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New Results on One Modulo N-Difference Mean Graphs

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Abstract

A graph G is said to be one modulo N-difference mean graph if there is an injective function f from the vertex set of G to the set $\{a / 0 \le a \le 2(q-1)N+1 \text{ and either } a \equiv 0 (\bmod N) \text{ or } a \equiv 1 (\bmod N)\}$, where N is the natural number and q is the number of edges of G and f induces a bijection f^* from the edge set of G to $\{a / 1 \le a \le (q-1)N+1 \text{ and } a \equiv 1 (\bmod N)\}$ given by $f^*(uv) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the function f is called a one modulo N-difference mean labeling of G. In this paper, we show that the graphs such as arbitrary union of paths, $M_2(P_n)(n \ge 2)$, ladder, slanting ladder, diamond snake, quadrilateral snake, alternately quadrilateral snake, $Jl_n(P_3)(n \ge 1)$, $C_4 \odot K_{1,n}(n \ge 1)$, $DUP_2(K_{1,n})$, $DUP_2(B_{n,n})$, friendship graph and $nC_4(n \ge 1)$ admit one modulo N-difference mean labeling.

Keywords

Skolem Difference Mean Labeling, One Modulo N-Graceful Labeling, One Modulo N-Difference Mean Labeling and One Modulo N-Difference Mean Graph

1. Introduction and Preliminaries

Here we consider only finite and simple graphs. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. The concept of mean labeling was introduced in [2]. Since then, several results have been published on mean labeling and its variations [3]. In 2014, the concept of skolem difference mean labeling, one of the variations of mean labeling was due to Murugan *et al.* [4]. A graph $G=(V,E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x\in V$ with distinct elements $f(x)$ from $\{1,2,3,\cdots,p+q\}$ in such a way that for each edge $e=uv$, let $f^*(e)=\left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the resulting labels of the edges are distinct and are $1,2,3,\cdots,q$. A graph that admits a skolem difference mean labeling is called skolem difference mean graph. The concept of one modulo N-graceful labeling was introduced by Ramachandran *et al.* [5]. A function f is called a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,2,\cdots,q\}$ such that, when each edge xy is assigned with the label $|f(x)-f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function φ from the vertex set of G to $\{0,1,N,(N+1),2N,(2N+1),\cdots,N(q-1),N(q-1)+1\}$ in such a way that 1) φ is 1-1; 2) φ induces a bijection φ^* from the edge set of G to $\{1,N+1,2N+1,\cdots,N(q-1)+1\}$ where $\varphi^*(uv)=|\varphi(u)-\varphi(v)|$.

Motivated by the concepts of skolem difference mean labeling and one modulo N-graceful labeling and the results in [4] [5], we introduced a new labeling namely "one modulo N-difference mean labeling" in [6] and established that the graphs $B_{m,n}$, $S_{m,n}$, $P_n@P_m$, $B(l,m,n)$, $T(n,m)$, shrub, caterpillar and $K_{1,n}$ are one modulo N-difference mean graphs. In addition, we showed that the graph C_3 is not a one modulo N-difference mean graph. In this paper, we further study on one modulo N-difference mean labeling and show that some more graphs admit one modulo N-difference mean labeling.

We use the following definitions in the subsequent sequel.

Definition 1.1. Let $G=(V,E)$ be a graph and $G'=(V',E')$ be the copy of G. Then the graph $M_2(G)$ of G is obtained from G and G' by joining each vertices in V to its corresponding vertices in V' by an edge.

Definition 1.2. The slanting ladder graph SL_n is obtained from two paths $u_1,u_2,u_3,\cdots,u_n$ and $v_1,v_2,v_3,\cdots,v_n$ by joining u_i with v_{i+1} for $1\le i\le n-1$.

Definition 1.3. Let $G=(V,E)$ be a bipartite graph with $V=V_1\cup V_2$. Let $G'=(V',E')$ be the copy of G with $V'=V_1'\cup V_2'$ such that V_1' and V_2' be the copies of V_1 and V_2. Then the graph $DUP_2(G)$ is obtained from G and G' such that $V(DUP_2(G))=V\cup V'$ and
$E(DUP_2(G))=E(G)\cup E(G')\cup\{v_i'v_j / v_iv_j\in E(G) \text{ where } v_i'\in V', v_j\in V\}$. That is, $DUP_2(G)$ is obtained from G and G' by joining each $v_i'\in V'$ to $v_j\in V$ if v_i is adjacent to v_j in G.

Definition 1.4. A quadrilateral snake graph Q_n is obtained from a path $u_1,u_2,\cdots,u_n$ by joining u_i and u_{i+1} to two new vertices x_i,y_i respectively and then joining x_i and y_i.

Definition 1.5. An alternate quadrilateral snake is obtained from a path

$u_1,u_2,\cdots,u_n$ by joining u_i and u_{i+1} to new vertices x_i and y_i respectively and then joining the vertices x_i and y_i for $i\equiv 1(\bmod 2)$ and $1\le i\le n-1$. That is, every alternate edge of a path is replaced by cycle C_4.

Definition 1.6. Let P_3 be a path of length 2 with vertices v_0,v_1,v_2. The graph $Jl_n(P_3)$ is obtained by taking *n* copies of P_3 and then identifying the left end vertices $v_0^i(1\le i\le n)$ with *u* and the right end vertices $v_2^i(1\le i\le n)$ with *v*.

Definition 1.7. Two graphs *G* and *H* are isomorphic (written $G\simeq H$) if there exists a one-to-one correspondence between their vertex sets which preserves adjacency.

Definition 1.8. The union of two graphs G_1 and G_2 is a graph $G_1\cup G_2$ with $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$.

Definition 1.9. The corona $G_1\odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with *p* vertices) and *p* copies of G_2 and then joining the *i*th vertex of G_1 to every vertex of the *i*th copy of G_2.

Definition 1.10. Let C_n be the cycle with vertices $v_1,v_2,\cdots,v_n$. The graph $C_n^{(t)}$ is obtained by taking *t* copies of C_n and then identifying the vertices $v_1^{(i)}$ for $1\le i\le t$.

2. Main Results

Theorem 2.1. The disjoint union of paths $\bigcup P_{n_i}$ ($n_i\ge 2$, is an integer) is a one modulo N-difference mean graph.

Proof. Let n_i be the vertices of the path P_{n_i} for $1\le i\le m$ and $n=n_1+n_2+\cdots+n_m$.

Define $f:V\left(\bigcup P_{n_i}\right)\to\{0,1,N,N+1,2N,2N+1,\cdots,2N(n-m-1)+1\}$ as follows:

$f(u_{i,j})=N\left[\sum_{k=1(n_k\text{ is odd})}^{i-1}(n_k-1)+\sum_{k=1(n_k\text{ is even})}^{i-1}(n_k-2)+i+j-2\right]$ if *j* is odd,

$f(u_{i,j})=\left[2(n-m-1)+i-j-1-\sum_{k=1(n_k\text{ is odd})}^{i-1}(n_k-1)-\sum_{k=1(n_k\text{ is even})}^{i-1}n_k\right]N+1$

if *j* is even.

Let $e_{i,j}=u_{i,j}u_{i,j+1}$ for $1\le i\le m$ and $1\le j\le n_i-1$.

The corresponding edge label f^* is

$f^*(e_{i,j})=N\left(n-m-\sum_{k=1}^{i-1}n_k+i-j-1\right)+1$ for $1\le i\le m$ and $1\le j\le n_i-1$.

Therefore, *f* is a one modulo N-difference mean labeling. Hence, $\bigcup P_{n_i}$ is a one modulo N-difference mean graph.

Figure 1 shows a one modulo N-difference mean labeling of $P_4\cup P_5\cup P_6\cup P_2\cup P_3\cup P_8$.

Theorem 2.2. The graph $M_2(P_n)(n\ge 2)$ is a one modulo N-difference mean graph.

Proof. Let $\{v_i,v_i':1\le i\le n\}$ be the vertices and $\{e_i,e_i',a_i=v_iv_i':1\le i\le n\}$ be the edges of the graph $M_2(P_n)$. Then the graph has $2n$ vertices and $3n-2$ edges.

Define $f:V(M_2(P_n))\to\{0,1,N,N+1,2N,2N+1,\cdots,2N(3n-3)+1\}$ by

$f(v_1)=0$,

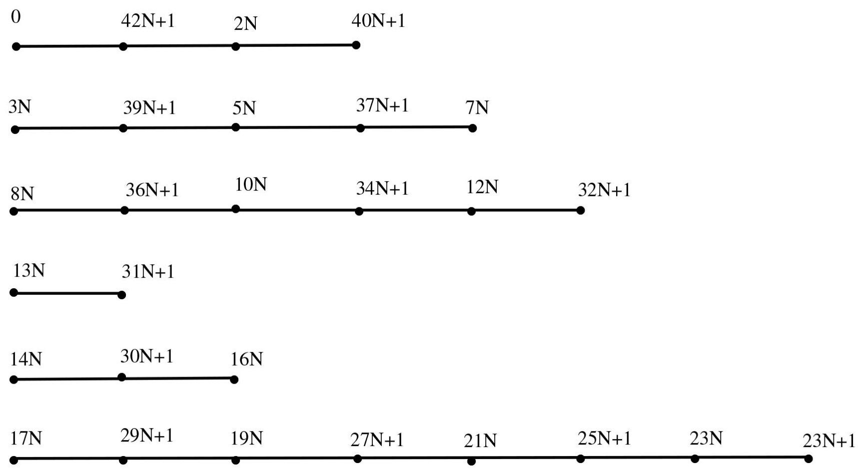

Figure 1. One modulo N-difference mean labeling of $P_4 \cup P_5 \cup P_6 \cup P_2 \cup P_3 \cup P_8$.

For $2 \le i \le n$,

$$f(v_i) = \begin{cases} (3i-5)N & \text{if } i \text{ is odd} \\ 3N(2n-i)+1 & \text{if } i \text{ is even} \end{cases}$$

For $1 \le i \le n$,

$$f(v_i') = \begin{cases} (6n-3i-2)N+1 & \text{if } i \text{ is odd} \\ 3iN & \text{if } i \text{ is even} \end{cases}$$

Then the induced edge labels are

$f^*(e_1) = 3(n-1)N+1$,

$f^*(e_i) = [3(n-i)+1]N+1$ for $2 \le i \le n$,

$f^*(e_i') = [3(n-i)-4]N+1$ for $1 \le i \le n$,

$f^*(v_1v_1') = [3(n-4)]N+1$,

$f^*(a_i) = [3(n-i)]N+1$ for $2 \le i \le n$.

Therefore, f is a one modulo N-difference mean labeling and hence $M_2(P_n)$ is a one modulo N-difference mean graph.

Figure 2 shows a one modulo N-difference mean labeling of $M_2(P_5)$.

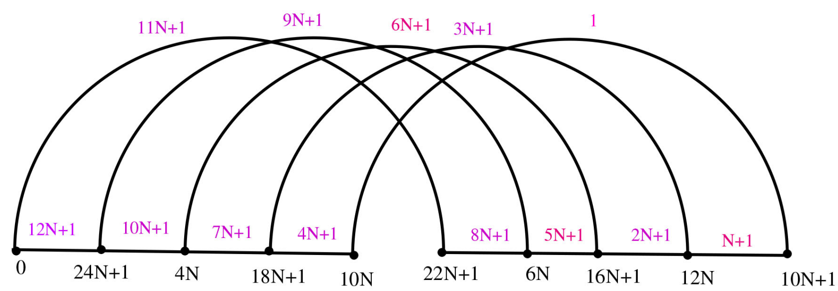

Figure 2. One modulo N-difference mean labeling of $M_2(P_5)$.

Corollary 2.3. *The ladder graph* $P_n \times P_2$ *is a one modulo N-difference mean graph.*

Theorem 2.4. *The slanting ladder* $SL_n(n \ge 2)$ *is a one modulo N-difference mean graph.*

Proof. Let $u_1, u_2, u_3, \cdots, u_n$ and $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path of length $n-1$.

Then $E(SL_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1} : 1 \le i \le n-1\}$.

Define $f : V(SL_n) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(3n-4)+1\}$ by

For $1 \le i \le n$,

$$f(u_i) = \begin{cases} 2(i-1)N & \text{if } i \text{ is odd} \\ 2N[3n-2(i+1)]+1 & \text{if } i \text{ is even} \end{cases}$$

$$f(v_1) = \begin{cases} 2(3n-4)N & \text{if } n \text{ is odd} \\ 2(3n-5)N & \text{if } n \text{ is even} \end{cases}$$

For $2 \le i \le n$,

$$f(v_i) = \begin{cases} 2(i-2)N & \text{if } i \text{ is odd} \\ 2N[3n-2i]+1 & \text{if } i \text{ is even} \end{cases}$$

Then the induced edge labels are

For $1 \le i \le n-1$,

$$f^*(u_i u_{i+1}) = \begin{cases} 3N(n-i-1)+1 & \text{if } i \text{ is odd} \\ N[3(n-i)-2]+1 & \text{if } i \text{ is even} \end{cases}$$

$$f^*(v_1 v_2) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ N+1 & \text{if } n \text{ is even} \end{cases}$$

For $2 \le i \le n-1$,

$$f^*(v_i v_{i+1}) = \begin{cases} 3N(n-i)+1 & \text{if } i \text{ is odd} \\ N[3(n-i)+1]+1 & \text{if } i \text{ is even} \end{cases}$$

$f^*(u_i v_{i+1}) = N[3(n-i)-1]+1$ for $1 \le i \le n-1$.

Therefore, *f* is a one modulo N-difference mean labeling and hence SL_n is a one modulo N-difference mean graph.

Figure 3 shows a one modulo N-difference mean labeling of SL_{10}.

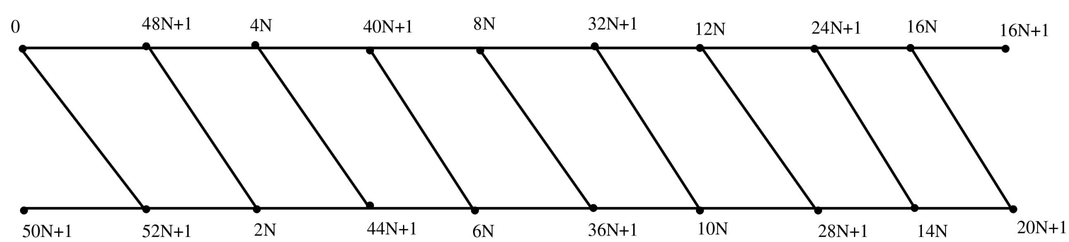

Figure 3. One modulo N-difference mean labeling of SL_{10}.

Theorem 2.5. *The diamond snake graph* $DS(n)(n \ge 1)$ *is a one modulo N-difference mean graph.*

Proof. Let $\{v_0, v_i, a_i, b_i : 1 \le i \le n\}$ be the vertices and $\{v_0 a_1, v_0 b_1, v_i a_{i+1}, a_i v_i, v_i b_{i+1}, b_i v_i : 1 \le i \le n\}$ be the edges of the diamond snake graph which has $4n-4$ vertices and $4n$ edges.

Define $f : V(DS(n)) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(4n-1)+1\}$ by

$f(v_0) = 0$,

$f(v_i) = 4iN$ for $1 \le i \le n$,

$f(a_i) = 2N(4n-2i+1)+1$ for $1 \le i \le n$,

$f(b_i) = 4N(2n-i)+1$ for $1 \le i \le n$.

Then the induced edge labels are

$f^*(v_0 a_1) = (4n-1)N+1$,

$f^*(v_i a_{i+1}) = (4n-4i-1)N+1$ for $1 \le i \le n-1$,

$f^*(a_i v_i) = (4n-4i+1)N+1$ for $1 \le i \le n$,

$f^*(v_0 b_1) = (4n-2)N+1$,

$f^*(v_i b_{i+1}) = (4n-4i-2)N+1$ for $1 \le i \le n-1$,

$f^*(b_i v_i) = (4n-4i)N+1$ for $1 \le i \le n$.

Therefore, *f* is a one modulo N-difference mean labeling and hence $DS(n)$ is a one modulo N-difference mean graph. A one modulo N-difference mean labeling of $DS(5)$ is shown in **Figure 4**.

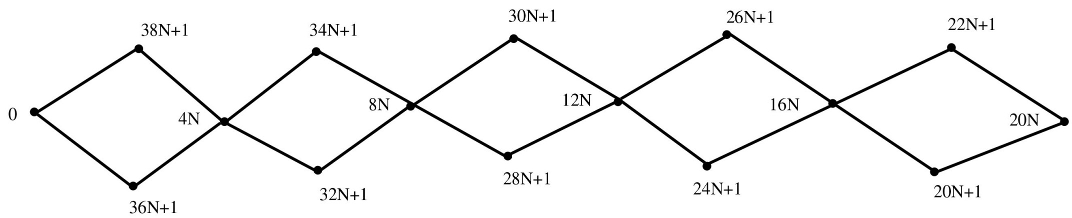

Figure 4. One modulo N-difference mean labeling of $DS(5)$.

Theorem 2.6. *The quadrilateral snake* $Q_n\ (n>1)$ *is a one modulo N-difference mean graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n of length $n-1$.

Then $\{u_i, x_j, y_j : 1 \le i \le n, 1 \le j \le n-1\}$ be the vertices of and $\{u_i u_{i+1}, u_i x_i, u_{i+1} y_i, x_i y_i : 1 \le i \le n-1\}$ be the edges of Q_n.

Define $f : V(Q_n) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(4n-5)+1\}$ by

$f(u_1) = 0$,

For $2 \le i \le n$,

$$f(u_i) = \begin{cases} (3i-1)N & \text{if } i \text{ is odd} \\ (8n-5i-2)N+1 & \text{if } i \text{ is even} \end{cases}$$

$f(x_1) = 2(4n-5)N+1$,

For $2 \le i \le n-1$,

$$f(x_i) = \begin{cases} (8n-5i-3)N+1 & \text{if } i \text{ is odd} \\ 3iN & \text{if } i \text{ is even} \end{cases}$$

For $1 \le i \le n-1$,

$$f(y_i) = \begin{cases} (3i+1)N & \text{if } i \text{ is odd} \\ (8n-5i-6)N+1 & \text{if } i \text{ is even} \end{cases}$$

Then the induced edge labels are

$f^*(u_1 u_2) = [4n-6]N+1$

For $2 \le i \le n-1$,

$$f^*(u_i u_{i+1}) = \begin{cases} [4(n-i)-3]N+1 & \text{if } i \text{ is odd} \\ [4(n-i)-2]N+1 & \text{if } i \text{ is even} \end{cases}$$

$f^*(x_1y_1) = (4n-7)N+1$,

For $2 \le i \le n-1$,

$$f^*(x_iy_i) = \begin{cases} [4(n-i)-2]N+1 & \text{if } i \text{ is odd} \\ [4(n-i)-3]N+1 & \text{if } i \text{ is even} \end{cases}$$

$f^*(u_ix_i) = [4(n-i)-1]N+1$ for $1 \le i \le n-1$,

$f^*(u_{i+1}y_i) = [4(n-i-1)]N+1$ for $1 \le i \le n-1$.

Therefore, f is a one modulo N-difference mean labeling and hence Q_n is a one modulo N-difference mean graph.

Figure 5 shows a one modulo N-difference mean labeling of Q_7.

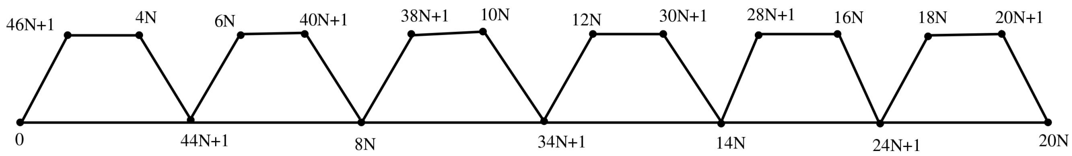

Figure 5. One modulo N-difference mean labeling of Q_7.

Theorem 2.7. *The alternately quadrilateral snake* $A(Q_n)(n>1)$ *is a one modulo N-difference mean graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n of length $n-1$.

Let $n = 2m$.

Then $\{u_i, x_j, y_j : 1 \le i \le n, 1 \le j \le m\}$ be the vertices of and $\{u_iu_{i+1}, u_ix_j, u_iy_j, x_jy_j : 1 \le i \le n, 1 \le j \le m\}$ be the edges of $A(Q_n)$.

Define $f : V(A(Q_n)) \to \left\{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N\left(2n+\frac{n}{2}-2\right)+1\right\}$ by

$f(u_1) = 0$,

For $2 \le i \le n$,

$$f(u_i) = \begin{cases} 2Ni & \text{if } i \text{ is odd} \\ (5n-3i)N+1 & \text{if } i \text{ is even} \end{cases}$$

$f(x_1) = (5n-4)N+1$,

$f(x_j) = (5n-6j+4)N+1$ for $2 \le j \le m$,

$f(y_j) = 4Nj$ for $1 \le j \le m$.

Then the induced edge labels are

For $1 \le i \le n-1$,

$$f^*(u_iu_{i+1}) = \begin{cases} \left[2n+\dfrac{n-5i-3}{2}\right]N+1 & \text{if } i \text{ is odd} \\ \left[2n+\dfrac{n-5i-2}{2}\right]N+1 & \text{if } i \text{ is even} \end{cases}$$

$f^*(x_jy_j) = (2n+m-5j+2)N+1$ for $1 \le j \le m$,

$f^*\left(u_ix_{\frac{i+1}{2}}\right) = \left[2n+\dfrac{n-5i+1}{2}\right]N+1$ if i is odd and $1 \le i \le n-1$,

$f^*\left(u_iy_{\frac{i}{2}}\right) = \left[2n+\dfrac{n-5i}{2}\right]N+1$ if i is even and $2 \le i \le n-2$.

Therefore, f is a one modulo N-difference mean labeling and hence $A(Q_n)$ is a one modulo N-difference mean graph.

Figure 6 shows a one modulo N-difference mean labeling of $A(Q_8)$.

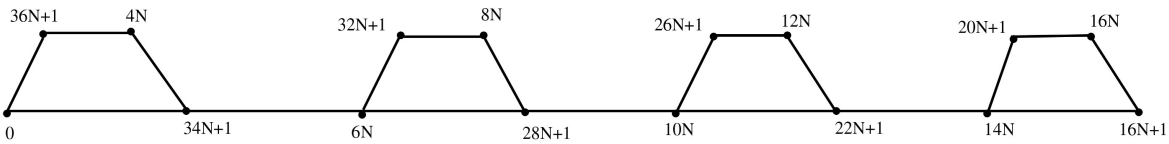

Figure 6. One modulo N-difference mean labeling of $A(Q_8)$.

Theorem 2.8. *The graph* $Jl_n(P_3)(n \geq 1)$ *is a one modulo N-difference mean graph.*

Proof. Let $v_0^i, v_1^i, v_2^i (1 \leq i \leq n)$ be the vertices of the n copies of the path P_3.

Then the graph $Jl_n(P_3)$ is obtained by identifying $v_0^i = u$ and $v_2^i = v$.

Define $f: V(Jl_n(P_3)) \rightarrow \{0,1,N,N+1,2N,2N+1,\cdots,2N(2n-1)+1\}$ as follows:

$f(u) = 0$,

$f(v) = 2N$,

$f(v_i^1) = (4i-2)N+1$ for $1 \leq i \leq n$.

Then the induced edge labels are

$f^*(x_1 u) = 0$,

$f^*(x_i v) = 2(i-1)N+1$ for $2 \leq i \leq n$,

$f^*(ux_i) = (2i-1)N+1$ for $1 \leq i \leq n$.

Therefore, f is a one modulo N-difference mean labeling and hence $Jl_n(P_3)$ is a one modulo N-difference mean graph.

Figure 7 shows a one modulo N-difference mean labeling of $Jl_n(P_3)$.

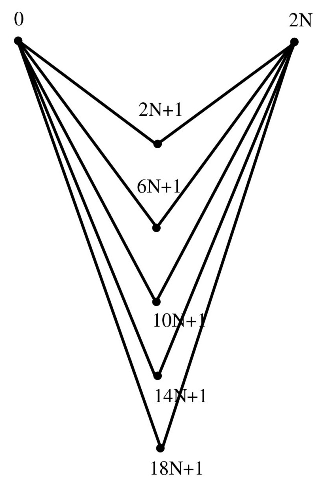

Figure 7. One modulo N-difference mean labeling of $Jl_5(P_3)$.

Theorem 2.9. *The corona graph* $C_4 \odot K_{1,n}(n \geq 1)$ *is a one modulo N-difference mean graph.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices of cycle C_4 and $\{v_i^j : 1 \le i \le n, 1 \le j \le 4\}$ be the vertices of the four stars $K_{1,n}$.

Define $f : V(C_n \odot K_{1,n}) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(4n+3)+1\}$ as follows:

We label the vertices of C_4 as follows:

$f(v_1) = 0$,

$f(v_2) = 2(4n+3)N + 1$,

$f(v_3) = 4N$,

$f(v_4) = 4(2n+1)N + 1$.

Now, we label the vertices of $K_{1,n}$ as follows:

$f(v_i^1) = 2(i-1)N + 1$ for $1 \le i \le n$,

$f(v_i^2) = 2N(3n - i + 4)$ for $1 \le i \le n$,

$f(v_i^3) = 2(2n + i + 1)N + 1$ for $1 \le i \le n$,

$f(v_i^4) = 2N(n - i + 3)$ for $1 \le i \le n$.

Let $e_i = \{v_i v_{i+1} : 1 \le i \le 3\}$ and $e_i^j = \{v_j v_i^j : 1 \le i \le n, 1 \le j \le 4\}$.

Then the induced edge labels are

$f^*(e_1) = (4n+3)N + 1$,

$f^*(e_2) = (4n+1)N + 1$,

$f^*(e_3) = 4nN + 1$,

$f^*(v_4 v_1) = (4n+2)N + 1$,

$f^*(e_i^j) = [(j-1)n + i - 1]N + 1$ for $1 \le i \le n$, $1 \le j \le 4$.

Therefore, f is a one modulo N-difference mean labeling. Hence, $C_4 \odot K_{1,n}$ is a one modulo N-difference mean graph.

Figure 8 shows a one modulo N-difference mean labeling of $C_4 \odot K_{1,3}$.

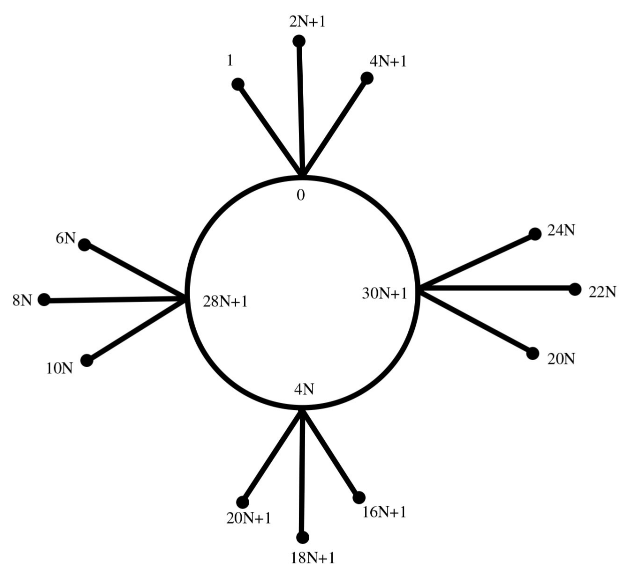

Figure 8. One modulo N-difference mean labeling of $C_4 \odot K_{1,3}$.

Theorem 2.10. *The graph* $DUP_2(K_{1,n}), n \ge 2$ *is a one modulo N-difference mean graph.*

Proof. Let $\{v, v_i(1 \le i \le n), u, u_i(1 \le i \le n)\}$ be the vertices and $\{vv_i, uu_i, v_iu : 1 \le i \le n\}$ be the edges of $DUP_2(K_{1,n})$.

Now, the vertex labels are defined as follows:

Define $f : V(DUP_2(K_{1,n})) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(3n-1)+1\}$ by

$f(v) = 2N(2n-1)$,

$f(u) = 0$,

$f(v_i) = 2N(2n-i)+1$ for $1 \le i \le n$,

$f(u_i) = 2N(3n-i)+1$ for $1 \le i \le n$.

Then the induced edge labels are

$f^*(vv_i) = N(i-1)+1$ for $1 \le i \le n$,

$f^*(uu_i) = N(3n-i)+1$ for $1 \le i \le n$,

$f^*(v_iu) = N(2n-i)+1$ for $1 \le i \le n$.

Therefore, f is a one modulo N-difference mean labeling. Hence, $DUP_2(K_{1,n})$ is a one modulo N-difference mean graph. **Figure 9** shows a one modulo N-difference mean labeling of $DUP_2(K_{1,7})$.

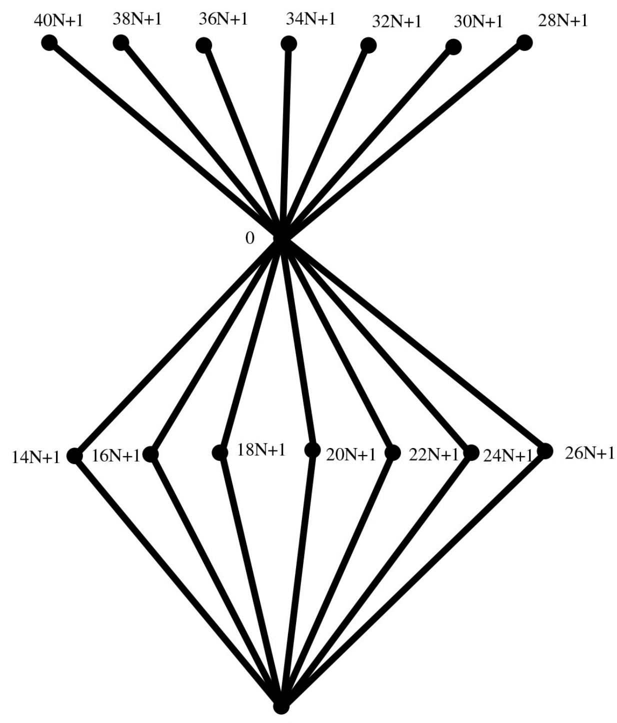

Figure 9. One modulo N-difference mean labeling of $DUP_2(K_{1,7})$.

Theorem 2.11. *The graph* $DUP_2(B_{n,n}), n \ge 2$ *is a one modulo N-difference mean graph.*

Proof. Let $\{v, v', v_i, v_i', u, u', u_i, u_i' : 1 \le i \le n\}$ be the vertices and $\{vv_i, uu_i, v_iu, v'v_i', u'u_i', v_i'u', vu', uu', uv' : 1 \le i \le n\}$ be the edges of $DUP_2(B_{n,n})$.

Now, the vertex labels are defined as follows:

Define $f : V(DUP_2(B_{n,n})) \to \{0, 1, N, N+1, 2N, 2N+1, \cdots, 2N(6n+2)+1\}$ by

$f(v) = 2N$,
$f(u) = 0$,
$f(v_i) = 4N(3n - i + 2) + 1$ for $1 \le i \le n$,
$f(u_i) = 2N(4n - i + 3) + 1$ for $1 \le i \le n$,
$f(v') = 4N + 1$,
$f(u') = 2N + 1$,
$f(v_i') = 2N(2n - i + 4)$ for $1 \le i \le n$,
$f(u_i') = 2N(3n - i + 4)$ for $1 \le i \le n$.

Then the induced edge labels are

$f^*(vv_i) = (6n - 2i + 3)N + 1$ for $1 \le i \le n$,
$f^*(uu_i) = (4n - i + 3)N + 1$ for $1 \le i \le n$,
$f^*(v_iu) = (6n - 2i + 4)N + 1$ for $1 \le i \le n$,
$f^*(v'v_i') = [2(n - i) + 3]N + 1$ for $1 \le i \le n$,
$f^*(u'u_i') = (3n - i + 3)N + 1$ for $1 \le i \le n$,
$f^*(v_i'u') = 2(n - i + 2)N + 1$ for $1 \le i \le n$,
$f^*(uu') = N + 1$, $f^*(vu') = 1$, $f^*(uv') = 2N + 1$.

Therefore, *f* is a one modulo N-difference mean labeling. Hence, $DUP_2(B_{n,n})$ is a one modulo N-difference mean graph. Figure 10 shows a one modulo N-difference mean labeling of $DUP_2(B_{5,5})$.

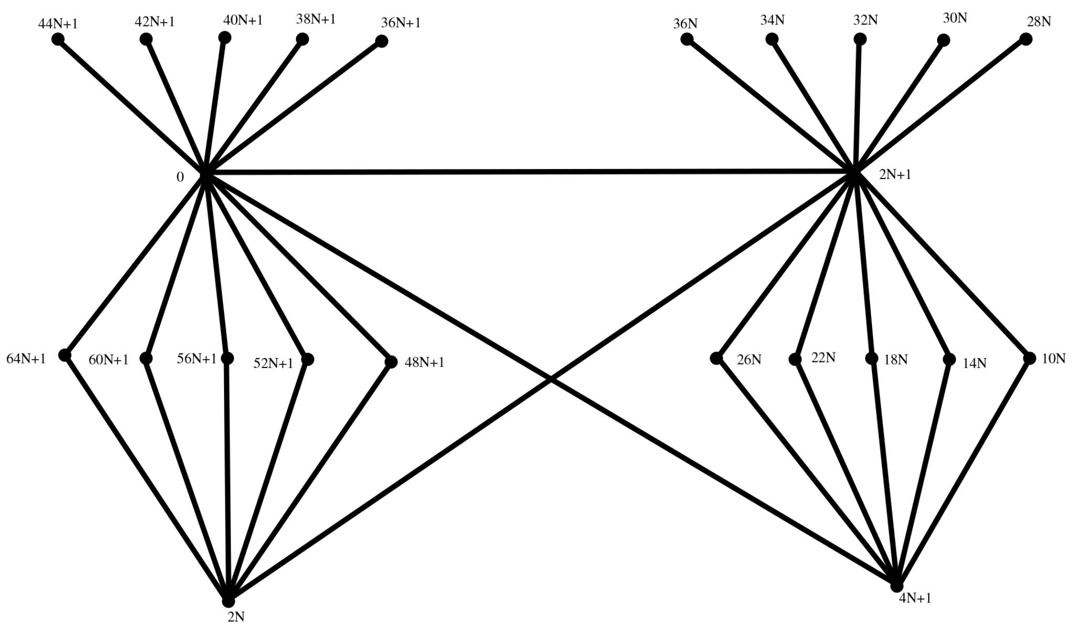

Figure 10. One modulo N-difference mean labeling of $DUP_2(B_{5,5})$.

Theorem 2.12. *The friendship graph* $C_4^{(n)}, n \ge 1$ *is a one modulo N-difference mean graph.*

Proof. Let $v_1^j, v_2^j, v_3^j, v_4^j \ (1 \le j \le n)$ be the vertices of the cycle C_4. Then the graph $C_4^{(n)}$ is obtained by identifying the vertices $v_1^j = v_1$ for $(1 \le j \le n)$.

Then $E(C_4^{(n)}) = \{v_i^j v_{i+1}^j, v_4^j v_1^j : 1 \le i \le 3, 1 \le j \le n\}$.

We label the vertices as follows:

Define $f:V\left(C_4^{(n)}\right)\to\{0,1,N,N+1,2N,2N+1,\cdots,2N(4n-1)+1\}$ by
$f(v_1)=0$,
$f\left(v_2^j\right)=2N(4n-2j+1)+1$ for $1\le j\le n$,
$f\left(v_3^1\right)=2N(4n-1)$,
$f\left(v_3^j\right)=4N\left[2(n-j)+1\right]$ for $1\le j\le n$,
$f\left(v_4^j\right)=2N(4n-2j)+1$ for $1\le j\le n$.
Then the induced edge labels are
$f^*\left(v_1v_2^j\right)=N(4n-2j+1)+1$ for $1\le j\le n$,
$f^*\left(v_2^1v_3^1\right)=1$, $f^*\left(v_3^1v_4^1\right)=N+1$,
$f^*\left(v_2^jv_3^j\right)=N(2j-1)+1$ for $2\le j\le n$,
$f^*\left(v_3^jv_4^j\right)=2N(j-1)+1$ for $2\le j\le n$,
$f^*\left(v_4^jv_1^j\right)=N(4n-2j)+1$ for $1\le j\le n$.

Therefore, f is a one modulo N-difference mean labeling. Hence, the graph $C_4^{(n)}$ is a one modulo N-difference mean graph. **Figure 11** shows a one modulo N-difference mean labeling of $C_4^{(5)}$.

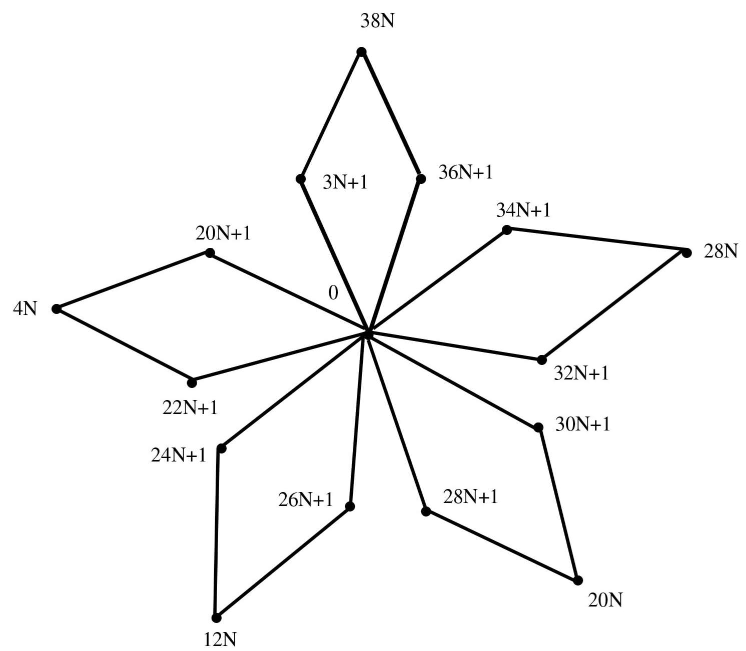

Figure 11. One modulo N-difference mean labeling of $C_4^{(5)}$.

Theorem 2.13. *The graph* $nC_4, n\ge 1$ *is a one modulo N-difference mean graph.*

Proof. Let $v_1^j, v_2^j, v_3^j, v_4^j\ (1\le j\le n)$ be the vertices of n copies of the cycle C_4.
Then $E(nC_4)=\left\{v_i^jv_{i+1}^j, v_4^jv_1^j : 1\le i\le 3, 1\le j\le n\right\}$.
We label the vertices as follows:
Define $f:V(nC_4)\to\{0,1,N,N+1,2N,2N+1,\cdots,2N(4n-1)+1\}$ by
$f\left(v_1^i\right)=(i-1)N$ for $1\le j\le n$,
$f\left(v_2^j\right)=N(8n-3j+1)+1$ for $1\le j\le n$,
$f\left(v_3^j\right)=N(8n-7j+3)$ for $1\le j\le n$,

$f\left(v_4^j\right) = N\left(8n-3j-1\right)+1$ for $1 \le j \le n$.

Then the induced edge labels are

$f^*\left(v_1^j v_2^j\right) = N\left(4n-2j+1\right)+1$ for $1 \le j \le n$,

$f^*\left(v_2^j v_3^j\right) = N\left(2j-1\right)+1$ for $2 \le j \le n$,

$f^*\left(v_3^j v_4^j\right) = 2N\left(j-1\right)+1$ for $2 \le j \le n$,

$f^*\left(v_4^j v_1^j\right) = N\left(4n-2j\right)+1$ for $1 \le j \le n$.

Therefore, f is a one modulo N-difference mean labeling. Hence, the graph nC_4 is a one modulo N-difference mean graph. **Figure 12** shows a one modulo N-difference mean labeling of $6C_4$.

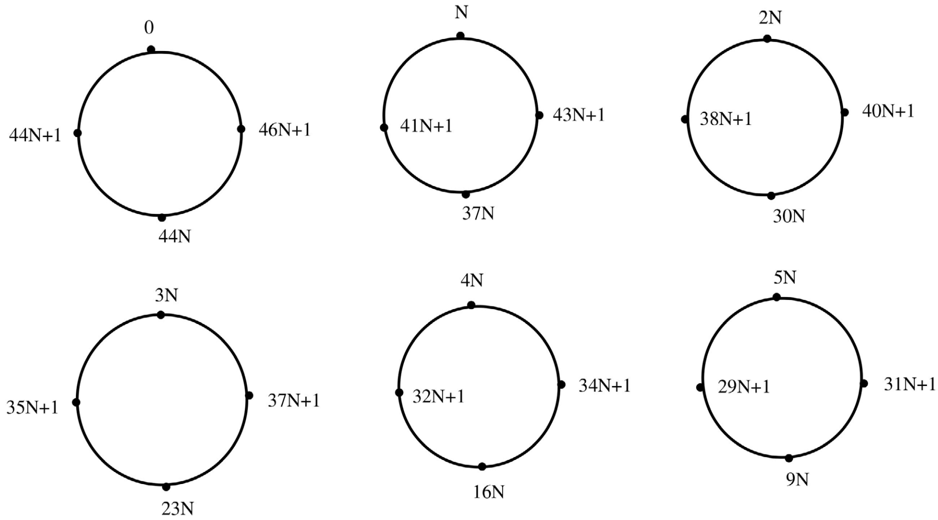

Figure 12. One modulo N-difference mean labeling of $6C_4$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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