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The Marshall-Olkin Right Truncated Fréchet-Inverted Weibull Distribution: Its Properties and Applications

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Abstract

In this paper, a new probability distribution is proposed by using Marshall and Olkin transformation. Some of its properties such as moments, moment generating function, order statistics and reliability functions are derived. The method of maximum likelihood is used to estimate the model parameters. The graphs of the reliability function and hazard rate function are plotted by taken some values of the parameters. Three real life applications are introduced to compare the behaviour of the new distribution with other distributions.

Keywords

Marshall and Olkin, Moment Generating Function, Density Function, Order Statistics, Reliability Function, Inverse Weibull Distribution, Maximum Likelihood

1. Introduction

In the past, researchers highlighted on the inversion of univariate probability models. They have applied the inverse technique for many distributions. For example, there are many examples such as inverted beta [1], inverse Rayleigh [2], inverse Gaussian [3], inverse Weibull [4], inverted Burr type XII also called Burr type III [5], inverted exponential [6] and many other distributions. The Weibull distribution is a continuous probability distribution which identified by Fréchet [7]. Inverse Weibull distribution has two parameters α and β with probability density function of a random variable X is denoted by

$$G(x) = e^{\left(-\alpha x^{-\beta}\right)}, x \geq 0, \alpha > 0, \beta > 0 \tag{1}$$

The basic objective in this paper is the study how the Marshall and Olkin right truncated Fréchet-Inverted Weibull distribution applied. Later, Haq [8] used this conversion to improve Marshall-Olkin length biased moment exponential distribution. Marshall and Olkin [9] proposed an ingenious approach for adding an additional shape parameter to the existing distribution. So, we find the cumulative distribution function $F(x)$ of right truncated Fréchet-inverted Weibull distribution (RTFIWD)

$$F(x)=\frac{G(x)}{G(b)}=\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)},0<x\le b,\alpha>0,\beta>0. \tag{2}$$

The pdf $f(x)$ of (RTFIWD) is

$$f(x)=\frac{\mathrm{d}F(x)}{\mathrm{d}x}=\frac{\alpha\beta}{x^{\beta+1}}\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)},0<x\le b,\alpha>0,\beta>0, \tag{3}$$

the reliability function $R(x)$ (the survival function) of the Marshall and Olkin (MO) family is defined by

$$R(x)=\frac{\gamma\bar{F}(X)}{1-\bar{\gamma}\bar{F}(X)},\gamma>0 \tag{4}$$

$$\bar{\gamma}=1-\gamma,$$

using (2) into (4), we obtain $R(x)$ by the Marshall and Olkin right truncated Fréchet-inverted Weibull distribution (MORTFIWD) as

$$R(x)=\frac{\gamma\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}{1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)},\gamma>0. \tag{5}$$

The pdf corresponding to (5) is given by

$$\begin{aligned}g(x)&=\frac{-\mathrm{d}R(x)}{\mathrm{d}x}=\frac{\gamma\bar{F}(X)}{1-\bar{\gamma}\bar{F}(X)},\gamma>0\\&=\frac{\gamma\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{x^{\beta+1}\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^2},0<x\le b,\alpha>0,\beta>0;\gamma>0,\end{aligned} \tag{6}$$

and its hazard function reduces to

$$h(x)=\frac{g(x)}{R(x)}=\frac{\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{x^{\beta+1}\left[1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right]\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]}. \tag{7}$$

The cumulative distribution function $F(x)$ of the Marshall and Olkin of the right truncated Fréchet-inverted Weibull distribution (MORTFIWD) is denoted by

$$F(x)=1-R(x), \tag{8}$$

by substituting (5) in (8), we find

$$F(x) = 1 - \frac{\gamma\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}$$
$$= \frac{\left(1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right)-\gamma\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}$$
$$= \frac{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)-\gamma\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)} \quad (9)$$
$$= \frac{1-\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)(\bar{\gamma}+\gamma)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}$$
$$= \frac{1-\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)(1-\gamma+\gamma)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}$$
$$= \frac{1-\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}$$
$$= \frac{e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)}.$$

Figure 1 and Figure 2 outlined the manner of the density function and interprets the susctability and elasticity of the pattern graphically. The pdf plot shows that for some different values β. Figure 3 and Figure 4 outlined the manner of

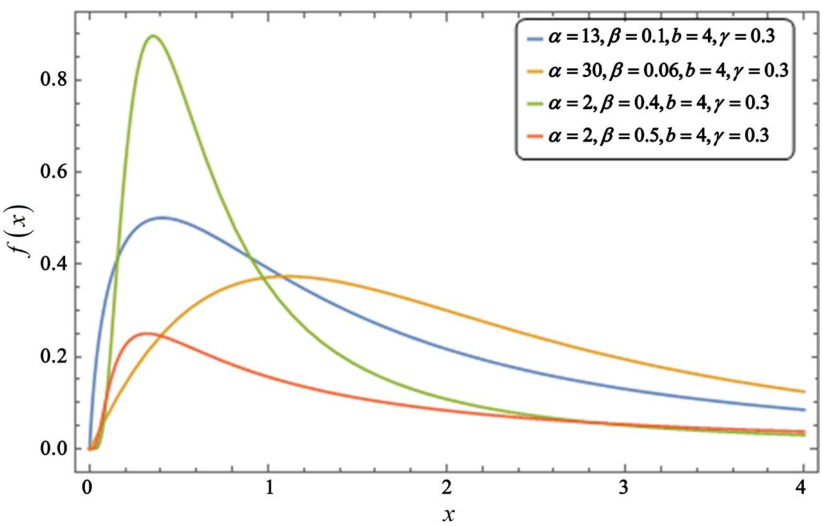

Figure 1. The density function for different values of the parameters.

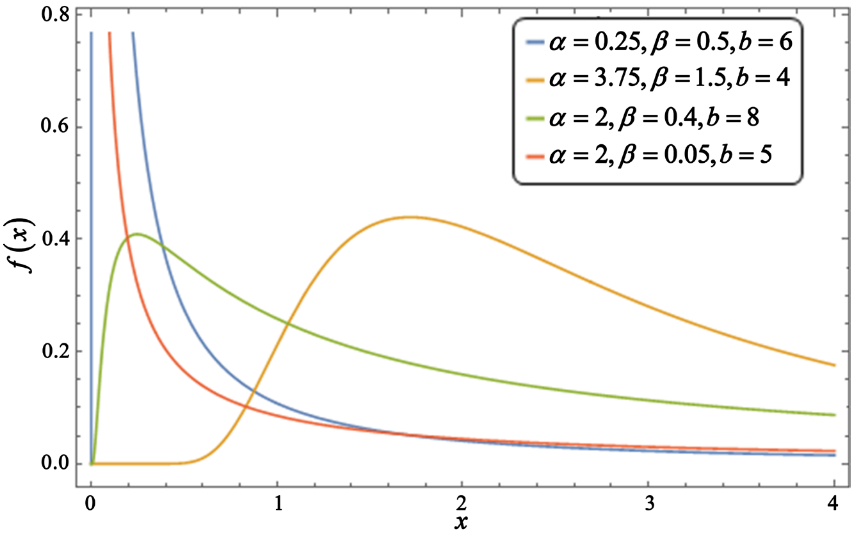

Figure 2. The density function for different values of the parameters.

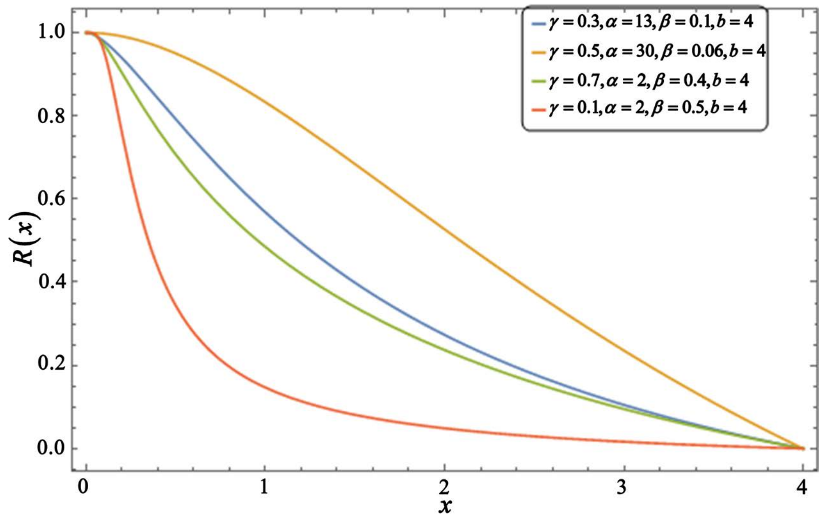

Figure 3. The hazard rate function for different values of the parameters.

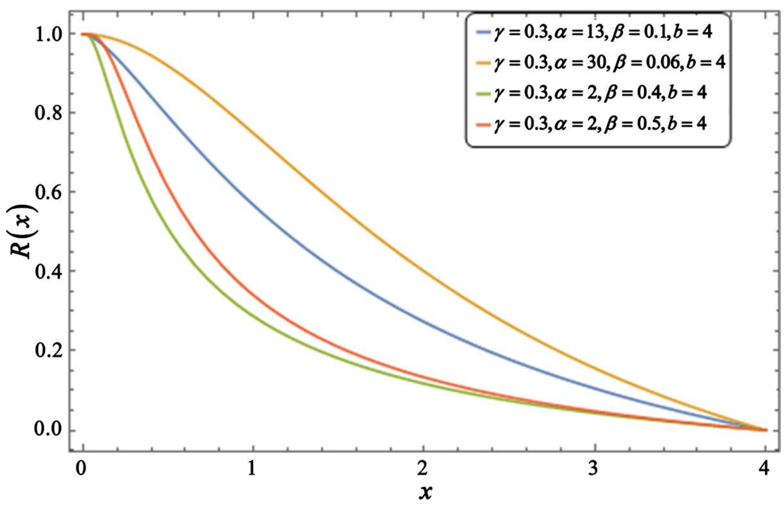

Figure 4. The hazard rate function for different values of the parameters.

hazard rate function. Figure 5 and Figure 6 outlined the manner of reserved hazard rate function which is monocular at different parameter combinations.

2. Moments

The rth moments μ_r' of MORTFIWD

$$\mu_r' = E\left(x^r\right) = \int_0^b x^r \cdot g\left(x\right)\mathrm{d}x$$

$$= \int_0^b x^r \cdot \frac{\gamma\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{x^{\beta+1}\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^2}\mathrm{d}x \tag{10}$$

$$= \int_0^b x^{r-\beta-1} \cdot \gamma\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}}\mathrm{d}x,$$

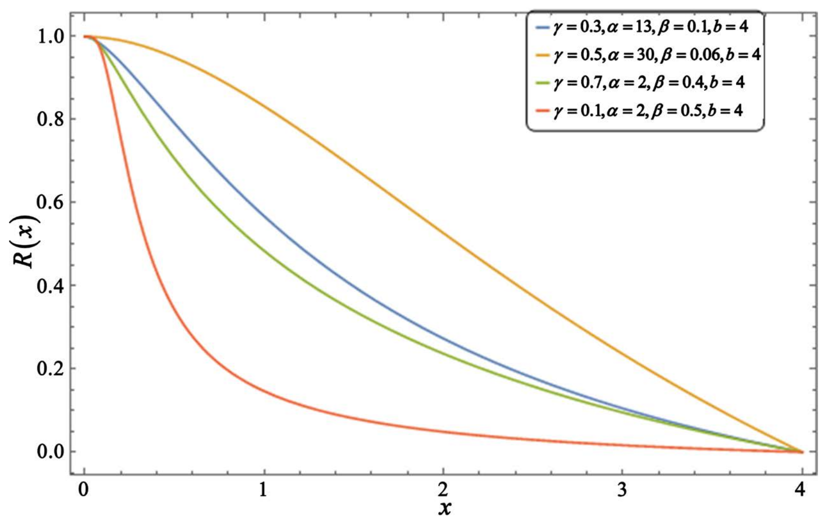

Figure 5. The reserved hazard rate function for different values of the parameters.

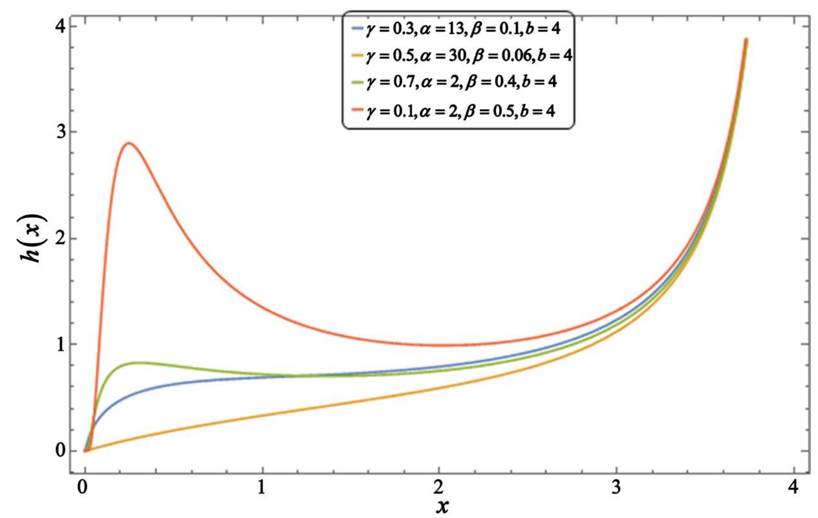

Figure 6. The reserved hazard rate function for different values of the parameters.

since $0<\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{2}<1$, for $x>0$ we have

$$\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}=\sum_{i=0}^{\infty}(i+1)\bar{\gamma}^{i}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{i}, \tag{11}$$

by substituting (11) in (10), we find

$$\begin{aligned}\mu_r' &= \int_0^b x^{r-\beta-1}\cdot\gamma\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)\left[1-\bar{\gamma}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}}\mathrm{d}x\\ &= \int_0^b x^{r-\beta-1}\cdot\gamma\alpha\beta\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\sum_{j=0}^{\infty}(j+1)\bar{\gamma}^{j}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{j}\mathrm{d}x\\ &= \sum_{j=0}^{\infty}(j+1)\bar{\gamma}^{j}\cdot\gamma\alpha\beta\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{j}\mathrm{d}x\\ &= \gamma\alpha\beta\sum_{i=0}^{\infty}(j+1)\bar{\gamma}^{j}\cdot\phi_{r,b}(x),\end{aligned} \tag{12}$$

where

$$\phi_{r,b}(x)=\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\left(1-\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{j}\mathrm{d}x \tag{13}$$

$$\phi_{r,b}(x)=\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{i}\mathrm{e}^{-\alpha i\left(x^{-\beta}-b^{-\beta}\right)}\mathrm{d}x \tag{14}$$

$$\begin{aligned}&=\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{i}\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha(i+1)\left(x^{-\beta}-b^{-\beta}\right)}\mathrm{d}x\\ &=\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{i}\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha(i+1)x^{-\beta}}\mathrm{e}^{\alpha(i+1)b^{-\beta}}\mathrm{d}x\\ &=\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{i}\mathrm{e}^{\alpha(i+1)b^{-\beta}}\int_0^b x^{r-\beta-1}\mathrm{e}^{-\alpha(i+1)x^{-\beta}}\mathrm{d}x.\end{aligned} \tag{15}$$

Setting

$$-\alpha(i+1)x^{-\beta}=t$$

$$x^{-\beta}=\frac{1}{\alpha(i+1)}t$$

$$x=\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}t^{\frac{-1}{\beta}}$$

$$\mathrm{d}x=\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\left(\frac{-1}{\beta}\right)t^{\frac{-1}{\beta}-1}\mathrm{d}t$$

$$x=0\to t=0$$

$$x=b\to t=\alpha(i+1)b^{-\beta},$$

we find (14) becomes as follows

$$\begin{aligned}
\phi_{r,b}(x) &= \sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\int_0^b x^{r-\beta-1}e^{-\alpha(i+1)x^{-\beta}}dx \\
&= \sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\int_0^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} t^{\frac{-1}{\beta}}\right)^{r-\beta-1} \\
&\quad\times e^{-t}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\left(\frac{-1}{\beta}\right)t^{\frac{-1}{\beta}-1}dt \\
&= \sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \\
&\quad\times\left(\frac{-1}{\beta}\right)\int_0^{\alpha(i+1)b^{-\beta}}\left(t^{\frac{-1}{\beta}}\right)^{r-\beta-1} t^{\frac{-1}{\beta}-1}e^{-t}dt \\
&= \sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \\
&\quad\times\left(\frac{-1}{\beta}\right)\int_0^{\alpha(i+1)b^{-\beta}} t^{\frac{-r+\beta}{\beta}-1}e^{-t}dt \\
&= \sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \\
&\quad\times\left(\frac{-1}{\beta}\right)\left(\alpha(i+1)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}\Gamma\left(\frac{-r+\beta}{\beta}\right)e^{\alpha(i+1)b^{-\beta}} \\
&\quad\times\sum_{k=0}^{\infty}\frac{\left(\alpha(i+1)b^{-\beta}\right)^k}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha(i+1)b^{-\beta}+1\right)},
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^{\alpha(i+1)b^{-\beta}} t^{\frac{-r+\beta}{\beta}-1}e^{-t}dt \\
&= \left(\alpha(i+1)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}\Gamma\left(\frac{-r+\beta}{\beta}\right)e^{\alpha(i+1)b^{-\beta}}\sum_{k=0}^{\infty}\frac{\left(\alpha(i+1)b^{-\beta}\right)^k}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha(i+1)b^{-\beta}+1\right)},
\end{aligned}$$

by using the definition of the incomplete gamma function

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt = x^s\Gamma(s)e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(s+x+1)}.$$

So we find the *r*th moments μ_r' of MORTFIWD

$$\mu_r' = \gamma\alpha\beta\sum_{i=0}^{\infty}(j+1)\bar{\gamma}^j\sum_{i=0}^{\infty}\binom{j}{i}(-1)^i e^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1}$$

$$\times\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\left(\frac{-1}{\beta}\right)\left(\alpha(i+1)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}\Gamma\left(\frac{-r+\beta}{\beta}\right)e^{\alpha(i+1)b^{-\beta}}$$
$$\times\sum_{k=0}^{\infty}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha(i+1)b^{-\beta}+1\right)} \quad (16)$$

$$\mu_r' = \gamma\alpha\beta\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\bar{\gamma}^{j}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{r-\beta-1}$$
$$\times\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\left(\alpha(i+1)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}$$
$$\times\Gamma\left(\frac{-r+\beta}{\beta}\right)e^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

3. Moment Generting Function

We need to compute the moment generting function of the Marshall and Olkin of the right truncated Fréchet-inverted Weibull distribution (MORTFIWD). If X a random variable has the distribution MORTFIWD where $\theta=(\alpha,\beta,b)$ and γ is a positive integer, then the moment generting function of MORTFIWD is denoted by

$$M_x(t)=E\left(e^{tx}\right)=\int_0^b e^{tx}\cdot g(x)\,dx$$
$$=\int_0^b e^{tx}\frac{\gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{x^{\beta+1}\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{2}}dx \quad (17)$$
$$=\int_0^b x^{-\beta-1}e^{tx}\gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}}dx.$$

From (11), we find

$$M_x(t)=\sum_{j=0}^{\infty}(j+1)\bar{\gamma}^{j}\cdot\gamma\alpha\beta\int_0^b x^{-\beta-1}e^{tx}e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{j}dx$$
$$=\sum_{j=0}^{\infty}(j+1)\bar{\gamma}^{j}\cdot\gamma\alpha\beta\Pi_j(x), \quad (18)$$

where

$$\Pi_j(x)=\int_0^b x^{-\beta-1}e^{tx}e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^{j}dx$$

$$\Pi_j(x)=\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{i}e^{\alpha(i+1)b^{-\beta}}\int_0^b x^{-\beta-1}e^{tx}e^{-\alpha(i+1)x^{-\beta}}dx$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \int_0^{\alpha(i+1)b^{-\beta}} \left(\left(\frac{1}{\alpha(i+1)} \right)^{\frac{-1}{\beta}} u^{-\frac{1}{\beta}} \right)^{-\beta-1}$$

$$\times e^{t\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} u^{\frac{-1}{\beta}}} e^{-u} \left(\frac{1}{\alpha(i+1)} \right)^{\frac{-1}{\beta}} \left(\frac{-1}{\beta} \right) u^{\frac{-1}{\beta}-1} du \tag{19}$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \frac{-1}{\alpha\beta(i+1)} \int_0^{\alpha(i+1)b^{-\beta}} e^{-u} e^{t\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} u^{\frac{-1}{\beta}}} du$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \frac{-1}{\alpha\beta(i+1)} \int_0^{\alpha(i+1)b^{-\beta}} e^{t\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} u^{\frac{-1}{\beta}} - u} du. \tag{20}$$

4. Inverse Moments

In this section, inverse moments

$$\mu'_{r^{-1}} = E\left(\frac{1}{x^r}\right) = \int_0^b \frac{1}{x^r} \cdot g(x) dx$$

$$= \int_0^b \frac{1}{x^r} \cdot \frac{\gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}}{x^{\beta+1}\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^2} dx \tag{21}$$

$$= \int_0^b x^{-r-\beta-1} \cdot \gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}} dx,$$

substituting by (11) in (10), we have

$$\mu'_{r^{-1}} = \int_0^b x^{-r-\beta-1} \cdot \gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)\right]^{-2}} dx \tag{22}$$

$$= \int_0^b x^{-r-\beta-1} \cdot \gamma\alpha\beta e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)} \sum_{j=0}^{\infty} (j+1)\bar{\gamma}^j \left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^j dx \tag{23}$$

$$= \sum_{j=0}^{\infty} (j+1)\bar{\gamma}^j \cdot \gamma\alpha\beta \int_0^b x^{-r-\beta-1} e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)} \left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^j dx \tag{24}$$

$$= \gamma\alpha\beta \sum_{i=0}^{\infty} (j+1)\bar{\gamma}^j \cdot \phi_{r,b}(x),$$

where

$$\phi_{r^{-1},b}(x) = \int_0^b x^{-r-\beta-1} e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)} \left(1-e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)}\right)^j dx \tag{25}$$

$$\phi_{r^{-1},b}(x) = \int_0^b x^{-r-\beta-1} e^{-\alpha\left(x^{-\beta}-b^{-\beta}\right)} \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{-\alpha i\left(x^{-\beta}-b^{-\beta}\right)} dx$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i \int_0^b x^{-r-\beta-1} e^{-\alpha(i+1)\left(x^{-\beta}-b^{-\beta}\right)} dx$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i \int_0^b x^{-r-\beta-1} e^{-\alpha(i+1)x^{-\beta}} e^{\alpha(i+1)b^{-\beta}} dx$$
$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \int_0^b x^{-r-\beta-1} e^{-\alpha(i+1)x^{-\beta}} dx, \tag{26}$$

setting

$$-\alpha(i+1)x^{-\beta} = t$$

$$x^{-\beta} = \frac{1}{\alpha(i+1)} t$$

$$x = \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} t^{\frac{-1}{\beta}}$$

$$dx = \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \left(\frac{-1}{\beta}\right) t^{\frac{-1}{\beta}-1} dt$$

$$x = 0 \to t = 0$$

$$x = b \to t = \alpha(i+1)b^{-\beta},$$

hence (14) becomes as follows

$$\phi_{r^{-1},b}(x) = \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \int_0^b x^{-r-\beta-1} e^{-\alpha(i+1)x^{-\beta}} dx$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \int_0^{\alpha(i+1)b^{-\beta}} \left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} t^{\frac{-1}{\beta}} \right)^{-r-\beta-1}$$
$$\times e^{-t} \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \left(\frac{-1}{\beta}\right) t^{\frac{-1}{\beta}-1} dt$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \right)^{-r-\beta-1} \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$
$$\times \left(\frac{-1}{\beta}\right) \int_0^{\alpha(i+1)b^{-\beta}} \left(t^{\frac{-1}{\beta}} \right)^{r-\beta-1} t^{\frac{-1}{\beta}-1} e^{-t} dt$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \right)^{-r-\beta-1} \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$
$$\times \left(\frac{-1}{\beta}\right) \int_0^{\alpha(i+1)b^{-\beta}} t^{\frac{-r+\beta}{\beta}-1} e^{-t} dt$$

$$= \sum_{i=0}^{\infty} \binom{j}{i} (-1)^i e^{\alpha(i+1)b^{-\beta}} \left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}} \right)^{-r-\beta-1} \left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$
$$\times \left(\frac{-1}{\beta}\right) \left(\alpha(i+1)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}$$

$$\times\Gamma\left(\frac{-r+\beta}{\beta}\right)\mathrm{e}^{\alpha(i+1)b^{-\beta}}\sum_{k=0}^{\infty}\frac{\left(\alpha\left(i+1\right)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha\left(i+1\right)b^{-\beta}+1\right)},\tag{27}$$

where

$$\int_{0}^{\alpha(i+1)b^{-\beta}} t^{\frac{-r+\beta}{\beta}-1}\mathrm{e}^{-t}\mathrm{d}t$$

$$=\left(\alpha\left(i+1\right)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}\Gamma\left(\frac{-r+\beta}{\beta}\right)\mathrm{e}^{\alpha(i+1)b^{-\beta}}\sum_{k=0}^{\infty}\frac{\left(\alpha\left(i+1\right)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha\left(i+1\right)b^{-\beta}+1\right)},$$

by using the definition of the incomplete gamma function

$$\gamma\left(s,x\right)=\int_{0}^{x}t^{s-1}\mathrm{e}^{-t}\mathrm{d}t=x^{s}\Gamma\left(s\right)\mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{x^{k}}{\Gamma\left(s+x+1\right)},$$

so the inverse *r*th moments $\mu'_{r^{-1}}$ about MORTFIWD

$$\mu'_{r^{-1}}=\gamma\alpha\beta\sum_{i=0}^{\infty}\left(j+1\right)\overline{\gamma}^{j}\sum_{i=0}^{\infty}\binom{j}{i}\left(-1\right)^{i}\mathrm{e}^{\alpha(i+1)b^{-\beta}}\left(\left(\frac{1}{\alpha\left(i+1\right)}\right)^{\frac{-1}{\beta}}\right)^{-r-\beta-1}\tag{28}$$

$$\times\left(\frac{1}{\alpha\left(i+1\right)}\right)^{\frac{-1}{\beta}}\left(\frac{-1}{\beta}\right)\left(\alpha\left(i+1\right)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}$$

$$\times\Gamma\left(\frac{-r+\beta}{\beta}\right)\mathrm{e}^{\alpha(i+1)b^{-\beta}}\sum_{k=0}^{\infty}\frac{\left(\alpha\left(i+1\right)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha\left(i+1\right)b^{-\beta}+1\right)}\tag{29}$$

$$=\gamma\alpha\beta\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(-1\right)^{i+1}\left(j+1\right)\overline{\gamma}^{j}\left(\left(\frac{1}{\alpha\left(i+1\right)}\right)^{\frac{-1}{\beta}}\right)^{-r-\beta-1}$$

$$\times\left(\frac{1}{\alpha\left(i+1\right)}\right)^{\frac{-1}{\beta}}\left(\alpha\left(i+1\right)b^{-\beta}\right)^{\frac{-r+\beta}{\beta}}\tag{30}$$

$$\times\Gamma\left(\frac{-r+\beta}{\beta}\right)\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha\left(i+1\right)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{-r+\beta}{\beta}+\alpha\left(i+1\right)b^{-\beta}+1\right)}.$$

5. The Mean Time to Failure

We need to compute the mean time to failure of the Marshall and Olkin of the right truncated Fréchet-inverted Weibull distribution (MORTFIWD). If X a random variable has the distribution MORTFIWD where $\theta=\left(\alpha,\beta,b\right)$ and γ is a positive integer, then the mean time to failure of MORTFIWD is given by

$$MTTF=E\left(X\right)=\int_{0}^{b}xg\left(x\right)\mathrm{d}x=\int R\left(x\right)\mathrm{d}x,$$

putting $r=1$ in Equation (16), we find

$$MTTF=\gamma\alpha\beta\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\gamma^{-j}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{-\beta}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$

$$\times\left(\alpha(i+1)b^{-\beta}\right)^{\frac{\beta-1}{\beta}}\Gamma\left(\frac{\beta-1}{\beta}\right)\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

$$=\gamma\alpha\beta\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\gamma^{-j}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{-\beta}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$

$$\times\left(\alpha(i+1)b^{-\beta}\right)^{\frac{\beta-1}{\beta}}\frac{1}{\beta}\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

$$=\gamma\alpha\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\gamma^{-j}\left(\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}\right)^{-\beta}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{-1}{\beta}}$$

$$\times\left(\alpha(i+1)b^{-\beta}\right)^{\frac{\beta-1}{\beta}}\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

$$=\gamma\alpha\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\gamma^{-j}\left(\frac{1}{\alpha(i+1)}\right)^{\frac{\beta-1}{\beta}}\alpha(i+1)b^{1-\beta}$$

$$\times\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\binom{j}{i}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

$$=\gamma\alpha\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}(j+1)\gamma^{-j}\binom{j}{i}\left(\alpha(i+1)\right)^{\frac{1}{\beta}}$$

$$\times b^{1-\beta}\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}$$

$$=\gamma\alpha\sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+1}\gamma^{-j}\binom{j+1}{i}\left(\alpha(i+1)\right)^{\frac{1}{\beta}}b^{1-\beta}$$

$$\times\mathrm{e}^{2\alpha(i+1)b^{-\beta}}\frac{\left(\alpha(i+1)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha(i+1)b^{-\beta}+1\right)}.$$

If X a random variable has the distribution MORTFIWD. We find the mean time to failure of the Marshall and Olkin of the right truncated Fréchet-inverted Weibull distribution (MORTFIWD)

$$MTTF = \gamma\alpha \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \left(-1\right)^{i+1} \bar{\gamma}^{j} \binom{j+1}{i} \left(\alpha\left(i+1\right)\right)^{\frac{1}{\beta}} b^{1-\beta} e^{2\alpha(i+1)b^{-\beta}} \times \frac{\left(\alpha\left(i+1\right)b^{-\beta}\right)^{k}}{\Gamma\left(\frac{\beta-1}{\beta}+\alpha\left(i+1\right)b^{-\beta}+1\right)}. \tag{31}$$

6. Maximum Likehood Method

The maximum likelihood assessment method is essentially used and extends the maximum acquaintance about the properties of the assessment parameters. Furthermore, the natural approximation of the estimators may truthfully and mathematically for huge sample theory. Accordingly, the Maximum likehood assessment has depended to guess the unknown parameters (α, β and *b*) of the MORTFIWD distribution. Let random variable *X* from the observed distribution and have the parameters (α, β and $b)^{\mathrm{T}}$ with size *n*. The sample likelihood function is

$$L\left(\alpha,\beta,b\right) = \prod_{i=1}^{n} g\left(x_i\right) = \prod_{i=1}^{n} \frac{\gamma\alpha\beta e^{-\alpha\left(x_i^{-\beta}-b^{-\beta}\right)}}{x_i^{\beta+1}\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x_i^{-\beta}-b^{-\beta}\right)}\right)\right]^2} = \frac{\gamma^n\alpha^n\beta^n e^{-\alpha\sum_{i=0}^{n}\left(x_i^{-\beta}-b^{-\beta}\right)}}{\prod_{i=1}^{n} x_i^{\beta+1}\prod_{i=1}^{n}\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x_i^{-\beta}-b^{-\beta}\right)}\right)\right]^2}.$$

The log-likehood function is

$$\log L\left(\alpha,\beta,b\right) = n\log\gamma + n\log\alpha + n\log\beta - \alpha\sum_{i=1}^{n}\left(x_i^{-\beta}-b^{-\beta}\right) \tag{32}$$

$$-\sum_{i=1}^{n}\log x_i^{\beta+1} - \sum_{i=1}^{n}\log\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x_i^{-\beta}-b^{-\beta}\right)}\right)\right]^2 \tag{33}$$

$$= n\log\gamma + n\log\alpha + n\log\beta - \alpha\sum_{i=1}^{n}\left(x_i^{-\beta}-b^{-\beta}\right) - \left(\beta+1\right)\sum_{i=1}^{n}\log x_i - 2\sum_{i=1}^{n}\log\left[1-\bar{\gamma}\left(1-e^{-\alpha\left(x_i^{-\beta}-b^{-\beta}\right)}\right)\right]. \tag{34}$$

7. Application

The potentiality and flexibility of the new distribution is introduced in this section. The new model has contrasted with some other existing life-time models such as:

(i) Hinkley [10] introduced the first real-life data. Data compounds on 30 notices of the March precipitation in Minneapolis/St Paul. The values are

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20	2.81	1.87
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81	1.18	1.35
4.75	2.48	0.96	1.89	0.90	2.05						

(ii) The second data also compounds on 30 values for the failure time which be repairable objects used by Murthy *et al.* [11]. The values are

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	1.97
0.74	1.23	0.94	4.36	0.40	1.74	4.73	2.23	0.45	1.86
0.70	1.06	1.46	0.30	1.82	2.37	0.63	1.23	1.24	1.17

(iii) Bhaumik [12] introduced the third real data life The values are

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6
0.9	0.4	2.0	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1.0	0.2
0.10	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2		

The new generated Marshall and Olkin of the right truncated Fréchet-inverted Weibull distribution (MORTFIWD) is compared with the right truncated Fréchet-inverted Weibull distribution, Marshall-Olkin extended inverted Kumaraswamy (MOEIK) distribution with the Generalized Inverted Kumaraswamy (GIK), Transmuted Exponentiated Inverse Rayleigh (TEIR), Logistic Weibull (LW), Transmuted Power Lindley (TPL), Marshall Olkin Frechet (MOFr) and Inverted Kumaraswamy (IK) distribution for these data sets.

The new distribution is compared with other distributions. By using some numerous fineness of fit measures such as the log likelihood function ($2l$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion(CAIC), Hanna-Quinn Information Criterion (HQIC). The assessment parameters and fineness of fit measures of the pervious real life data sets. The fineness of fit measures such as AIC, BIC, CAIC, HQIC [13].

The guess of maximum likehood method for parameters with the fineness of fit measures (March precipitation).

Models	Estimates			−2l	AIC	BIC	CAIC	HQIC
MORTFIWD (α, β, b)	13.320	1.185136	4.75	75.0588	81.0589	81.982	85.2625	82.3036
RTFIWD (α, β)	1.38587	1.14316	-	75.7754	81.7754	82.6985	85.975	83.1202
MOEIK (α, β, λ)	4.3228	6.5798	6.9226	76.69	82.69	86.89	84.79	84.03
GIK (α, β, γ)	1.9552	3.9501	1.4202	78.63	84.63	88.84	86.77	85.98
TEIR (α, θ, λ)	6.5630	0.0958	−0.6700	84.20	90.20	94.41	92.30	91.55
LW (α, β, λ)	2.7709	0.3635	1.0061	77.86	83.86	88.06	85.96	85.21
TPL (α, θ, λ)	1.5965	0.4801	0.5812	77.30	83.31	89.05	86.18	86.19
MOFr (α, β, σ)	42.598	2.6975	0.3548	77.60	83.60	87.80	85.70	84.94
IK (α, β)	2.9872	8.5899	-	78.85	83.85	87.65	85.25	84.74

The guess of maximum likehood method for parameters with the fineness of fit measures (failure time data).

Models	Estimates			−2l	AIC	BIC	CAIC	HQIC
MORTFIWD (α, β, b)	30.3832	0.0614346	4.73	76.9488	82.9488	87.5279	83.7488	84.5104
RTFIWD (α, β)	1.39561	0.680631	–	79.1164	85.1164	89.32	86.0395	86.4611
MOEIK (α, β, λ)	3.8031	2.1861	9.6185	79.60	85.60	89.80	87.70	86.94
GIK (α, β, γ)	1.1623	1.4462	1.8481	81.64	87.64	91.84	89.74	88.98
TEIR (α, θ, λ)	7.2940	0.0254	−0.8803	117.0	123.0	127.2	125.1	124.3
LW (α, β, λ)	2.8927	0.5531	0.07775	80.83	86.83	91.03	88.93	88.17
TPL (α, θ, λ)	1.3380	0.6301	0.5356	80.73	86.73	90.93	88.83	88.07
MOFr (α, β, σ)	63.165	2.1070	0.1669	81.53	87.53	91.73	89.63	88.87
IK (α, β)	2.4609	4.1716	–	82.48	86.48	90.28	88.88	87.37

The guess of maximum likehood method for parameters with the fineness of fit measures (chloride data).

Models	Estimates			−2l	AIC	BIC	CAIC	HQIC
MORTFIWD (α, β, b)	2.27479	0.401845	8	106.265	112.265	113.065	116.844	113.826
RTFIWD (α, β)	1.09489	0.602397	–	106.279	112.28	113.08	116.859	113.841
MOEIK (α, β, λ)	2.1193	1.7026	2.2471	110.9	116.9	121.5	119.2	118.5
GIK (α, β, γ)	2.0236	2.6369	0.8858	111.5	117.5	122.1	119.8	119.1
TEIR (α, θ, λ)	6.9784	0.0138	−0.7798	170.8	176.8	181.4	179.1	178.3
LW (α, β, λ)	2.7709	0.7056	0.5526	111.9	117.9	122.5	120.2	119.5
TPL (α, θ, λ)	0.9265	0.7453	0.4094	111.2	117.2	122.5	120.5	119.7
MOFr (α, β, σ)	29.053	1.4730	0.1124	111.4	117.4	121.9	119.7	119.3
IK (α, β)	1.7409	2.1058	–	112.5	117.5	122.6	120.1	119.6

8. Conclusion

A new probability distribution is introduced by using Marshall and Olkin transformation. Some of its properties such as moments, moment generating function, order statistics and reliability functions are derived. The method of maximum likelihood is used to estimate the model parameters.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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