Open Journal of Optimization, 2020, 9, 129-137
https://www.scirp.org/journal/ojop
ISSN Online: 2325-7091
ISSN Print: 2325-7105

On the Stable Method Computing Values of Unbounded Operators

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How to cite this paper: Van Kinh, N. (2020) On the Stable Method Computing Values of Unbounded Operators. *Open Journal of Optimization*, **9**, 129-137.
https://doi.org/10.4236/ojop.2020.94009

Received: November 10, 2020
Accepted: December 6, 2020
Published: December 9, 2020

Abstract

Unbounded operators can transform arbitrarily small vectors into arbitrarily large vectors—a phenomenon known as instability. Stabilization methods strive to approximate a value of an unbounded operator by applying a family of bounded operators to rough approximate data that do not necessarily lie within the domain of unbounded operator. In this paper we shall be concerned with the stable method of computing values of unbounded operators having perturbations and the stability is established for this method.

Keywords

The Stable Method, Ill-Posed Problem, Regularization, Tikhonov Method, Unbounded Linear Operator

1. Introduction

The stable computation of values of unbounded operators is one of the most important problems in computational mathematics. Indeed, let A be a linear operator from X into Y with domain $D(A)\subset X$ and range $R(A)\subset Y$, where X and Y are normed spaces and A is unbounded, that is, there exists a sequence of elements $x_n \in D(A), n=1,2,\cdots$, such that $\|Ax_n\| \to +\infty$ as $n\to\infty$. Let $x_0 \in D(A)$ and $y_0 = Ax_0$. We put $x_{n,\delta} = x_0 + \delta x_n$, where δ is an arbitrarily small number. Let $y_{n,\delta} = Ax_{n,\delta}$. Then

$$\|y_{n,\delta} - y_0\| = \delta \|Ax_n\| \to +\infty, \forall \delta > 0,$$

while $\|x_{n,\delta} - x_0\| = \delta$ may be arbitrarily small.

Therefore, the problem of computing values of an operator in the considered case is unstable [1]. Moreover, if we bear in mind arbitrarily δ-approximation to the element x_0 in X, that is the elements $x_\delta \in X$ with $\|x_\delta - x_0\| \le \delta$, we can see that the values of the operator A may not even be defined on the elements x_δ, that is, $x_\delta \notin D(A)$ and if $x_\delta \in D(A)$, it may happen $Ax_\delta \nrightarrow Ax_0$ as $\delta \to 0$, since the operator A is unbounded.

In the case, where A is a closed densely defined unbounded linear operator from a Hilbert space X into a Hilbert space Y, V. A. Morozov has studied a stable method for approximating the value Ax_0 when only approximate data x_δ is available [2]. This method takes as an approximation to $y_0 = Ax_0$ the element $y_\alpha^\delta = Az_\alpha^\delta$, where z_α^δ minimizes the parametric functional

$$\Phi_\alpha^\delta(z) = \|z - x_\delta\|^2 + \alpha\|Az\|^2, z \in D(A), \alpha > 0. \quad (1)$$

He shows that, if $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$, in such a way that $\frac{\delta}{\sqrt{\alpha}} \to 0$, then $y_\alpha^\delta \to Ax_0$ as $\delta \to 0$. Moreover, the order of convergence results for $\{y_\alpha^\delta\}$ have been established [2]-[7].

In the another case, where A is a monotone operator from a real strictly convex reflexive Banach space X into its dual X^*, an approximation to $y_0 = Ax_0$ is the element $y_\alpha^\delta = -U(x_\alpha^\delta - x_\delta)/\alpha$, where x_α^δ is the unique solution of the equation

$$\alpha Ax + U(x - x_\delta) = 0,$$

where $U : X \to X^*$ is the dual mapping in X [8] [9]. Then the sequence $\{y_\alpha^\delta\}$ for $\frac{\delta}{\alpha} \to 0, \alpha \to 0$, in the norm of X^*, to a generalized value y_0 of the operator A at x_0 [10].

We now assume that both the operator A and $x_0 \in D(A)$ are only given approximately by A_h and $x_\delta \in X$, which satisfy

$$\|x_\delta - x_0\| \le \delta, \text{and } \|A_h x - Ax\| \le h, \forall x \in D = D(A_h) \cap D(A), h, \delta > 0, \quad (2)$$

where A_h is also an operator from X into Y. We should approximate values of A when we are given the approximations A_h and x_δ. Until now, this problem is still an open problem.

In this paper we shall be concerned with the construction of a stable method of computing values of the operator A for the perturbations (2).

2. The Stable Method of Computing Values of Closed Densely Defined Unbounded Linear Operators

In this section, we assume that $A : D(A) \subset X \to Y$ is a closed densely defined unbounded linear operator from a Hilbert space X into a Hilbert space Y with domain $D(A) \subset X$ and $x_0 \in D(A)$. (A, x_0) is called an exact data.

Instead of the exact data (A, x_0), we have an approximation (A_h, x_δ), which satisfies (1.2), where A_h is also a closed densely defined unbounded linear operator from X into Y with domain $D(A_h) = D(A), \forall h > 0$.

First, we define the regularization functional

$$\Phi_\Delta(z) = \|z - x_\delta\|^2 + \alpha\|A_h z\|^2, \forall z \in D(A_h), \quad (1)$$

where $\alpha > 0$ is called the regularization parameter, $\Delta = (h, \delta, \alpha)$.

We shall take as an approximation to $y_0 = Ax_0$ the element $y_\Delta = A_h z_\Delta$, where z_Δ minimizes the regularization functional $\Phi_\Delta(z)$ over $D(A_h)$.

Theorem 2.1. [5] *For any* $\Delta = (h, \delta, \alpha)$ *the minimization problem* (1) *has a unique solution*

$$z_\Delta = \left(I + \alpha A_h^* A_h\right)^{-1} x_\delta. \tag{2}$$

Hence

$$y_\Delta = A_h \left(I + \alpha A_h^* A_h\right)^{-1} x_\delta. \tag{3}$$

To establish the convergence of (3), it will be convenient to reformulate (3) as

$$y_\Delta = A_h \breve{A}_h \left[\alpha I + (1-\alpha)\breve{A}_h\right]^{-1} x_\delta, \tag{4}$$

where $\breve{A}_h = \left(I + A_h^* A_h\right)^{-1}$.

$\breve{A}_h, A_h \breve{A}_h$ are known to be bounded everywhere defined linear operators and $\breve{A}_h$ is a self-adjoint with spectrum $\sigma(\breve{A}_h) \subset [0,1]$ ([4], p. 38).

To further simplify the presentation, we introduce the functions

$$T_\alpha(t) = \left[\alpha + (1-\alpha)t\right]^{-1}, \alpha > 0, t \in [0,1].$$

We then have

$$y_\Delta = A_h \breve{A}_h T_\alpha\left(\breve{A}_h\right) x_\delta. \tag{5}$$

We also denote

$$y_{h,\alpha} = A_h \breve{A}_h T_\alpha\left(\breve{A}_h\right) x_0. \tag{6}$$

The following lemma will be used in the proof of Theorem 2.2.

Lemma 2.1. *Under the stated assumption, we obtain*

$$A_h \breve{A}_h = \hat{A}_h A_h,$$

where $\hat{A}_h = \left(I + A_h A_h^*\right)^{-1}$.

Proof. We denote

$$G(A_h) = \left\{(x, A_h x) : x \in D(A_h)\right\}$$

$$VG\left(A_h^*\right) = \left\{\left(-A_h^* y, y\right) : y \in D\left(A_h^*\right)\right\}.$$

Since A_h is a closed densely defined linear operator then $G(A_h)$ and $VG(A_h^*)$ are complementary orthogonal subspaces of the Hilbert space $X \times Y$ ([11], p. 307). Hence, for any $z \in X$, we have the uniquely determined decomposition

$$(z, 0) = (x, A_h x) + \left(-A_h^* y, y\right), \text{with } x \in D(A_h), y \in D\left(A_h^*\right). \tag{7}$$

Thus

$$z = x - A_h^* y, 0 = A_h x + y. \tag{8}$$

Therefore, $x \in D\left(A_h^* A_h\right)$ and $x + A_h^* A_h x = z$. Because of the uniqueness of decomposition (7), *x* is uniquely determined by *z*, and so the everywhere defined inverse $\left(I+A_h^*A_h\right)^{-1}$ exists.

In a similar way as above, the everywhere defined inverse $\left(I+A_hA_h^*\right)^{-1}$ exists. It follows from (8) that

$$A_h\left(I+A_h^*A_h\right)^{-1}=\left(I+A_hA_h^*\right)^{-1}A_h,$$

that means $A_h\breve{A}_h=\hat{A}_hA_h$. Moreover, $\breve{A}_h,\hat{A}_h$ are bouned operators and

$$\left\|\breve{A}_h\right\|\le 1,\left\|\hat{A}_h\right\|\le 1.$$

([11], p. 308).

Theorem 2.2. *If* $D\left(A_hA_h^*A_h\right)=D\left(AA^*A\right),\forall h>0$ *and* $x_0\in D\left(AA^*A\right)$, *and* $\alpha=\alpha\left(h,\delta\right)\to 0$, $\delta^2/\alpha\to 0$ *as* $h,\delta\to 0$, *then* $\left\{y_\Delta\right\}$ *converges to* Ax_0.

Proof. Let $\omega=\left(I+A_hA_h^*\right)A_hx_0$. Then $A_hx_0=\hat{A}_h\omega$. Since $A_h\breve{A}_h=\hat{A}_hA_h$ (Lemma 2.1) and $A_hx_0=\hat{A}_h\omega$, we have

$$\begin{aligned} y_{h,\alpha}-A_hx_0&=A_h\left(\breve{A}_h-\left[\alpha I+\left(1-\alpha\right)\breve{A}_h\right]\right)\left(\alpha I+\left[\left(1-\alpha\right)\breve{A}_h\right]\right)^{-1}x_0\\ &=\alpha\left(\hat{A}_h-I\right)T_\alpha\left(\hat{A}_h\right)\hat{A}_h\omega. \end{aligned}$$

Since $\left\|T_\alpha\left(\hat{A}_h\right)\hat{A}_h\right\|\le\frac{1}{\alpha}$ and $\left\|\hat{A}_h-I\right\|\le 2$, for all $h>0$, we obtain

$$\lim_{\alpha\to 0}y_{h,\alpha}=A_hx_0,\forall h>0.$$

On the other hand we have

$$\begin{aligned} \left\|y_\Delta-y_{h,\alpha}\right\|^2&=\left\langle A_h\breve{A}_hT_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right),A_h\breve{A}_hT_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right)\right\rangle\\ &=\left\langle A_h^*A_h\breve{A}_hT_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right),\breve{A}_hT_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right)\right\rangle\\ &=\left\langle\left(I-\breve{A}_h\right)T_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right),\breve{A}_hT_\alpha\left(\breve{A}_h\right)\left(x_\delta-x_0\right)\right\rangle\\ &=\left\|I-\breve{A}_h\right\|\frac{\delta^2}{\alpha}, \end{aligned}$$

since $\left\|T_\alpha\left(\breve{A}_h\right)\right\|\le 1$.

Hence

$$\left\|y_\Delta-y_{h,\alpha}\right\|\to 0,\text{ as }\alpha\left(h,\delta\right)\to 0,\frac{\delta^2}{\alpha}\to 0.$$

We have

$$\begin{aligned} \left\|y_\Delta-Ax_0\right\|&\le\left\|y_\Delta-y_{h,\alpha}\right\|+\left\|y_{h,\alpha}-A_hx_0\right\|+\left\|A_hx_0-Ax_0\right\|\\ &\le\left\|y_\Delta-y_{h,\alpha}\right\|+\left\|y_{h,\alpha}-A_hx_0\right\|+h. \end{aligned}\tag{9}$$

It follows from (9) that

$$y_\Delta\to Ax_0,\text{ as }h,\delta\to 0.$$

The theorem is proved.

We shall call y_Δ the approximate values of the operator A at x_0.

3. The Stable Method of Computing Values of Hemi-Continuous Monotone Operators

Let X be a real strictly convex reflexive Banach space with the dual X^* be an E-space. Suppose that $A: X \to X^*$ is a hemi-continuous monotone operator from X into X^* with domain $D(A) \subset X$ (possibly multi-valued) and y is a given element in X^*. We consider the following three problems

1) To solve the equation

$$Ax = y, \tag{1}$$

2) To solve the variational inequality

$$\langle Ax - y, x - z \rangle \geq 0, \forall x \in D(A), \tag{2}$$

3) To compute values of the operator A at x_0 in X with x_0 given approximately.

These problems are important objects of investigation in the theory unstable problems. In [7] [10] [12]-[17] a class of monotone operators was singled out and, as an approximate method, the operator-regularization method was used.

As it is known [17], a solution of (1) is understood to be an element $\tilde{x} \in D(A)$ such that $A\tilde{x} = y$ if A is a single-valued, and $y \in A\tilde{x}$ if A is a maximal monotone (possibly multi-valued). If A is an arbitrary monotone operator, we follow [15] and understand a solution of (1) to be an element $\overline{x} \in X$ such that

$$\langle Ax - y, x - \overline{x} \rangle \geq 0, \forall x \in D(A), \tag{3}$$

where $\langle Ax - y, x - \overline{x} \rangle$ values of the linear functional $Ax - y$ at $x - \overline{x}$.

We shall call $\overline{x}$ a generalized solution of Equation (1). We note that, if A is hemi-continuous and $D(A)$ is open or everywhere dense in X, or if A is maximal monotone, then a generalized solution $\overline{x}$ coincides with the corresponding solution $\tilde{x}$, and (3) is equivalent to the inclusion $y \in A\overline{x}$ [17].

We now deal with the stable method of computing values of the operator A at x_0 when only the approximations A_h, x_δ as in (2) are given, where A_h is also a hemi-continuous monotone operator from X into X^* with domain $D(A_h) = D(A) = X$.

We denote the set values of A at x_0

$$R_{x_0} = \{ y \in X^* : y \in Ax_0 \}.$$

In X^* we consider the set

$$M_{x_0} = \{ y \in X^* \mid \langle Ax - y, x - x_0 \rangle \geq 0, \forall x \in X \},$$

and we call M_{x_0} the set of generalized values of A at x_0. It is easy to show that $R_{x_0} \subset M_{x_0}$.

Lemma 3.1. [5] The set M_{x_0} is convex and closed in X^*, moreover, there is a unique element $y_0 \in M_{x_0}$ such that

$$\|y_0\| = \min_{y \in M_{x_0}} \|y\|.$$

Under the above hypotheses, there exist the dual mappings

$$U: X \to X^*, V: X^* \to X,$$

being strictly monotone, single-valued, homogeneous, hemi-continuous and such that

$$VUx = x, \forall x \in X; UVy = y, \forall y \in X^*,$$

(see [8] [9] [18]).

We consider the equation

$$\alpha A_h x + U(x - x_\delta) = 0, \alpha > 0. \tag{4}$$

The following theorem asserts the existence and uniqueness of generalized solution of (4).

Theorem 3.1. Under hypotheses as above, Equation (4) has a unique solution x_Δ, for any $\Delta = (h, \delta, \alpha)$.

Proof. Let $\tilde{A}_h$ be the maximal monotone extension of A_h (such an extension exists by virtue of Zorn's lemma). Therefore, the operator $\alpha \tilde{A}_h x + U(x - x_\delta)$ is maximal monotone [19] and Browder's theorem [14] implies that Equation (4) has a unique solution $\tilde{x}_\Delta$, *i.e.*, $0 \in \tilde{A}_h \tilde{x}_\Delta + U(\tilde{x}_\Delta - x_\delta)$. In view of the preceding remark, this follows that

$$\langle \alpha A_h x + U(x - x_\delta), x - \tilde{x}_\Delta \rangle \ge 0, \forall x \in X.$$

Thus, $\tilde{x}_\Delta$ coincides with the generalized solution of Equation (2). Therefore, (2) has a unique solution $x_\Delta = \tilde{x}_\Delta$, for any $\Delta = (h, \delta, \alpha)$. We now consider the sequence

$$y_\Delta = -U(x_\Delta - x_\delta)/\alpha. \tag{5}$$

The uniqueness of x_Δ implies that y_Δ is uniquely determined. It is easy to show that $y_\Delta \in \tilde{A}_h x_\Delta$.

y_Δ is call approximate value of A at x_0 for the given approximation (A_h, y_δ).

Theorem 3.2. Under the stated assumption, if $\alpha(h, \delta) \to 0$, $\delta/\alpha \to 0$, as $h, \delta \to 0$, then the sequence $\{y_\Delta\}$ converges to the generalized value $y_0 \in M_{x_0}$ of the operator A at x_0.

Proof. By applying the dual mapping $V: X^* \to X$ to (5), we obtain

$$\alpha V y_\Delta + (x_\Delta - x_\delta) - 0. \tag{6}$$

Let $M_{x_0}^h$ denote the set of generalized values of A_h at x_0, *i.e.*

$$M_{x_0}^h = \{ y_h \in X^* \mid \langle A_h x - y_h, x - x_0 \rangle \ge 0, \forall x \in X \}.$$

By using [10] we obtain $M_{x_0}^h \ne \varnothing$. It follows from (6) that

$$\langle y_\Delta - y_h, x_\Delta - x_0 \rangle + \langle y_\Delta - y_h, x_0 - x_\delta \rangle + \alpha \langle y_\Delta - y_h, V y_\Delta \rangle = 0, \forall y_h \in M_{x_0}^h. \tag{7}$$

It is easy to show that $(x_0, y_h) \in gr\tilde{A}_h$ and hence

$$\langle y_\Delta - y_h, x_\Delta - x_0 \rangle \ge 0. \tag{8}$$

It follows from (7) and (8), that

$$\langle y_\Delta - y_h, x_0 - x_\delta \rangle + \alpha \langle y_\Delta - y_h, Vy_\Delta \rangle \le 0,$$

implies

$$\alpha \|Vy_\Delta\|^2 - \alpha \|y_h\| \|Vy_\Delta\| - \|y_\Delta - y_h\| \|x_0 - x_\delta\| \le 0,$$

consequently

$$\alpha \|y_\Delta\|^2 - (\alpha \|y_h\| + \delta) \|y_\Delta\| - \delta \|y_h\| \le 0, \forall y_h \in M_{x_0}^h. \quad (9)$$

It follows from (9), that

$$\|y_\Delta\| \le \|y_h\| + 2\delta/\alpha, \forall y_h \in M_{x_0}^h.$$

In view of preceding remark and (2) we obtain

$$\|y_h - y\| \le h, \forall y \in M_{x_0}, \forall y_h \in M_{x_0}^h.$$

Hence,

$$\|y_\Delta\| \le \|y\| + 2\delta/\alpha + h, \forall y \in M_{x_0},$$

implies

$$\|y_\Delta\| \le \|y_0\| + 2\delta/\alpha + h, \forall h, \delta > 0. \quad (10)$$

Since X^* is an *E*- space and from (10) and by using [10] we see that the sequence $\{y_\Delta\}$ converges to y_0 as $\alpha(h,\delta) \to 0$, $\delta/\alpha \to 0$, $h, \delta \to 0$.

The theorem is proved.

4. Applications

As a simple concrete example of this type of approximation, consider differentiation in $L^2(\mathbb{R})$. That is, the operator A is defined on $H^1(\mathbb{R})$, the Sobolev space of functions possessing a weak derivative in $L^2(\mathbb{R})$ by

$$Ax = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

For a given data function $x_\delta \in L^2(\mathbb{R})$ and a given data operator A_h is defined on $H^1(\mathbb{R})$ possessing a weak derivative in $L^2(\mathbb{R})$, by $A_h x_\delta = \frac{\mathrm{d}x_\delta}{\mathrm{d}t}$ satisfying

$$\|x^\delta - x_0\| \le \delta, \|A_h x - Ax\| \le h, \forall x \in H^1(\mathbb{R}). \quad (1)$$

The stabilized approximate derivative (3) is easily seen (using Fourier transform analysis) to be given by

$$y_\Delta(s) = \int_{-\infty}^{+\infty} \sigma_{\alpha,h}(s-t) x_\delta(t), \quad (2)$$

where the kernel $\sigma_{\alpha,h}$ is given by

$$\sigma_{\alpha,h}(t) = -\frac{\operatorname{sign}(t)}{2\alpha} \exp\left(-|t|/\sqrt{\alpha}\right). \quad (3)$$

Then $y_\Delta(s)$ in (2) is the approximate value of the operator A at x_0 for this method.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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