepsilon > 0$ such that

$$\langle z_0^*, y' - z_0 \rangle > 0 \text{ for all } y' \in B_\varepsilon(y).$$

By the pseudo-convexity of $f$, it follows that $y$ is a global minimum of $f$. Hence, $z_0$ is also a global minimum of $f$. Thus, $0 \in \partial f(z_0)$ and this is a contradiction with (17).

## 4. Conclusion

We extended the relationships between convex functions and corresponding monotone maps to pseudo-convexity and the corresponding pseudo-monotonicity of their sub-differentiable maps. We characterized the lower semi-continuous Clarke-Rockafeller sub-differentiable pseudo-convex functions by the corresponding monotonicity of their Clarke-Rockafeller sub-differentials $\partial f$, and have shown that if a lower semi-continuous Clarke-Rockafeller sub-differentiable function $f : X \to \mathbb{R} \cup \{+\infty\}$ is radially continuous, then $f$ is pseudo-convex if and only if the sub-differential map $\partial f$ is pseudo-monotone.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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