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Computational Analysis for Solving the Linear Space-Fractional Telegraph Equation

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Abstract

Over the last few years, there has been a significant increase in attention paid to fractional differential equations, given their wide array of applications in the fields of physics and engineering. The recent development of using fractional telegraph equations as models in some fields (e.g., the thermal diffusion in fractal media) has heightened the importance of examining the method of solutions for such equations (both approximate and analytic). The present work is designed to serve as a valuable contribution to work in this field. The key objective of this work is to propose a general framework that can be used to guide quadratic spline functions in order to create a numerical method for obtaining an approximation solution using the linear space-fractional telegraph equation. Additionally, the Von Neumann method was employed to measure the stability of the analytical scheme, which showed that the proposed method is conditionally stable. What's more, the proposal contains a numerical example that illustrates how the proposed method can be implemented practically, whilst the error estimates and numerical stability results are discussed in depth. The findings indicate that the proposed model is highly effective, convenient and accurate for solving the relevant problems and is suitable for use with approximate solutions acquired through the two-dimensional differential transform method that has been developed for linear partial differential equations with space- and time-fractional derivatives.

Keywords

Fractional Differential Equations, Quadratic Spline Functions, Linear Space-Fractional Telegraph Equation, Von Neumann Stability

1. Introduction

Fractional calculus was a hot topic amongst researchers in the 1980s when its explicit applications in many different fields started to emerge. Such fields included physics, processing, control theory, fluid mechanics quantum evolution of complex systems, viscoelastic mechanics and chemical and industrial mathematics [1]. Nowadays, communications systems are vital in the modern world and thus it is crucial to examine the fractional telegraph equation. This is a typical fractional diffusion-wave equation that is applied to signal analysis to facilitate transmission, electrical signal propagation, modelling reaction-diffusion and examining the random walk of suspension flows [2] [3]. In recent times, many researchers have turned their attention towards investigating the telegraph equations of fractional order. For instance, Odibat and Momani (2008) put forward a new generalisation of the two-dimensional differential transform technique, in which its use was expanded in order to apply it linear partial differential equations using space- and time-fractional derivatives [4]. On the other hand, the generalised differential transformation method was employed by Garg *et al.* (2011) to solve the space-time fractional telegraph equation [5]. Subsequently, the fractional difference method was employed by Zhao *et al.* (2012) to measure the temporal direction, whilst the finite element method was used to measure spatial direction. The key purpose of this was to numerically solve the time-space fractional-order telegraph equation [6]. Moreover, at a later date, Aguilar and Baleanu (2014) examined the fractional differential equation and its function in the transmission line without losses regarding the Caputo fractional derivative (CFD) [7]. Meanwhile, in order to calculate the approximate solutions for space- and time-fractional telegraph equations, Varaki *et al.* (2005) employed the Homotopy Analysis Method (HAM) [8]. Lopushanska and Rapita (2005) also developed a unique solution for an inverse issue relating to the semi-linear fractional telegraph equation using regularised fractional derivatives, in which they carefully considered time on a bounded cylindrical domain [9]. Similarly, Khan *et al.* (2018) worked hard to create a new analytical technique for the space-fractional telegraph equation (FTE), which proved to be highly efficient. This new analytical approach involved the use of a fractional Sumudu decomposition method (SDM) [10]. Meanwhile, Kamran *et al.* (2018) also developed a local meshless method, which they combined with the Laplace transformation technique to solve a time-fractional telegraph equation [11]. In the same year, Wei *et al.* (2018) developed and analysed a flexible numerical technique to solve the time-fractional telegraph equation, whereby a new, finite differentiation method in time and a local, discontinuous Galerkin method in space were employed [12]. Ru Liu approximate the solution to time-fractional telegraph equation by two kinds of difference methods: the Grünwald formula and Caputo fractional difference in 2018 [13]. The following year, Mohammadian *et al.* (2019) put forward the Generalised Differential Transformational method (GDTM) to develop a semi-analytical solution for fractional partial differential equations. The Riesz space fractional derivative is a key part of this process [14]. On the other hand, Akram *et al.* proposed a finite difference scheme which is essentially a combination of the extended cubic B-spline method and Caputo's fractional derivative for numerically solving time-fractional telegraph equations [15]. Kumar *et al.* (2019) proposed a finite difference technique to solve the Generalised Time-Fractional Telegraph Equation (GTFTE), in which the equation was defined using GFD terms [16]. Meanwhile, a new, iterative approach was developed by Ali *et al.* (2019) to address the two-dimensional hyperbolic telegraph fractional differential equation (2D-HTFDE), and this is critical in the mathematical modelling of transmission lines that facilitate a direct, unique relationship between voltage and current waves covering a specific time and distance [17]. Hosseininia and Heydari examined a new version of the nonlinear 2D telegraph equation, in which variable-order (V-O) time-fractional derivatives were applied in the Atangana-Baleanu-Caputo sense. In 2019, this was combined with Mittag-Leffler non-singular kernel [18]. On the other hand, to demonstrate the existence, uniqueness and stability of the integral solution to the nonlocal telegraph equation using the conformable time-fractional derivative, Bouaouid *et al.* (2019) employed the cosine family of linear operators. They also presented a key solution based on the classical trigonometric functions [19]. Furthermore, Mohammadian *et al.* (2020) attempted to provide semi-analytical solutions for fractional partial differential equations based on Riesz space fractional derivatives using the fractional reduced differential transform method (FRDTM) [20]. Nonetheless, to solve time-fractional telegraph equations, Wu and Yang (2020) employed pure alternating segment explicit-implicit (PASE-I) and implicit-explicit (PASI-E) parallel difference methods [21]. A new model was proposed by Hamada (2020) to solve the time-dependent Boltzmann transport equation. In this process, new terms such as the time derivative of reactivity and fractional integral of the neutron density were added [22]. In recent times, Devi and Jakhar (2021) applied an adapted decomposition method called the Sumudu-Adomian Decomposition Method (SADM) to solve fractional-order telegraph equations [23]. Meanwhile, Hamza *et al.* (2021) employed a double Sumudu matching transformation approach to identify and obtain accurate numerical solutions to linear space-time matching telegraph fractional equations [24], whilst Azhar *et al.* (2021) turned their attention towards to natural transform decomposition approach when they used non-singular kernel derivatives to investigate fractional-order telegraph equations. To solve the problem [25], they applied natural transformations to fractional telegraph equations, after which an inverse natural transformation was achieved. However, Ibrahim and Bijiga (2021) decided to portray the time-fractional telegraph equation as an optimisation issue [26], which they solved using a neural network approach. The Cauchy problem associated with the time-fractional equation of distributed order in Rn R+ was studied carefully by Vieira *et al.* (2021), who used Fourier, Laplace, and Mellin transformations to present the equation solution based on the Fox H-Function [27]. Moreover, in order to justify the matching of analytical and approximate solutions, Khater *et al.* (2021) used five numerical methods to examine various numerical solutions to the fractional nonlinear telegraph equation. These techniques were Adomian decomposition (AD), cubic B-spline (CBS), El Kalla (EK), extended cubic B-spline (ECBS), and exponential cubic B-spline (ExCBS) [28]. Meanwhile, Nikan *et al.* (2021) based their numerical solution on the nonlinear time-fractional telegraph equation on the Caputo sense. In this model, the neutron transport process that occurs inside nuclear reactors is effectively described [29].

In this context, this paper proposes a quadratic-polynomial spline-based method to obtain the numerical solution of the time-space fractional-order telegraph equation in the form:

$$\frac{\partial^\alpha u}{\partial x^\alpha}=\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u,\ \ x>0,\ 1<\alpha\le 2, \tag{1}$$

subject to boundary conditions

$$u(a,t)=\beta_1(t),\ u(b,t)=\beta_2(t),\ t>0 \tag{2}$$

and the initial conditions

$$u(x,0)=f_1(x),\ \frac{\partial u(x,0)}{\partial t}=f_2(x),\ a\le x\le b \tag{3}$$

The space-fractional partial derivative of order α in Equation (1) is considered in the Caputo sense, defined by [5] [6],

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x,t_j)=\frac{1}{\Gamma(n-\alpha)}\int_g^x\frac{\partial^n u(s,t_j)}{\partial x^n}(x-s)^{n-\alpha-1}\mathrm{d}s,\ n-1<\alpha\le n. \tag{4}$$

2. Derivation of the Method

To set up the quadratic polynomial spline method, select an integer $N>0$ and time-step size $k>0$. With $h=\frac{b-a}{N}$, then mesh points (x_i,t_j) are $x_i=a+ih$, for each $i=0,1,\cdots,N$, and $t_j=jk,\ k=\Delta t$ for each $j=0,1,\cdots$.

Let Z_i^j be an approximation to $u(x_i,t_j)$ obtained by the segment $P_i(x,t_j)$ of the spline function passing through the points (x_i,Z_i^j) and (x_{i+1},Z_{i+1}^j). Each segment has the form [30]

$$P_i(x,t_j)=a_i(t_j)(x-x_i)^2+b_i(t_j)(x-x_i)+c_i(t_j) \tag{5}$$

for each $i=0,1,\cdots,N-1$. To obtain expressions for the coefficients of (5) in terms of $Z_{i+1/2}^j$, D_i^j, and $S_{i+1/2}^j$, we first define

$$P_i(x_{i+1/2},t_j)=Z_{i+1/2}^j \tag{6}$$

$$P_i^{(1)}(x_i,t_j)=D_i^j \tag{7}$$

$$P_i^{(\alpha)}(x_{i+1/2},t_j)=\frac{\partial^\alpha}{\partial x^\alpha}P_i(x_{i+1/2},t_j)=S_{i+1/2}^j,\ 1<\alpha\le 2,\ x_i<x_{i+1/2}\le x_{i+1} \tag{8}$$

where $a_i\equiv a_i(t_j),b_i\equiv b_i(t_j),c_i\equiv c_i(t_j),d_i\equiv d_i(t_j)$ and $\theta=\omega h$. Equations

(5), (6) and (7), give

$$\frac{h^2}{4}a_i+\frac{h}{2}b_i+c_i=Z^j_{i+1/2} \tag{9}$$

$$b_i=D^j_i \tag{10}$$

Using Equations (4), (5), and (8), we obtain

$$\frac{\partial^\alpha}{\partial x^\alpha}u\left(x_{i+1/2},t_j\right)=\frac{1}{\Gamma(2-\alpha)}\int_{x_i}^{x_{i+1/2}}\frac{\partial^2 P_i\left(s,t_j\right)}{\partial x^2}\left(x_{i+1/2}-s\right)^{1-\alpha}\mathrm{d}s=S^j_{i+1/2}.$$

This equation can be simplified as:

$$\mu a_i=S^j_{i+1/2} \tag{11}$$

where $\mu=\frac{2}{\Gamma(3-\alpha)}\left(\frac{h}{2}\right)^{2-\alpha}$. By solving Equations (9), (10), and (11), we obtain the following expressions:

$$a_i=\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha-2}S^j_{i+1/2},$$

$$b_i=D^j_i,$$

$$c_i=-\frac{1}{2}\Gamma(3-\alpha)\left(\frac{h}{2}\right)^{\alpha}S^j_{i+1/2}-\frac{h}{2}D^j_i+Z^j_{i+1/2} \tag{12}$$

Spline Relations

Using the following continuity conditions at $x=x_i$

$$P_i\left(x_i,t_j\right)=P_{i-1}\left(x_i,t_j\right)\Rightarrow c_i=h^2a_{i-1}+hb_{i-1}+c_{i-1} \tag{13}$$

$$P^{(1)}_i\left(x_i,t_j\right)=P^{(1)}_{i-1}\left(x_i,t_j\right)\Rightarrow b_i=2ha_{i-1}+b_{i-1} \tag{14}$$

Using expressions in Equation (12), Equations (13) and (14) become

$$\begin{aligned}&Z^j_{i+1/2}-Z^j_{i-1/2}-\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}\left(S^j_{i+1/2}-S^j_{i-1/2}\right)-\frac{h}{2}\left(D^j_i-D^j_{i-1}\right)\\&=\frac{4\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}S^j_{i-1/2}+hD^j_{i-1}\end{aligned} \tag{15}$$

$$D^j_i-D^j_{i-1}=\frac{4\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha-1}S^j_{i-1/2} \tag{16}$$

By solving for D^j_{i-1}

$$hD^j_{i-1}=\left(Z^j_{i+1/2}-Z^j_{i-1/2}\right)-\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}S^j_{i+1/2}-\frac{7\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}S^j_{i-1/2} \tag{17}$$

Similarly,

$$hD^j_{i}=\left(Z^j_{i+3/2}-Z^j_{i+1/2}\right)-\frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}S^j_{i+3/2}-\frac{7\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}S^j_{i+1/2} \tag{18}$$

Using expressions in Equations (17) and (18), then Equation (16) becomes

$$Z^j_{i+3/2}-2Z^j_{i+1/2}+Z^j_{i-1/2}=\delta\left(S^j_{i+3/2}+6S^j_{i+1/2}+S^j_{i-1/2}\right),\ i=1,2,\cdots,N-2 \tag{19}$$

where $\delta = \frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}$. As $\alpha \to 2$, system (19) reduces to

$$Z_{i+3/2}^{j} - 2Z_{i+1/2}^{j} + Z_{i-1/2}^{j} = \frac{h^2}{8}\left(S_{i+3/2}^{j} + 6S_{i+1/2}^{j} + S_{i-1/2}^{j}\right), \quad i = 1, 2, \cdots, N-2. \tag{20}$$

Remark

The truncation error for Equation (19), that is,

$$T^{*j}_{i} = \left(u_{i-1/2}^{j} + u_{i+3/2}^{j}\right) - 2u_{i+1/2}^{j} - \delta\left(D_x^2 u_{i-1/2}^{j} + D_x^2 u_{i+3/2}^{j}\right) - 6\delta D_x^2 u_{i+1/2}^{j}$$

can be obtained by expanding this equation in Taylor series in terms of $u\left(x_{i+1/2}, t_j\right)$ and its derivatives as follows

$$T^{*j}_{i} = \left(h^2 - 8\delta\right) D_x^2 u_{i+1/2}^{j} + \left(\frac{h^4}{12} - \delta h^2\right) D_x^4 u_{i+1/2}^{j} + \left(\frac{h^6}{360} - \frac{\delta h^4}{12}\right) D_x^6 u_{i+1/2}^{j} + \cdots$$

Since $\delta = \frac{\Gamma(3-\alpha)}{2}\left(\frac{h}{2}\right)^{\alpha}$ then the last expression can be simplified as

$$T^{*j}_{i} = h^{\alpha}\left(h^{2-\alpha} - 8\theta\right) D_x^2 u_{i+1/2}^{j} + h^{2+\alpha}\left(\frac{h^{2-\alpha}}{12} - \theta\right) D_x^4 u_{i+1/2}^{j}$$
$$+ h^{4+\alpha}\left(\frac{h^{2-\alpha}}{360} - \frac{\theta}{12}\right) D_x^6 u_{i+1/2}^{j} + \cdots$$

where $\theta = \frac{\Gamma(3-\alpha)}{2^{\alpha+1}}$. From this expression of the local truncation error, our scheme is $O\left(h^{\alpha}\right), 1 < \alpha \le 2$.

$$S_i^j = \frac{\partial^{\alpha} Z_i^j}{\partial x^{\alpha}} = \frac{\partial^2 Z_i^j}{\partial t^2} + \frac{\partial Z_i^j}{\partial t} + Z_i^j \tag{21}$$

$$S_i^j = \frac{\partial^{\alpha} Z_i^j}{\partial x^{\alpha}} \approx \frac{Z_i^{j+1} - 2Z_i^j + Z_i^{j-1}}{k^2} + \frac{Z_i^{j+1} - Z_i^{j-1}}{2k} + Z_i^j \tag{22}$$

which can be discretised as follows:

$$\begin{aligned}
S_{i-1/2}^{j} &= \frac{\partial^{\alpha} Z_{i-1/2}^{j}}{\partial x^{\alpha}} \approx \frac{Z_{i-1/2}^{j+1} - 2Z_{i-1/2}^{j} + Z_{i-1/2}^{j-1}}{k^2} + \frac{Z_{i-1/2}^{j+1} - Z_{i-1/2}^{j-1}}{2k} + Z_{i-1/2}^{j} \\
S_{i+1/2}^{j} &= \frac{\partial^{\alpha} Z_{i+1/2}^{j}}{\partial x^{\alpha}} \approx \frac{Z_{i+1/2}^{j+1} - 2Z_{i+1/2}^{j} + Z_{i+1/2}^{j-1}}{k^2} + \frac{Z_{i+1/2}^{j+1} - Z_{i+1/2}^{j-1}}{2k} + Z_{i+1/2}^{j} \\
S_{i+3/2}^{j} &= \frac{\partial^{\alpha} Z_{i+3/2}^{j}}{\partial x^{\alpha}} \approx \frac{Z_{i+3/2}^{j+1} - 2Z_{i+3/2}^{j} + Z_{i+3/2}^{j-1}}{k^2} + \frac{Z_{i+3/2}^{j+1} - Z_{i+3/2}^{j-1}}{2k} + Z_{i+3/2}^{j}
\end{aligned} \tag{23}$$

Using Formulas (23) in (19) gives the following useful systems

$$\begin{aligned}
&AZ_{i-1/2}^{j+1} + BZ_{i+1/2}^{j+1} + AZ_{i+3/2}^{j+1} \\
&= A^{*}Z_{i-1/2}^{j} + B^{*}Z_{i+1/2}^{j} + A^{*}Z_{i+3/2}^{j} + \hat{A}Z_{i-1/2}^{j-1} + \hat{B}Z_{i+1/2}^{j-1} + \hat{C}Z_{i+3/2}^{j-1}
\end{aligned} \tag{24}$$

where

$$A = \frac{\delta}{k^2} + \frac{\delta}{2k}, \quad A^{*} = 1 + \frac{2\delta}{k^2} - \delta \quad \text{and} \quad \hat{A} = \frac{-\delta}{k^2} + \frac{\delta}{2k}$$

$$B=\frac{6\delta}{k^2}+\frac{3\delta}{k},\ \ B^*=-2+\frac{12\delta}{k^2}-6\delta\ \text{ and }\ \widehat{B}=\frac{-6\delta}{k^2}+\frac{3\delta}{k} \tag{25}$$

System (24) consists of $N-2$ equations in N unknowns. To get a solution to this system, we need 2-additional equations. Using the boundary conditions (2), that are $Z_0^j=\beta_1(t), Z_{N+1}^j=\beta_2(t)$, we can obtain the following equations: Suppose that $Z_{1/2}^j$ is linearly interpolated between Z_0^j and $Z_{3/2}^j$

$$-3Z_{1/2}^j+Z_{3/2}^j=-2Z_0^j=-2\beta_1,\ \ j\geq 0 \tag{26}$$

In a similar manner,

$$Z_{N-3/2}^j-3Z_{N-1/2}^j=-2Z_N^j=-2\beta_2,\ \ j\geq 0 \tag{27}$$

Equation (24) implies that the ($j+1$)st time step requires values from the (j)st and ($j-1$)st time steps. This produces a minor starting problem since values for $j=0$ are given by the first part in Equation (3)

$$Z_i^0=u(x_i,0)=f_1(x_i),\ \ i=1,\cdots,N \tag{28}$$

but values for $j=0$, which are needed in Equation (25) to compute Z_i^1, must be obtained from the first part in (3)

$$\frac{\partial Z_i^0}{\partial t}=u_t(x_i,0)=f_2(x_i),\ \ i=1,\cdots,N.$$

One approach is to replace $\frac{\partial Z_i^0}{\partial t}$ by a forward-difference approximation

$$f_2(x_i)=\frac{\partial Z_i^0}{\partial t}=\frac{Z_i^1-Z_i^0}{k}+o(k) \tag{29}$$

which gives us

$$Z_i^1=Z_i^0+kf_2(x_i),\ \ i=1,\cdots,N. \tag{30}$$

3. Stability Analysis

The Von Neumann technique will be carried out to investigate the stability of systems (23) and (24). The key part of Von Neumann analysis is to assume a solution of the form [31]

$$Z_i^j=\zeta_j \mathrm{e}^{(q\phi h)} \tag{31}$$

where ϕ is the wave number, $q=\sqrt{-1}$, h is the element size, and ζ is the amplification factor of the scheme. The use of Equations (31) and (24) gives us the characteristic equation in the form

$$\begin{aligned}&\zeta^{j+1}\left\{A\mathrm{e}^{((i-1)q\phi h)}+B\mathrm{e}^{(iq\phi h)}+A\mathrm{e}^{((i+1)q\phi h)}\right\}\\ &=\zeta^{j}\left\{A^*\mathrm{e}^{((i-1)q\phi h)}+B^*\mathrm{e}^{(iq\phi h)}+A^*\mathrm{e}^{((i+1)q\phi h)}\right\}\\ &\quad+\zeta^{j-1}\left\{\widehat{A}\mathrm{e}^{((i-1)q\phi h)}+\widehat{B}\mathrm{e}^{(iq\phi h)}+\widehat{A}\mathrm{e}^{((i+1)q\phi h)}\right\}\end{aligned} \tag{32}$$

Dividing both sides of the last equation by $\mathrm{e}^{(iq\phi h)}$ and canceling the common term, which is ζ^{j-1}, Equation (32) becomes:

$$\zeta^2\left\{Ae^{(-q\phi h)}+B+Ae^{(q\phi h)}\right\}-\zeta\left\{A^*e^{(-q\phi h)}+B^*+A^*e^{(q\phi h)}\right\}$$
$$-\left\{\hat{A}e^{(-q\phi h)}+\hat{B}+\hat{A}e^{(q\phi h)}\right\}=0 \tag{33}$$

where

$$A=\frac{\delta}{k^2}+\frac{\delta}{2k},\quad A^*=1+\frac{2\delta}{k^2}-\delta \quad \text{and} \quad \hat{A}=\frac{-\delta}{k^2}+\frac{\delta}{2k}$$

$$B=\frac{6\delta}{k^2}+\frac{3\delta}{k},\quad B^*=-2+\frac{12\delta}{k^2}-6\delta \quad \text{and} \quad \hat{B}=\frac{-6\delta}{k^2}+\frac{3\delta}{k}$$

This equation can be rewritten in the simple form

$$a\zeta^2+b\zeta+c=0 \tag{34}$$

where

$$a=\left(Ae^{(-q\phi h)}+B+Ae^{(q\phi h)}\right),\ b=-\left(A^*e^{(-q\phi h)}+B^*+A^*e^{(q\phi h)}\right)$$

and

$$c=-\left(\hat{A}e^{(-q\phi h)}+\hat{B}+\hat{A}e^{(q\phi h)}\right)$$

Or

$$a=B+2A\cos\varphi,\ \ b=-B^*-2A^*\cos\varphi,\ \ c=-\hat{B}-2\hat{A}\cos\varphi,\ \ \varphi=h\phi$$

Or

$$a=\frac{\delta}{k}(3+\cos\varphi)\left(\frac{2}{k}+1\right),$$
$$b=2(1-\cos\varphi)-2\delta(3+\cos\varphi)\left(\frac{2}{k^2}-1\right),$$
$$c=\frac{\delta}{k}(3+\cos\varphi)\left(\frac{2}{k}-1\right)$$

Equation (34) is a quadratic in ζ and, hence, will have two roots, that is

$$\varsigma_{\pm}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\varsigma_{\pm}=\sqrt{\frac{c}{a}}\left(-\psi\pm\sqrt{\psi^2-1}\right),\psi=\frac{b}{2\sqrt{ac}}$$

For the stability, we must have $|\zeta_{\pm}|\le 1$. So, we have three cases.

Case 1: The discriminant of the Quadratic equation (34) is zero, that is $\psi^2-1=0$, in that case $\varsigma_{\pm}=\pm\sqrt{\frac{c}{a}}=\pm\sqrt{\frac{2-k}{2+k}}$, $0<k<1$ and the stability condition, $|\zeta_{\pm}|\le 1$, is satisfied.

Case 2: Discriminant is less than zero, that is $\psi^2-1<0$, in this case

$$\varsigma_{\pm}=\sqrt{\frac{c}{a}}\left(-\psi\pm q\sqrt{1-\psi^2}\right)=\sqrt{\frac{2-k}{2+k}}\left(-\psi\pm q\sqrt{1-\psi^2}\right)\Rightarrow$$

the stability condition, $|\zeta_{\pm}|\le 1$, is satisfied.

Case 3: The discriminant is greater than zero. This means that one of ζ_{+}

and ζ_{-} does not satisfy the stability condition.

Thus, for stability we must have $\psi^2 - 1 \le 0$

$$-1 \le \psi \le 1 \tag{35}$$

$$-1 \le \frac{b}{2\sqrt{ac}} \le 1$$

Since

$$\sqrt{ac} > 0 \Rightarrow -2\sqrt{ac} \le b \le 2\sqrt{ac}$$

$$-\frac{2\delta}{k^2}(3+\cos\varphi)\sqrt{4-k^2} \le 2(1-\cos\varphi) - 2\delta(3+\cos\varphi)\left(\frac{2}{k^2}-1\right)$$
$$\le \frac{2\delta}{k^2}(3+\cos\varphi)\sqrt{4-k^2}$$

The right above inequality takes the form:

$$2(1-\cos\varphi) \le \frac{2\delta}{k^2}(3+\cos\varphi)\left(\sqrt{4-k^2}+2-k^2\right)$$

Which is satisfied for $k \ll \delta$, where h is small enough.

But the left above inequality takes the form:

$$-2(1-\cos\varphi) \le \frac{2\delta}{k^2}(3+\cos\varphi)\left(\sqrt{4-k^2}-2+k^2\right)$$

Which is satisfied for $k \ll \delta$, where h is small enough, and the method is then conditionally stable.

4. Numerical Example

In this section, a numerical example is included to illustrate the practical implementation of the proposed method.

Consider the following linear space-fractional telegraph equation [4]

$$\frac{\partial^{1.5}u}{\partial x^{1.5}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad x > 0 \tag{36}$$

Subject to the initial condition

$$u(x,0) = 0 \tag{37}$$

and boundary conditions

$$u(0.0125,t) \approx \exp(-t)(1+0.0125) + \frac{0.0125^{1.5}}{\Gamma(5/2)} + \frac{0.0125^{2.5}}{\Gamma(7/2)} + \frac{0.0125^{3}}{\Gamma(4)} + \frac{0.0125^{4}}{\Gamma(5)} + \cdots \tag{38}$$

and

$$u(1.0125,t) \approx \exp(-t)(1+1.0125) + \frac{1.0125^{1.5}}{\Gamma(5/2)} + \frac{1.0125^{2.5}}{\Gamma(7/2)} + \frac{1.0125^{3}}{\Gamma(4)} + \frac{1.0125^{4}}{\Gamma(5)} + \cdots \tag{39}$$

Then the exact solution is

$$u(x,t) \approx \exp(-t)\left(1 + x + \frac{x^{1.5}}{\Gamma(5/2)} + \frac{x^{2.5}}{\Gamma(7/2)} + \frac{x^3}{\Gamma(4)} + \frac{x^4}{\Gamma(5)} + \cdots\right). \quad (40)$$

Tables 1-3 illustrate the comparison between our method, developed in Section 3 and other existing methods [4] and [23] with $k = 0.00005$, $h = 0.025$, $t = 0.05, 0.1, 0.15$ and $\alpha = 1.5$.

Table 4 and Table 5 illustrate the comparison between our method, developed in Section 3 and other existing methods [4] and [23] with $k = 0.00005$, $h = 0.025$, $t = 0.05, 0.1$ and $\alpha = 1.75$.

Using Tables 1-5 and Figures 1-6, it can be seen that the obtained approximate numerical solutions are in good agreement with the approximate solutions obtained using methods [4] and [23] for all values of x and t.

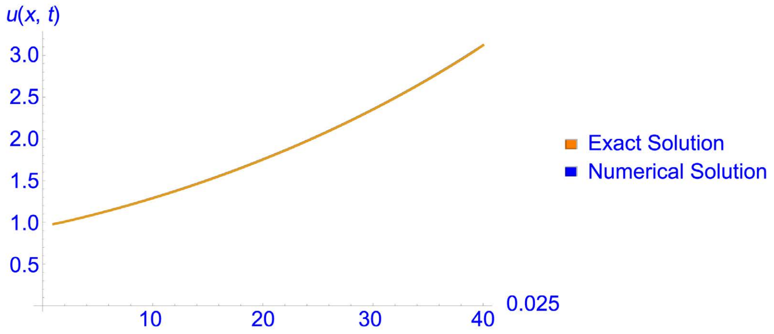

Figure 1. The comparison between our method and method [4] and [23] when $t = 0.05$, $k = 0.000005$, and $h = 0.025$ and $\alpha = 1.5$.

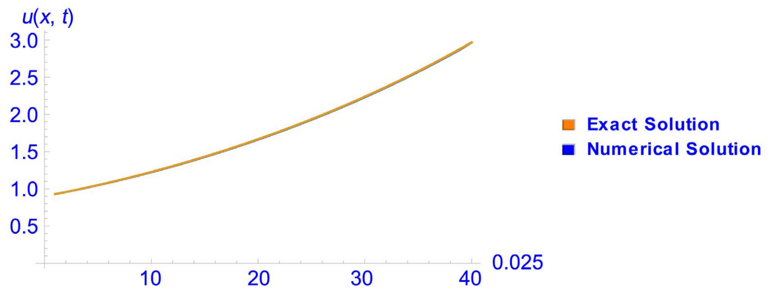

Figure 2. The comparison between our method and method [4] and [23] when $t = 0.1$, $k = 0.000005$, and $h = 0.025$ and $\alpha = 1.5$.

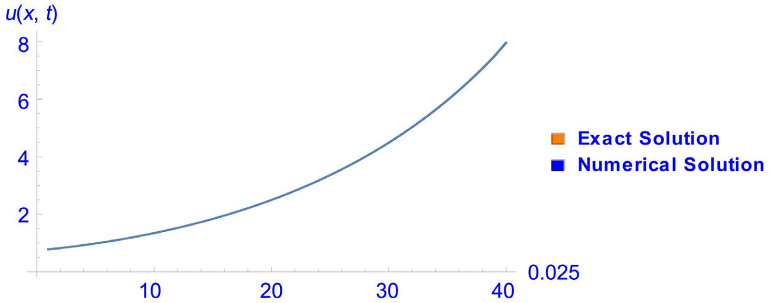

Figure 3. The comparison between our method and methods [4] and [23] when $t = 0.15$, $k = 0.000005$, and $h = 0.025$ and $\alpha = 1.5$.

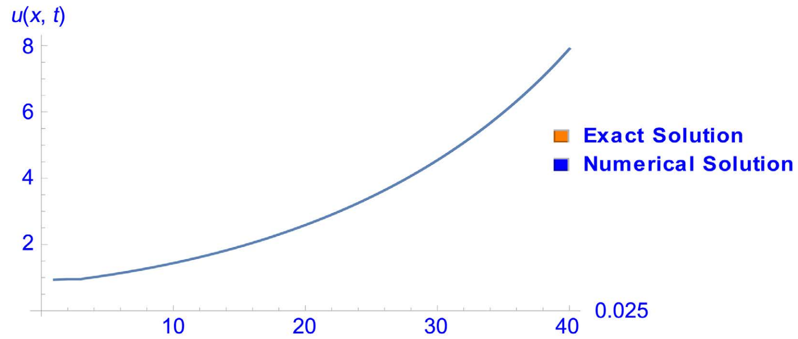

Figure 4. Comparison between our numerical method and methods [4] and [23] when $t = 0.05$, $k = 0.00005$, and $h = 0.025$, $\alpha = 1.75$.

Table 1. Comparison between our numerical method and methods [4] and [23] when $t = 0.05$, $k = 0.000005$, and $h = 0.025$ and $\alpha = 1.5$.

x	Our Method	Methods [4] and [23]
0.1	1.0689295078552934	1.0700487208006241
0.2	1.2105809555003169	1.2119422776213813
0.3	1.3713891366890514	1.3729692926753612
0.4	1.5513544745941052	1.5531657691651113
0.5	1.7514676554067925	1.7535302292267538
0.6	1.9732140327892282	1.9755523719324375
0.7	2.2184148686049125	2.2210569535598963
0.8	2.4891614801652615	2.4921386344095513
0.9	2.7877830486712691	2.7911317256723061
1.0	3.1196537361432172	3.1205956925765412

Table 2. Comparison between our numerical method and methods [4] and [23] when $t = 0.1$, $k = 0.000005$, and $h = 0.025$ and $\alpha = 1.5$.

x	Our Method	Methods [4] and [23]
0.1	1.0139125288963764	1.0178618288749026
0.2	1.1480559941970572	1.1528351552698712
0.3	1.3004476445553682	1.3060087901287358
0.4	1.4710342588062175	1.4774169807571376
0.5	1.6607354930479643	1.6680095507919713
0.6	1.8709521529921287	1.8792035458243128
0.7	2.1034072173406435	2.1127347277180891
0.8	2.3600811054248516	2.3705955989853926
0.9	2.6431880681647740	2.6550066251169517
1.0	2.9649893224621964	2.9684024447489910

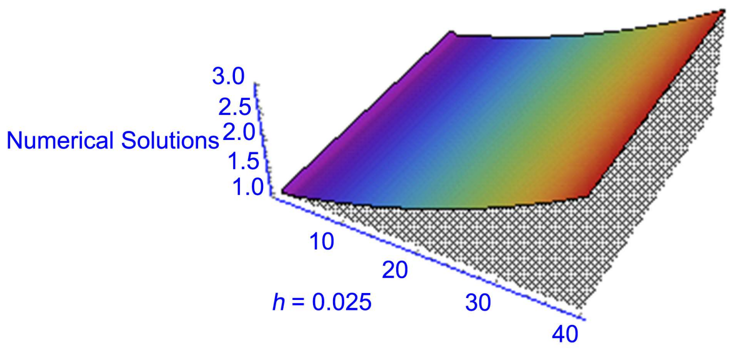

Figure 5. The 3-D behavior of the numerical solutions from $t = 0.0005$ to $t = 0.05$, $k = 0.0005$, and $h = 0.025$, $\alpha = 1.5$.

Table 3. Comparison between our numerical method and methods [4] and [23] when $t = 0.15$, $k = 0.000005$, and h = 0.025 and $\alpha = 1.5$.

x	Our Method	Methods [4] and [23]
0.1	0.9596371905024942	0.9682201217019183
0.2	1.0862170007301811	1.0966107212915508
0.3	1.2302168726205736	1.2423139898270312
0.4	1.3914776194134673	1.4053625043531945
0.5	1.5708357143835627	1.5866597650615402
0.6	1.7696035239597443	1.7875537074141623
0.7	1.9894042815310826	2.0096954391699512
0.8	2.2321069494266474	2.2549802873468003
0.9	2.4998194340024652	2.5255204240555812
1.0	2.8167740246016137	2.8236317492050944

Table 4. Comparison between our numerical method and methods [4] and [23] when $t = 0.05$, $k = 0.00005$, and $h = 0.025$, $\alpha = 1.75$.

x	Our Method	Methods [4] and [23]
0.1	0.9591480315753628	1.0572785260392232
0.2	1.0867445813092063	1.1797304399209085
0.3	1.2307489176630863	1.3176660343983828
0.4	1.3920372585030207	1.4716641084951654
0.5	1.5714366396546564	1.6428130928171421
0.6	1.7702550524339082	1.8325254529378252
0.7	1.9901139102638157	2.0424754500651385
0.8	2.2328814918984262	2.2745763567746264
0.9	2.5005404809568174	2.5309742073637795
1.0	2.8087858338014686	2.8140496901826731

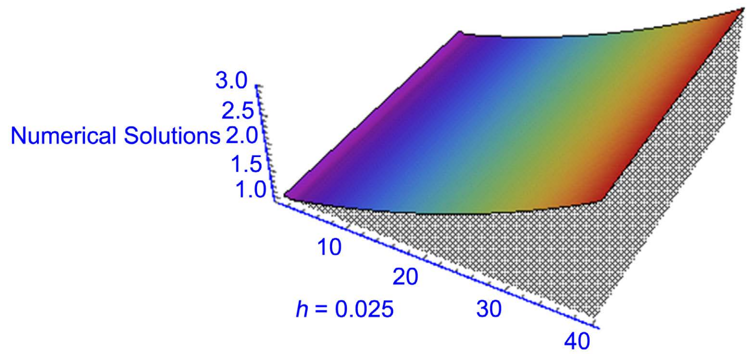

Figure 6. The 3-D behavior of the numerical solutions from $t = 0.0005$ to $t = 0.05$, $k = 0.0005$, and $h = 0.025$, $\alpha = 1.75$.

Table 5. Comparison between our numerical method and methods [4] and [23] when $t = 0.1$, $k = 0.00005$, and $h = 0.025$, $\alpha = 1.75$.

x	Our Method	Methods [4] and [23]
0.1	0.9597673920033094	1.0057144438612532
0.2	1.0867461330050554	1.1221943074319412
0.3	1.2307489144320851	1.2534027035849116
0.4	1.3920372584910621	1.3998902029822125
0.5	1.5714366396549502	1.5626921528426882
0.6	1.7702550524472353	1.7431521319809578
0.7	1.9901139126215512	1.9428627469222985
0.8	2.2328811392604192	2.1636439588376586
0.9	2.4997581380145126	2.4075371386967985
1.0	2.6931301646725034	2.6768068673088763

5. Conclusion

In the present work, a numerical approach to solving the linear space-fractional telegraph equation has been proposed based on the quadratic polynomial spline. Von-Neumann stability analysis was performed, with the findings revealing that the model has high conditional stability. The numerical example is effective and supports the theoretical analysis that the numerical approach is accurate and effective in solving the time-space fractional-order telegraph equation. For the values of x and t, the approximate numerical solutions identified in this work are in line with approximate solutions acquired using the [4] and [23] methods. Additionally, it is important to note that the local truncation error of our proposed model is $O\left(h^{\alpha}\right), 1<\alpha \leq 2$. It is reasonable to conclude that the proposed method is efficient and effective in identifying approximate solutions for many different linear partial differential equations of fractional order.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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