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AAK Theorem and a Design of Multidimensional BIBO Stable Filters

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Abstract

Rational approximation theory occupies a significant place in signal processing and systems theory. This research paper proposes an optimal design of BIBO stable multidimensional Infinite Impulse Response filters with a realizable (rational) transfer function $H \in \mathbb{C}[z_1, \cdots, z_n]$ thanks to the Adamjan, Arov and Krein (AAK) theorem. It is well known that the one dimensional AAK results give the best approximation of a polynomial as a rational function in the Hankel semi norm. We suppose that the Hankel matrix associated to the transfer function has a finite rank.

Keywords

Multidimensional Filter, Design, BIBO Stability, Optimal, AAK Theorem, Transfer Function, Hankel Matrix

1. Introduction

Multidimensional filters are central elements in digital signal processing and control systems.

Consider a n dimensional (nD) filter S with an analytic transfer function

$$H(z_1, \cdots, z_n) = \sum_{\alpha \in \mathbb{Z}^n} h_\alpha z^\alpha. \tag{1}$$

The sequence $\{h_\alpha\}_{\alpha \in \mathbb{Z}^n}$ is the so called impulse response of the filter.

To avoid burst in a filter, stability is a necessary condition. For instance, the filter S is BIBO stable if for any bounded input signal x, the output y is bounded. A fundamental theorem states that S is BIBO stable if and only if

$$\sum_{\alpha \in \mathbb{Z}^n} h_\alpha < \infty. \tag{2}$$

However, only a rational filter (filter having a rational transfer function) is physically realizable. Under some assumptions, the filter S is rational and BIBO stable if and only if all poles of H lie in the unit polydik

$\mathbb{D}^n = \left\{ z \in \mathbb{C}^n, |z_i| < 1, i = 1, 2, \cdots, n \right\}$ (see [1] [2] for example).

Note that BIBO stability has also been studied in the consensus coordination of control systems [3] [4].

Given the significance of rational approximation theory in signal processing and systems theory, during the last years several mathematicians have been researching the problem of approximating multivariate functions (see for example [5] [6] [7]). One can also cite the works of Avilov and all [8] through the method of Padé-type extended to several variables and recently Austin and all [9] by discrete least-squares methods.

We propose in this paper to introduce a possibility to design multidimensional BIBO stable Infinite Impulse Response (IIR) filters as realizable filters thanks to the one dimensional AAK (Adamyan, Arov and Kreĭn) theorem [2]. To our knowledge such approach does not exist in the literature. We are concerned with filters S such that the multidimensional transfer function $H \in \mathbb{C}\left[z_1, \cdots, z_n \right]$ may be written as a combination (sum or product) of separable polynomials. We assume that the Hankel operator associated to H has a finite rank.

It is known that the AAK theorem gives the best approximation of a univariate polynomial as a rational function in the Hankel semi-norm. A remarkable contribution of AAK's result is its significance to engineering. One of its important applications is the problem of system reduction which consists of finding a lower-dimensional linear system to approximate a given high-dimensional linear system in a certain optimal sense. The AAK result provides a wonderful characterization of this problem in sense that the Hankel norm of the error between the given high-order transfer function and the approximant is minimized over all transfer functions of the same (lower) order (see [2] for an overview on the topic). The origins of the AAK method can be founded in the papers by Adamyan-Arov-Kreĭn [10] [11]. One can cite also the following paper [12].

Outline

This paper is organized as follows. Section 2 provides some explanations on the system reduction problem and preliminaries on AAK theorem for one dimensional (1D) filters. In Section 3, we propose rational optimal approximations of the transfer functions of a multidimensional (nD) filter by our approach based on AAK results.

2. Preliminaries on AAK Theorem on 1D Filters

Consider $H(z) = \sum_{n=0}^{\infty} h_n z^{-n}$, the transfer function of an 1D filter. The Hankel matrix corresponding to $H(z)$ is the infinite matrix $\Gamma_H = \left[h_{\ell+m-1} \right]_{\ell, m=1,2,\cdots}$, namely

$$\Gamma_H = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & \cdots & \cdots \\ h_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{3}$$

The matrix Γ_H can be viewed as an operator on the space of square summable sequences ℓ^2. The operator norm of Γ_H is given by

$$\|\Gamma_H\|_s := \sup_{\|x\|_{\ell^2}=1} \|\Gamma_H x\| \tag{4}$$

and one defines the Hankel semi-norm of $H(z)$ by

$$\|H(z)\|_\Gamma = \|\Gamma_H\|_s. \tag{5}$$

If $h_0 = 0$, then $\|\Gamma_H\|_s$ is the spectral norm of the operator Γ_H.

Note that the terms

$$H_a(z) = h_0 \text{ and } H_s = \sum_{n=1}^{\infty} h_n z^{-n}$$

are called the analytic and the singular part of $H(z) = \sum_{n=0}^{\infty} h_n z^{-n}$.

For a positive integer *m*, let

$$\mathcal{H}_m = \{\text{Hankel matrix } \Gamma; \|\Gamma\| < \infty, \text{ rank } \Gamma \le m\}. \tag{6}$$

The problem stated as follows:

$$\|\Gamma_H - \hat{\Gamma}_m\|_s = \inf_{\Gamma_m \in \mathcal{H}_m} \|\Gamma_H - \Gamma_m\|_s \tag{7}$$

is called system reduction for the filter [2] and is equivalent to the extremal problem

$$\|H - \hat{r}_m\|_\Gamma = \inf_{r_m \in \mathcal{R}_m^s} \|H - r_m\| \tag{8}$$

with

$$\mathcal{R}_m^s := \left\{ r_d(z) = \frac{p_1 z^{d-1} + \cdots + p_d}{z^d + q_1 z^{d-1} + \cdots + q_d}; \text{ all poles of } r_d(z) \text{ lie in } |z| < 1, d \le m \right\}. \tag{9}$$

The following theorem makes a connection between the system reduction problem and the approximation of a polynomial as a rational function.

Theorem 1 ([2], Kronecker's theorem). The infinite Hankel matrix Γ_H has finite rank *m* if and only if the singular part H_s is a strictly proper rational function in *z i.e.*

$$H(z) = \sum_{n=0}^{\infty} h_n z^{-n} = h_0 + \frac{p_1 z^{m-1} + \cdots + p_m}{z^m - c_m z^{m-1} - \cdots - c_1}. \tag{10}$$

Moreover,

Corollary 1 (Kronecker Nehari result, [2]). The function

$$H(z) = \sum_{n=1}^{\infty} h_n z^{-n}$$

is in $\mathcal{R}_m^s$ if and only if its corresponding Hankel matrix Γ_H is in $\mathcal{H}_m$.

Definition 1. Let Γ be a Hankel matrix and Γ^* its adjoint.

The *s*-numbers (or singular values) of Γ are the eigenvalues of $|\Gamma| = \Gamma^*\Gamma$.

Any pair (ξ_m, η_m) of elements in ℓ^2 that satisfies

$$\Gamma\xi_m = s_m\eta_m \quad \text{and} \quad \Gamma\eta_m = s_m\xi_m$$

is called a Schmidt pair of Γ.

One denotes by $L^\infty(|z|=1)$ the espace of essentially bounded functions on the unit circle $|z|=1$. Then

Theorem 2 ([2], Adamjan, Arov, and Krein theorem). Let $f(z)$ be a given function in $L^\infty(|z|=1)$ such that the Hankel matrix Γ_f associated to f is a compact operator with *s*-numbers $s_1 \geq s_2 \geq \cdots \geq s_\infty = 0$ and let $\left(\xi_m = \left\{u_i^{(m)}\right\}_{i=1,2,\cdots}, \eta_m = \left\{v_i^{(m)}\right\}_{i=1,2,\cdots}\right)$, the Schmidt pair corresponding to s_m. Then a solution to the extremal problem

$$\|f - \hat{r}_m\|_\Gamma = \inf_{r \in \mathcal{R}_m^s} \|f - r\|$$

is given by the singular part $\left[\hat{h}(z)\right]_s$ of

$$\hat{h}(z) = f(z) - s_{m+1}\frac{\sum_{i=1}^{\infty} v_i^{(m+1)} z^i}{\sum_{i=1}^{\infty} u_i^{(m+1)} z^{i-1}} \tag{11}$$

A detailed proof of this theorem has been made in [2].

3. Approximation of the Transfer Function of an (*n*D) BIBO Stable Filter

Definition 2: We say that a polynomial $r \in \mathbb{C}[z_1, \cdots, z_n]$ is stable if

$$r(z_1, \cdots, z_n) = 0 \text{ only for } |z_1| < 1, \cdots, |z_n| < 1.$$

Let $\mathcal{R}_m^n$ be the set of rational polynomials in $\mathbb{C}[z_1, \cdots, z_n]$ with stable denominator and such that the degrees of the numerator and denominator do not exceed *m*.

One has the following result.

Proposition 1: Suppose that

$$H(z) = \sum_{\alpha \in \mathbb{N}^n - \{0,\cdots,0\}} h_\alpha z^\alpha \tag{12}$$

is the transfer function of a multidimensional (*n*D) BIBO stable filter such that

1) $H(z_1, \cdots, z_n) = H_1(z_1) \times \cdots \times H_n(z_n)$,

2) For each $i = 1, 2, \cdots, n$, H_i is a polynomial in $L^\infty(|z|=1)$,

3) The operators Γ_{H_i} are compact operators with finite ranks $R_i \leq m$.

Then an optimal approximation of *H* as a rational function in $\mathcal{R}_m^n$ is given by

$$\left[\hat{h}(z_1, \cdots, z_n)\right]_s = \left[\hat{h}_1(z_1)\right]_s \times \cdots \times \left[\hat{h}_n(z_n)\right]_s, \tag{13}$$

the elements $\left[\hat{h}_i(z_i)\right]_s$ being the singular parts of a certain polynomial function $\hat{h}_i$.

Remark 1: If the hypothesis one in the proposition 1 is replaced by

$$H(z_1,\cdots,z_n)=H_1(z_1)+\cdots+H_n(z_n),$$

then

$$\left[\hat{h}(z_1,\cdots,z_n)\right]_s=\left[\hat{h}_1(z_1)\right]_s+\cdots+\left[\hat{h}_n(z_n)\right]_s.$$

Proof. The existence of such approximation of the (nD) transfer function H (in $\mathcal{R}_m^n$) given in Equation (12) is certified by the Kronecker-Nehari result applied to each H_i. The Formula (13) is given by the application of the AAK theorem to each polynomial H_i and for each $i=1,2,\cdots,n$

$$\hat{h}_i(z_i)=H_i(z_i)-s_{m+1}^i\frac{\sum_{j=1}^{\infty}v_j^{(m+1)}z_i^j}{\sum_{j=1}^{\infty}u_j^{(m+1)}z_i^{j-1}} \tag{14}$$

with s_{m+1}^i the $(m+1)^{\text{th}}$ s-number of Γ_{H_i}. □

Moreover one has.

Corollary 2 (Lemma 2.1, [2]). If for each $i=1,\cdots,n$

$$\left[\hat{h}_i(z_i)\right]_s=\frac{p_1z_i^{m-1}+\cdots+p_m}{z_i^m-c_mz_i^{m-1}-\cdots-c_1} \tag{15}$$

and

$$\Gamma_{H_i}^m=\begin{pmatrix} h_1^i & \cdots & h_m^i \\ \vdots & & \vdots \\ h_m^i & \cdots & h_{2m-1}^i \end{pmatrix}.$$

the principal minor of order m in Γ_{H_i}, then

$$\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}=\left(\Gamma_{H_i}\right)^{-m}\cdot\begin{pmatrix} h_{m+1}^i \\ \vdots \\ h_{2m}^i \end{pmatrix} \tag{16}$$

and

$$p_1=h_1^i,\ p_2=h_2^i-c_1h_1^i,\cdots,\ p_m=h_m^i-h_{m-1}^ic_1-\cdots-h_1^ic_{m-1}. \tag{17}$$

Proposition 2: Suppose that

$$H(z_1,\cdots,z_n)=\frac{\sum_{i=1}^{n}P_i(z_i)}{\sum_{i=1}^{n}Q_i(z_i)}$$

belongs to $\mathcal{R}_m^n$. Then for each $i=1,2,\cdots,n$ the coefficients of P_i and Q_i can be computed thanks to corollary 2. More precisely if

$$P_i(z_i)=p_1z_i^{m-1}+\cdots+p_m,\ Q_i(z_i)=z_i^m-c_mz_i^{m-1}-\cdots-c_1$$

and $H_i(z_i):=H(1,\cdots,1,z_i,1,\cdots,1)$ then

1) The coefficients of P_i and Q_i except p_m and c_1 are computed by using (16) and (17);

2) The rank of Γ_{H_i}, $R_i \leq m$.

Proof. The item 1 is a consequence of the corollary 2.

The contraposition of the theorem of DeCarlo *et al* [13] allows to say that H_i belongs to $\mathcal{R}_m^s$. The item 2 is then a consequence of **Corollary 1**. □

4. Conclusion

This paper has proposed an optimal design of a multidimensional BIBO stable filter with a rational transfer function $H \in \mathbb{C}[z_1, \cdots, z_n]$. The approach is based on the one dimensional AAK theorem. Two propositions have been developed and require that the Hankel operators Γ_{H_i} associated with the polynomial function H are compact with finite ranks. We hope that in further works, comparisons (based on numerical tests) with other methods that exist in the literature on the approximation of multivariate functions will be made.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter in this manuscript.

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