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Applications of Analytic Continuation to Tables of Integral Transforms and Some Integral Equations with Hyper-Singular Kernels

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Abstract

Analytic continuation of some classical formulas with respect to a parameter is discussed. Examples are presented. The validity of these formulas is greatly expanded. Application of these results to solving some integral equations with hyper-singular kernels is given.

Keywords

Analytic Continuation, Integral Equations with Hyper-Singular Kernels

1. Introduction

In [1] one finds several formulas of integral transforms the validity of which can be greatly expanded by analytic continuation with respect to a parameter. This is of interest per se, but also is important in applications. Analytic continuation with respect to the parameter can be used in a study of integral equations with hyper-singular kernels. This is done in Section 3. The examples of the integral equations are chosen to demonstrate that some integral equations, which do not make sense classically (that is, from the classical point of view), can be understood using the analytic continuation. Moreover, they can be solved analytically and the properties of their solutions can be studied. In Sections 1 and 2 examples of the formulas from tables of integral transforms are discussed. The number of such examples can be increased greatly. The author wants to emphasize the principle based on the analytic continuation. The choice of the parameter $\lambda=-\frac{1}{4}$ is motivated by the role playing by the corresponding integral equations in the Navier-Stokes problem, see [2] [3] [4]. The choice of the parameter

$\lambda = -\frac{1}{2}$ is motivated by the novel feature in the investigation, the pole in the Laplace transforms the solution.

Example 1. In [1] Formula (1) in Section 2.3. is given in the form:

$$\int_0^\infty x^{-\nu} \sin(xy)\,\mathrm{d}x = y^{\nu-1}\Gamma(1-\nu)\cos\left(\frac{\nu\pi}{2}\right), \quad y>0,\ 0<\mathrm{Re}\,\nu<2. \tag{1}$$

Here and below $\Gamma(\nu)$ is the Gamma function. Classically the integral on the left in (1) diverges if $\mathrm{Re}\,\nu<0$ or $\mathrm{Re}\,\nu>2$. On the other hand, in many applications, one has to consider ν outside the region specified in (1). The right side of Formula (1) admits analytic continuation with respect to ν. Indeed, if $y>0$, which we assume throughout, then $y^{\nu-1} = \mathrm{e}^{(\nu-1)\ln y}$ is an entire function of ν. The $\Gamma(z)$ is an analytic function of z on the complex plane z except for a discreet set of points $z = 0,-1,-2,\cdots$, at which it has simple poles with known residues, see [5]. Therefore $\Gamma(1-\nu)$ is an analytic function of ν except for the points $\nu = 1,2,3,\cdots$. The function $\cos\left(\frac{\nu\pi}{2}\right)$ is an entire function of ν. Therefore, the right side of Formula (1) admits analytic continuation on the complex plane ν except for the points $\nu = 2,3,4,\cdots$. The function $\cos\left(\frac{\nu\pi}{2}\right) = 0$ if $\nu = n + \frac{1}{2}$, where n is an integer. Therefore, the zeros of $\cos\left(\frac{\nu\pi}{2}\right)$ do not eliminate the poles of the $\Gamma(1-\nu)$. We have proved the following theorem.

Theorem 1. *Formula* (*1*) *remains valid by analytic continuation with respect to* ν *for all complex* $\nu \neq 2,3,4,\cdots$. □

2. More Examples

Consider two more examples of a similar nature. The number of such examples can be increased. In [1], Formula (7) in Section 2.4. is:

$$\begin{aligned}&\int_0^\infty x^{-\nu}\mathrm{e}^{-ax}\sin(xy)\,\mathrm{d}x \\ &= \Gamma(\nu)\left(a^2+y^2\right)^{-\frac{\nu}{2}}\sin\left[\nu\,\mathrm{arctg}(y/a)\right], \ \mathrm{Re}\,a>0,\ \mathrm{Re}\,\nu>-1.\end{aligned} \tag{2}$$

In [1], Formula (4) in Section 2.5. is:

$$\begin{aligned}&\int_0^\infty x^{-\nu}\ln x\sin(xy)\,\mathrm{d}x \\ &= \frac{\pi y^{-\nu}}{2\Gamma(1-\nu)\cos(\nu\pi/2)}\left[\psi(\nu)+0.5\pi\,\mathrm{ctg}(\nu\pi/2)-\ln y\right], \quad |\mathrm{Re}\,\nu|<1,\end{aligned} \tag{3}$$

where $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$.

We leave it for the reader to discuss the analytic continuation of Formulas (2) and (3) with respect to ν.

3. Some Applications

Consider an integral equation

$$q(t)=h(t)+\int_0^t (t-s)^{-\frac{5}{4}} q(s)\,\mathrm{d}s. \tag{4}$$

More generally, consider the equation

$$q(t)=h(t)+\int_0^t (t-s)^{\lambda-1} q(s)\,\mathrm{d}s. \tag{5}$$

This equation has a hyper-singular kernel: the integral in this equation *diverges* if $\operatorname{Re}\lambda \le 0$ classically (that is, from the point of view of classical analysis).

Our goal is to give sense to this equation and solve it analytically. One knows that

$$L\left(t^{\lambda-1}\right)=\Gamma(\lambda)p^{-\lambda}, \tag{6}$$

where $L(h):=\int_0^\infty \mathrm{e}^{-pt}h(t)\,\mathrm{d}t$ is the Laplace transform. For $\operatorname{Re}\lambda>0$ Formula (6) is known classically. For $\operatorname{Re}p>0$ the function $p^{-\lambda}=\mathrm{e}^{-\lambda\ln p}$ is an entire function of λ on the complex plane $\mathbb{C}$ of λ. The function $\Gamma(\lambda)$ is analytic on $\mathbb{C}$ except for the points $0,-1,-2,\cdots$. Therefore, Formula (6) is valid by analytical continuation with respect to λ from the region $\operatorname{Re}\lambda>0$ to the $\mathbb{C}$ except for the points $0,-1,-2,\cdots$.

In the region $\operatorname{Re}\lambda>0$ one can take the Laplace transform of Equation (5) classically and get

$$L(q)=L(h)+\Gamma(\lambda)p^{-\lambda}L(q). \tag{7}$$

From (7) one gets

$$L(q)=\frac{L(h)}{1-\Gamma(\lambda)p^{-\lambda}}. \tag{8}$$

The question is: under what conditions the right side of Formula (8) is the Laplace transform of a function q from some functional class? The answer to this question depends on λ and h.

Equation (7) makes sense by an analytic continuation with respect to λ for $\lambda\in\mathbb{C}$ except for the points $\lambda=0,-1,-2,\cdots$. Therefore, Equation (4) can be considered as a particular case of Equation (5) with $\lambda=-1/4$. At this value of λ this equation is well defined by analytic continuation, although classically its kernel is hyper-singular and the integral in (4) diverges classically.

To solve Equation (4), we apply the Laplace transform and continue analytically the result with respect to λ, as was explained above. For $\lambda=-1/4$ Equation (7) yields:

$$L(q)=\frac{L(h)}{1-\Gamma\left(-\frac{1}{4}\right)p^{1/4}}. \tag{9}$$

One has $\Gamma(z+1)=z\Gamma(z)$. For $z=-\frac{1}{4}$, one has

$$\Gamma\left(-\frac{1}{4}\right)=\frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{4}}=-4\pi^{1/2}:=-b,\quad b>0. \tag{10}$$

Let $\mathbb{R}_+ = [0,\infty)$. We assume for simplicity that $h(t)$ is a smooth rapidly decaying function. Then

$$|L(h)| \le \frac{c}{(1+|p|)^{-1}}, \tag{11}$$

where $c>0$ does not depend on *p*.

From Formulas (9) and (10) the following theorem follows.

Theorem 2. *Assume that* (*11*) *holds. Then* Equation (*4*) *has a solution* $q(t)$ *in* $C(\mathbb{R}_+)$, $q(0)=0$, *this solution is unique in* $C(\mathbb{R}_+)$ *and can be calculated by the formula*

$$q(t) = L^{-1}\left(\frac{L(h)}{1+bp^{1/4}}\right), \tag{12}$$

where $b>0$ is defined in (10).

Proof. To prove Theorem 2 it is sufficient to check that the expression $\frac{L(h)}{1+bp^{1/4}}$ is the Laplace transform of a function $q(t)\in C(\mathbb{R}_+)$. We also prove that $q(0)=0$.

Consider the function

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ist}\frac{L(h)}{1+bp^{1/4}}\mathrm{d}s, \quad p = is. \tag{13}$$

This is the inverse of the Laplace transform of $\frac{L(h)}{1+bp^{1/4}}$ since $\mathrm{d}p = i\mathrm{d}s$. The integral (13) converges absolutely under our assumptions since the integrand is $O\left(\frac{1}{|p|^{5/4}}\right)$ for $|p|\gg 1$. Therefore, $q\in C(\mathbb{R}_+)$. To prove that $q(0)=0$, let us check that

$$\int_{-\infty}^{\infty} \mathrm{e}^{ist}\frac{L(h)}{1+bp^{1/4}}\mathrm{d}s\Big|_{t=0} = 0. \tag{14}$$

The function $\frac{L(h)}{1+bp^{1/4}}$ is analytic in $\operatorname{Re} p>0$ and is $O\left(\frac{1}{|p|^{5/4}}\right)$ for $|p|\gg 1$. One checks that $\frac{1}{1+bp^{1/4}}$ with $b>0$ is a uniformly bounded analytic function of *p* in the half-plane $\operatorname{Re} p \ge 0$.

Let L_n be a closed contour, oriented counterclockwise, consisting of the segment $[in,-in]$ and half a circle $\gamma_n = \{n\mathrm{e}^{i\phi}\}$, $-\frac{\pi}{2}\le\phi\le\frac{\pi}{2}$. By the Cauchy theorem,

$$\int_{L_n}\frac{L(h)}{1+bp^{1/4}}\mathrm{d}p = 0, \tag{15}$$

and

$$\lim_{n\to\infty}\int_{\gamma_n}\frac{L(h)}{1+bp^{1/4}}\mathrm{d}p=0. \tag{16}$$

Consequently, from (15) and (16) it follows that

$$\lim_{n\to\infty}\int_{-in}^{in}\frac{L(h)}{1+bp^{1/4}}\mathrm{d}p=0. \tag{17}$$

This relation is equivalent to (14).

Theorem 2 is proved. □

Consider Equation (5) with $\lambda=-\frac{1}{2}$ and with the minus sign in front of the integral. By Formula (7) with the minus sign in front of the $\Gamma(\lambda)$ the Laplace transform of the solution is:

$$L(q)=\frac{L(h)}{1-ap^{1/2}},\quad a=\left|\Gamma\left(-\frac{1}{2}\right)\right|>0. \tag{18}$$

The new feature, compared with Theorem 2, is the existence of the singularity at $p=\frac{1}{a^2}$. We assume for simplicity that $h(t)\in C(\mathbb{R}_+)$ has compact support. In this case $L(h)$ is an entire function of p and the behavior for large t of the solution $q(t)$, found in Theorem 3 (see below), is easy to estimate.

Let us investigate the function

$$\frac{1}{1-ap^{1/2}}=-a^{-1}\left[\frac{a^{-1}+p^{1/2}}{p-a^{-2}}\right]=-a^{-2}\frac{1}{p-a^{-2}}-a^{-1}\frac{p^{1/2}}{p-a^{-2}}. \tag{19}$$

One has:

$$L^{-1}\left(-a^{-2}\frac{1}{p-a^{-2}}\right)=-a^{-2}\mathrm{e}^{a^{-2}t}. \tag{20}$$

So, using the known formula $L(f)L(g)=L\left(\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right)$, we derive:

$$L^{-1}\left(-L(h)a^{-2}\frac{1}{p-a^{-2}}\right)=-a^{-2}\int_0^t h(\tau)\mathrm{e}^{a^{-2}(t-\tau)}\mathrm{d}\tau. \tag{21}$$

Consider the last term in (19). In [1] Formula (22) in Section 5.3 is:

$$(p-\beta)^{-1}p^{1/2}=L\left((\pi t)^{-\frac{1}{2}}+\beta^{1/2}\mathrm{e}^{\beta t}Erf\left(\beta^{1/2}t^{1/2}\right)\right),\quad Erf(x):=2\pi^{-\frac{1}{2}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t. \tag{22}$$

Therefore, taking $\beta=a^{-2}$, one derives:

$$L^{-1}\left(-L(h)a^{-1}\frac{p^{1/2}}{p-a^{-2}}\right)=-a^{-1}\int_0^t h(\tau)g(t-\tau)\mathrm{d}\tau, \tag{23}$$

where

$$g(t):=(\pi t)^{-\frac{1}{2}}+\beta^{1/2}\mathrm{e}^{\beta t}Erf\left(\beta^{1/2}t^{1/2}\right)\quad \beta=a^{-2}. \tag{24}$$

From Formulas (18), (19), (21), (23) it follows that

$$q(t)=-a^{-2}\int_0^t h(\tau)\mathrm{e}^{a^{-2}(t-\tau)}\mathrm{d}\tau-a^{-1}\int_0^t h(\tau)g(t-\tau)\mathrm{d}\tau. \tag{25}$$

We have proved the following theorem.

Theorem 3. *Assume that h is compactly supported. Then Equation* (*5*) *with* $\lambda = -\frac{1}{2}$ *is uniquely solvable and its solution is given by Formula* (*25*), *where* $g(t)$ *is given by Formula* (*24*).

The behavior of the solution for $t \to \infty$ depends on λ, on the sign in front of the integral Equation (5) and on *h*. Theorems 2 and 3 are examples of a study of the solution to Equation (5).

4. Conclusion

It is proved in this paper that the validity of some formulas in the tables of integral transforms can be greatly expanded by the analytic continuation with respect to parameters. This idea is used for the investigation of some integral equations with hyper-singular kernels. Such an equation, see (4), plays a crucial role in the author's investigation of the Navier-Stokes problem, see [2] [3] [4].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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