Scientific Research Publishing

Open Journal of Optimization, 2022, 11, 7-14
https://www.scirp.org/journal/ojop
ISSN Online: 2325-7091
ISSN Print: 2325-7105

On the Regularization Method for Solving Ill-Posed Problems with Unbounded Operators

Nguyen Van Kinh

Faculty of Applied Science, Ho Chi Minh University of Food Industry, Ho Chi Minh City, Vietnam
Email: nguyenvankinh58@gmail.com

How to cite this paper: Van Kinh, N. (2022) On the Regularization Method for Solving Ill-Posed Problems with Unbounded Operators. *Open Journal of Optimization*, **11**, 7-14.
https://doi.org/10.4236/ojop.2022.112002

Received: April 16, 2022
Accepted: June 11, 2022
Published: June 15, 2022

Abstract

Let $A: D(A) \subset X \to Y$ be a linear, closed, and densely defined unbounded operator, where *X* and *Y* are Hilbert spaces. Assume that *A* is not boundedly invertible. Suppose the equation $Au = f$ is solvable, and instead of knowing exactly *f* only know its approximation f_δ satisfies the condition: $\| f_\delta - f \| \le \delta, 0 < \delta \to 0.$ In this paper, we are interested a regularization method to solve the approximation solution of this equation. This approximation is a unique global minimizer $u_{\alpha,\delta}$ of the functional $F_\delta(u) := \|Au - f_\delta\|^2 + \alpha \|u\|^2$, for any $f_\delta \in Y$, defined as follows: $u_{\alpha,\delta} = A^*\left(AA^* + \alpha I_X\right)^{-1} f_\delta$. We also study the stability of this method when the regularization parameter is selected a priori and a posteriori. At the same time, we give an application of this method to the weak derivative operator equation in Hilbert space $H = L^2[0,1]$.

Keywords

Ill-Posed Problem, Regularization Method, Unbounded Linear Operator

1. Introduction

Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined unbounded operator, where *X* and *Y* are Hilbert spaces. Consider the equation

$$Au = f \tag{1}$$

Problem-solving solution of Equation (1) is called ill-posed [1] if *A* is not boundedly invertible. This may happen if the null space $N(A) = \{u : Au = 0\}$ is not trivial, *i.e.* *A* is not injective, or if *A* is injective but A^{-1} is unbounded, *i.e.* the range of *A*, $R(A)$ is not closed [2].

If $\|A\| < \infty$, problem-solving stable solution of Equation (1) has been extensively studied in the literature in detail ([2] [3] [4] [5] [6] and references therein).

If f_δ, the noisy data, are given

$$\| f_\delta - f \| \le \delta \quad (2)$$

is a stable approximation to the unique minimal norn solution to Equation (1) was constructed by several methods (variational regularization, quasi solution, iterative regularization, ... [2] [3] [4] [5] [6] and references therein).

If A is a linear, closed, densely defined unbounded operator, problem (1) has been some recent research [2] [7] [8] [9] [10] [11], however, there are still many open problems such as parameter choice rules of regularization method with the linear closed, densely defined unbounded operator $A: D(A) \subset X \to Y$.

Our aim is to study problem-solving stable approximation solution of Equation (1) when operator A is a linear, closed, and densely defined from space Hilbert X into space Hilbert Y. We shall present the regularization method for solving the problem (1), we shall present a priori and a posteriori parameter choice rules of regularization; at the same time give an application to the weak derivative operator equation.

The paper structure consists of 3 sections: Section 1 the introduction briefly summarizes the recent research results and come up with the problem that needs to be studied; Section 2 presents some main results; Section 3 presents an application of this method.

2. Some Main Results

Lemma 1. [2] Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces, then

1) the operators $T = A^*A$ and $Q = AA^*$ are densely defined, self-adjoint;

2) A^* is closed, densely defined and $A^{**} = A$;

3) the operators $\tilde{A} := \left(I_X + A^*A\right)^{-1} : X \to Y$, $A\tilde{A}: X \to Y$ are both defined on all of X and are bounded, $\sigma\left(\tilde{A}\right) \subseteq [0,1]$. Also, $\tilde{A}$ is self-adjoint;

4) the operator $\hat{A} := \left(I_Y + AA^*\right)^{-1} : Y \to X$ is bounded and self-adjoint and $A^*\hat{A}: Y \to X$ is bounded.

Lemma 2. Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces. If $f = Ay$, $y \perp N(A)$ then y is unique.

Proof. Suppose y_1, and y_2 satisfy $f = Ay_1$, $y_1 \perp N(A)$, and $f = Ay_2$, $y_2 \perp N(A)$ then $A(y_1 - y_2) = 0$. Thus $y_1 - y_2 \in N(A)$. There exits $u \in N(A)$ such that $y_1 - y_2 = u$ imply $\langle y_1 - y_2, u \rangle = \langle y_1, u \rangle - \langle y_2, u \rangle = 0 = \langle u, u \rangle$. Thus $u = 0$, it follows that $y_1 = y_2$.

Theorem 1. For any $f \in Y$, the problem

$$F(u) = \|Au - f\|^2 + \alpha \|u\|^2 \to \min, \alpha = \text{const} > 0, \quad (3)$$

has a unique solution $u_\alpha = A^*\left(AA^* + \alpha I_Y\right)^{-1} f$, where I_Y is the identity operator on Y.

Proof. Consider the equation

$$\left(AA^* + \alpha I_Y\right) w_\alpha = f, \alpha = \text{const} > 0 \quad (4)$$

which is uniquely solvable $w_\alpha = \left(AA^* + \alpha I_Y\right)^{-1} f$ (Lemma 1). Let $u_\alpha = A^* w_\alpha$ then

$$\begin{aligned} Au_\alpha &= AA^*\left(AA^* + \alpha I_Y\right)^{-1} f \\ &= \left(AA^* + \alpha I_Y\right)\left(AA^* + \alpha I_Y\right)^{-1} f - \alpha I_Y \left(AA^* + \alpha I_Y\right)^{-1} f \\ &= f - \alpha w_\alpha, \end{aligned}$$

or

$$Au_\alpha - f = -\alpha w_\alpha.$$

We have

$$F(u+v) = \|Au - f\|^2 + \alpha\|u\|^2 + \|Av\|^2 + \alpha\|v\|^2 + 2\,\mathrm{Re}\left[(Au - f, Av) + \alpha(u,v)\right], \quad (5)$$

for any $v \in D(A)$. If $u = u_\alpha$, then

$$\begin{aligned} (Au_\alpha - f, Av) + \alpha(u_\alpha, v) &= -\alpha(w_\alpha, Av) + \alpha(u_\alpha, v) \\ &= -\alpha\left(A^* w_\alpha, v\right) + \alpha(u_\alpha, v) = 0. \end{aligned} \quad (6)$$

Thus Equation (6) implies

$$F(u_\alpha + v) = F(u_\alpha) + \|Av\|^2 + \alpha\|v\|^2 \ge F(u_\alpha) \quad (7)$$

and $F(u_\alpha + v) = F(u_\alpha)$ if and only if $v = 0$, so u_α is the unique minimier of $F(u)$.

Theorem 1 is proved.

Theorem 2. If $f = Ay$, $y \perp N(A)$ then

$$\lim_{\alpha \to 0}\|u_\alpha - y\| = 0, u_\alpha = A^*\left(AA^* + \alpha I\right)^{-1} f. \quad (8)$$

Proof. It follows from Lemma 2, y is unique. Write Equation (4) as $A\left(A^* w_\alpha - y\right) = -\alpha w_\alpha$. Apply A^*, which is possible because $w_\alpha \in D\left(A^*\right)$, we obtain

$$A^* A(u_\alpha - y) = -\alpha u_\alpha. \quad (9)$$

Multiply Equation (9) by $u_\alpha - y$, we obtain

$$\left(A^* A(u_\alpha - y), u_\alpha - y\right) = -\alpha(u_\alpha, u_\alpha - y)$$

or

$$\|A(u_\alpha - y)\|^2 = -\alpha\left(\|u_\alpha\|^2 - (u_\alpha, y)\right). \quad (10)$$

Since $\alpha > 0$ this implies

$$\|u_\alpha\|^2 \le (u_\alpha, y),$$

so

$$\|u_\alpha\| \le \|y\|, \forall \alpha > 0.$$

Therefore one may assume (taking a subsequence) that u_α weakly converges to an element z, $u_n := u_{\alpha_n} \dagger\ z$, as $\alpha_n \to 0$.

It follows from Equation (10) that

$$\lim_{n\to\infty}\left\|A\left(u_n-y\right)\right\|=0,\ i.e.\ \lim_{n\to\infty}\left\|Au_n-f\right\|=0.$$

We shall prove that $z=y$.

Let γ run through the set such that $\left\{A^*A\gamma\right\}$ is dense in $N^\perp$, where $N:=N(A)$. Note that $N(T)=N(A)$, where $T=A^*A$. Because of the formulas $X=\overline{R(T)}\oplus N(T)$ the $\{\gamma\}=D(T)$ is dense in X, and the set $\{T\gamma\}$ is dense in $N^\perp$.

Multiply the equation $T\left(u_\alpha-y\right)=-\alpha u_\alpha$ by γ and pass to the limit $\alpha\to 0$. We obtain

$$\left(z-y,T\gamma\right)=0.$$

We have assume $y\perp N$. If $z\perp N$, then $z-y\perp N$ and $z-y\perp N^\perp$, so $z-y=0$.

One may always assume that $z\perp N$ because $Tu_\alpha=Tu_\alpha$, where u_α is the orthogonal projection of u_α onto $N^\perp$.

Thus, we have $u_n:=u_{\alpha_n}\dagger\ z$, $\|u_n\|\le\|y\|$. Thus implies $\lim_{n\to\infty}\|u_n-y\|=0$.

For convenience for the reader we prove this claim. Since $u_n:=u_{\alpha_n}\dagger\ z$, one gets $\|y\|\le\underline{\lim}_{n\to\infty}\|u_n\|$. The inequality $\|u_n\|\le\|y\|$ implies $\overline{\lim}_{n\to\infty}\|u_n\|\le\|y\|$. Therefore $\lim_{n\to\infty}\|u_n\|=\|y\|$. This and the weakly converge $u_n:=u_{\alpha_n}\dagger\ z$ imply strong convergence

$$\|u_n-y\|^2=\|u_n\|^2+\|y\|^2-2\operatorname{Re}\left(u_n-y\right)\to 0,\ \text{as } n\to\infty.$$

Theorem 2 is proved.

Theorem 3. If $\|f_\delta-f\|\le\delta$, $f=Ay$, $y\perp N(A)$ and

$$F_\delta(u)=\|Au-f_\delta\|^2+\alpha\|u\|^2=\min, \tag{11}$$

then there exists a unique global minimier $u_{\alpha,\delta}$ to (11) and $\lim_{\delta\to 0}\|u_\delta-y\|=0$, where $u_\delta:=u_{\alpha(\delta),\delta}$ and $\alpha(\delta)$ is properly chosen, in particular $\lim_{\delta\to 0}\alpha(\delta)=0$.

Proof. It follows from Lemma 2, *y* is unique. The existence and uniqueness of the minimizer $u_{\alpha,\delta}$ of $F_\delta(u)$ follows from Theorem 1 and $u_{\alpha,\delta}=A^*\left(Q+\alpha I_Y\right)^{-1}f_\delta$. We have

$$\|u_{\alpha,\delta}-y\|\le\|u_{\alpha,\delta}-u_\alpha\|+\|u_\alpha-f\|.$$

By Theorem 2, $\|u_\alpha-f\|:=\eta(\alpha)\to 0$, as $\alpha\to 0$.

Let us estimate

$$\|u_{\alpha,\delta}-u_\alpha\|=\left\|A^*\left(Q+\alpha I_Y\right)^{-1}\left(f_\delta-f\right)\right\|\le\delta\left\|A^*\left(Q+\alpha I_Y\right)^{-1}\right\|.$$

By the polar decomposition theorem [12], one has $A^*=UQ^{1/2}$, where *U* is a partial isometry, so $\|U\|\le 1$. One has,

$$\left\|A^*\left(Q+\alpha I_Y\right)^{-1}\right\|=\left\|UQ^{1/2}\left(Q+\alpha I_Y\right)^{-1}\right\|\le\left\|Q^{1/2}\left(Q+\alpha I_Y\right)^{-1}\right\|$$
$$=\max_{\lambda\ge 0c}\frac{\lambda^{1/2}}{\lambda+\alpha}=\frac{1}{2\sqrt{\alpha}},$$

where the spectral representation for Q was used.

Thus

$$\left\|u_{\alpha,\delta}-y\right\|\le\frac{\delta}{2\sqrt{\alpha}}+\eta(\alpha). \tag{12}$$

For a fixed small $\delta>0$, choose $\alpha=\alpha(\delta)$ which minimizes the right side of Equation (12). Then $\lim_{\delta\to 0}\alpha(\delta)=0$ and $\lim_{\delta\to 0}\left(\frac{\delta}{2\sqrt{\alpha(\delta)}}+\eta(\alpha(\delta))\right)=0.$

Theorem 3 is proved.

Remark 1. We can also choose $\alpha(\delta)=c\delta^k$, with any $k<2$ and $c=\text{const}>0$. The constant c can be arbitrary.

We can also choose $\alpha(\delta)$ by a descrepancy principle. For example, consider the equation for finding $\alpha(\delta)$:

$$\left\|Au_{\alpha,\delta}-f_\delta\right\|=c\delta,\ c=\text{const}>1.$$

We assume that $\|f_\delta\|>c\delta$.

That is the content of the following theorem.

Theorem 4. The equation

$$\left\|Au_{\alpha,\delta}-f_\delta\right\|=c\delta,\ c=\text{const}>1,\ \|f_\delta\|>c\delta, \tag{13}$$

has a unique solution $\alpha=\alpha(\delta)>0$, $\lim_{\delta\to 0}\alpha(\delta)=0$, and if $u_\delta:=u_{\alpha(\delta),\delta}$, then $\lim_{\delta\to 0}\|u_\delta-y\|=0$.

Proof. Let us prove that Equation (13) has a unique root $\alpha(\delta)>0$, $\lim_{\delta\to 0}\alpha(\delta)=0$. Indeed, using the spectial theorem [12], one gets

$$\left\|Au_{\alpha,\delta}-f_\delta\right\|^2=\left\|\left[AA^*(Q+\alpha I)\right]^{-1}f_\delta\right\|^2=\int_0^\infty\left|\frac{s}{s+\alpha}-1\right|^2 d\left(E_s,f_\delta,f_\delta\right)$$
$$=\alpha^2\int_0^\infty\frac{d\left(E_s,f_\delta,f_\delta\right)}{(s+\alpha)^2}:=g(\alpha,\delta),$$

where E_s is the resolution of the identity of Q.

One has $g(\infty,\delta)=\|f_\delta\|^2>c^2\delta^\delta$, and $g(+0,\delta)=\left\|P_{N^*}f_\delta\right\|^2$, where P_{N^*} is the orthoprojector onto the subspace $N^*=N(Q)=N\left(A^*\right)=R(A)^\perp$.

Since $f\in R(A)$ and $\|f_\delta-f\|\le\delta$, it follows that $\left\|P_{N^*}f_\delta\right\|\le\delta$, so $g(+0,\delta)\le\delta^2$. The function $g(\alpha,\delta)$ for a fixed $\delta>0$ is a continuous strictly increasing function of α on $[0,\infty)$. Therefore there exists a unique $\alpha=\alpha(\delta)>0$ which solves Equation (13) if $\|f_\delta\|>c\delta$ and $c>1$. Clearly $\lim_{\delta\to 0}\alpha(\delta)=0$, because $\lim_{\delta\to 0}c\alpha(\delta)=0$ and the relation $\lim_{\delta\to 0}\alpha^2(\delta)\int_0^\infty\frac{d\left(E_s,f_\delta,f_\delta\right)}{\left(s+\alpha(\delta)\right)^2}=0$ implies $\lim_{\delta\to 0}\alpha(\delta)=0$. The function $\alpha=\alpha(\delta)$ is a monotonically growing function of

δ with $\alpha(+0)=0$.

Let us prove that $\lim_{\delta\to0}\|u_\delta - y\|=0$, where $u_\delta := u_{\alpha(\delta),\delta}$, and $\alpha(\delta)$ solves Equation (13). By the definition of u_δ, we get

$$\|Au_\alpha - f_\delta\|^2 + \alpha(\delta)\|u_\delta\|^2 \le \|Ay - f_\delta\|^2 + \alpha(\delta)\|y\|^2 = \delta^2 + \alpha(\delta)\|y\|^2.$$

Since $\|Au_\alpha - f_\delta\|^2 = c^2\delta^2 > \delta^2$, it follows that $\|u_\delta\| \le \|y\|$. Thus u_δ † z, and, as in the proof of Theorem 2, we obtain $z=y$ and $\lim_{\delta\to0}\|u_\delta - y\|=0$.

Theorem 4 is proved.

Remark 2. Theorems 1 - 4 are well known in the case of a bounded operator *A*.

If *A* is bounded, then a necessary condition for the minimum of the functional $f(u)=\|Au-f\|^2+\alpha\|u\|^2$ is the equation

$$A^*Au + \alpha u = A^* f. \tag{14}$$

Hence in this case conditions are required $f \in D(A^*)$.

If *A* is unbounded, then *f* does not necessarily belong to $D(A^*)$, so Equation (14) may have no sence. Therefore, some changes in the usual theory are necessary. The changes are given in this paper. We prove, among other things, that for any $f \in Y$, in particular for $f \notin D(A^*)$, the element $u_\alpha = A^*(AA^* + \alpha I_Y)^{-1} f$ is well defined for any $\alpha = \text{const} > 0$, provided that *A* is a closed, linear, densely defined operator in Hilbert space (Theorem 1).

3. Applications

As a simple concrete example of this type of approximation, consider differentiation in $H = L^2[0,1]$.

We define the operator $A: D(A) \subset H \to H$ as follows

$$Af = \frac{\mathrm{d}f}{\mathrm{d}x},\ f \in D(A),$$

with $D(A) = \{ f \in H : f \text{ is absolutely continuous on } [0,1] \text{ and } f'(x) \in H\}$.

Then $D(A)$ is dense in *H* since it contains the complete orthonormal set $\{\sin n\pi x\}_{n=1}^{\infty}$.

Clearly, *A* is a linear operator.

We show that *A* is a closed operator in Hilbert space *H*. Indeed, for suppose $\{f_n\} \subset D(A)$ and $f_n \to f$ and $f_n' \to g$, in each case the convergence being in the $L^2[0,1]$ norm. Since

$$f_n(x) = f_n(0) + \int_0^x f_n'(t)\,\mathrm{d}t,$$

we see that the sequence of constant functions $\{f_n(0)\}$ converges in $L^2[0,1]$ and hence the numerical sequence $\{f_n(0)\}$ converges to some real number *C*.

Now define $h \in D(A)$ by $h(x) = C + \int_0^x g(t)\,\mathrm{d}t$. Then, for any $x \in [0,1]$, we have by of the Cauchy-Schwarz inequality

$$\begin{aligned} |f_n(x) - h(x)| &= \left| f_n(0) - C + \int_0^x (f_n'(t) - g(t))\,\mathrm{d}t \right| \\ &\le |f_n(0) - C| + \int_0^x |f_n'(t) - g(t)|\,\mathrm{d}t \\ &\le |f_n(0) - C| + \|f_n' - g\| \end{aligned}$$

and hence $f_n \to h$ uniformly. Therefore, $f = h \in D(A)$ and $Af = f' = h' = g$, verifying that the operator A is closed, linear, densely defined in $L^2[0,1]$.

Let

$$D^* = \{g \in D(A) : g(0) = g(1) = 0\}.$$

Then for $f \in D(A)$ and $g \in D^*$, we have

$$\langle Af, g\rangle = \int_0^1 f'(t) g(t) \mathrm{d}t = f(t) g(t)\big|_0^1 - \int_0^1 f(t) g'(t) \mathrm{d}t = \langle f, -g'\rangle$$

Therefore $D^* \subset D(A^*)$ and $A^* g = -g'$, for $g \in D^*$.

On the other hand, if $g \in D(A^*)$, let $g^* = A^* g$. Then

$$\langle Af, g\rangle = \langle f, g^*\rangle$$

for all $f \in D(A)$. In particular, for $f \equiv 1$, we find that $\int_0^1 g^*(t) \mathrm{d}t = 0$.

Now let

$$h(t) = -\int_0^t g^*(s) \mathrm{d}s.$$

Then $h \in D^*$ and $A^* h = g^* = A^* g$ and hence $h - g \in N(A^*)$. Therefore, $\langle Af, h - g\rangle = 0$, for all $f \in D(A)$. But $R(A)$ contains all continuous function and hence $g = h \in D^*$.

We conclude that

$$D(A^*) = D^*, \text{ and } A^* g = -g'.$$

According to Theorem 1, for any $f \in Y = L^2[0,1]$, the problem

$$F(u) = \|Au - f\|^2 + \alpha \|u\|^2 \to \min, \alpha = \text{const} > 0,$$

has a unique solution $u_\alpha = A^*(AA^* + \alpha I)^{-1} f$, where I is the identity operator on $Y = L^2[0,1]$. $f \in Y = L^2[0,1]$ does not necessarily belong to $D(A^*)$.

It follows from Theorem 2, that if $f = Ay$, $y \perp N(A)$ then

$$\lim_{\alpha \to 0} \|u_\alpha - y\| = 0, u_\alpha = A^*(AA^* + \alpha I)^{-1} f.$$

It follows from Theorem 3, that if $\|f_\delta - f\| \le \delta$, $f = Ay$, and

$$F_\delta(u) = \|Au - f_\delta\|^2 + \alpha \|u\|^2 = \min, \tag{15}$$

then there exists a unique global minimier $u_{\alpha,\delta}$ to Equation (15) and $\lim_{\delta \to 0} \|u_\delta - y\| = 0$, where $u_\delta := u_{\alpha(\delta),\delta}$ and $\alpha(\delta)$ is properly chosen, in particular $\lim_{\delta \to 0} \alpha(\delta) = 0$.

It follows from Theorem 4, that the equation

$$\|Au_{\alpha,\delta} - f_\delta\| = c\delta, c = \text{const} > 1, \|f_\delta\| > c\delta,$$

has a unique solution $\alpha = \alpha(\delta) > 0$, $\lim_{\delta \to 0} \alpha(\delta) = 0$, and if $u_\delta := u_{\alpha(\delta),\delta}$, then $\lim_{\delta \to 0} \|u_\delta - y\| = 0$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Tikhonov, A.N. and Arsenin, V.Ya. (1978) Solution of Ill-Posed Problems. John Wiley & Sons, Hoboken.

[2] Groetsch, C.W. (2006) Stable Approximate Evualuation of Unbounded Operators. Springer, Berlin. https://doi.org/10.1007/3-540-39942-9

[3] Bakushinsky, A. and Goncharsky, A. (1994) Ill-Posed Problems: Theory and Applications. Springer, Berlin. https://doi.org/10.1007/978-94-011-1026-6

[4] Engl, H.W., Hanke, M. and Neubauer, A. (2000) Regularization of Inverse Problems. Kluwer, Alphen aan den.

[5] Morozov, V.A. (1984) Methods for Solving Incorrectly Posed Problems. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-5280-1

[6] Ramm, A.G. (2005) Inverse Problems. Springer-Verlag, New York.

[7] Ramm, A.G. (2007) Ill-Posed Problems with Unbounded Operators. *Journal of Mathematical Analysis and Applications*, **325**, 490-495. https://doi.org/10.1016/j.jmaa.2006.02.004

[8] Van Kinh, N., Chuong, N.M. and Gorenflo, R. (1996) Regularization Method for Nonlinear Variational Inequalities. *Proceedings of the First National Workshop* "*Optimization and Control*", Quinhon, May 27 - June 1, 1996, 53-64. (Preprint: Nr. A-89-28, Freie Universitat, Berlin).

[9] Van Kinh, N. (2007) Lavrentiev Regularization Method for Nonlinear Ill-Posed Problems. Quy Nhon Uni. *Journal of Science*, No. 1, 13-28.

[10] Van Kinh, N. (2014) On the Stable Method of Computing Values of Unbounded Operators. *Journal Science of Ho Chi MInh City University of Food Industry*, No. 2, 21-30.

[11] Van Kinh, N. (2020) On the Stable Method Computing Values of Unbounded Operators. *Open Journal of Optimization*, **9**, 129-137. https://doi.org/10.4236/ojop.2020.94009

[12] Zwart, K. (2018) The Spectral Theorem for Unbounded Self-Adjoint Operators and Nelson's Theorem. Bachelor Thesis, Universiteit Utrecht, Utrecht.