

Acute Triangulations of the Surface of Circular Cone

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Abstract

In this paper, we prove that the surface of any circular cone S can be triangulated into 8 non-obtuse and 20 acute triangles. Furthermore, we also show that the bounds are both the best possible.

Keywords

Acute Triangulation, Surface of Circular Cone, Gauss-Bonnet Formula

1. Introduction

By a *triangulation* of a two-dimensional space, we mean a set of (full) triangles covering the space, in such a way that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments, i.e.*, shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. In rather general two-dimensional spaces, like Alexandrov surfaces, two geodesics starting at the same point determine a well defined angle.

An *acute* (*non-obtuse*) triangulation of a two-dimensional space is a geodesic triangulation such that the angles of all geodesic triangles are smaller (respectively, not greater) than $\frac{\pi}{2}$. The number of triangles in a triangulation is called its *size*.

The discussion of acute triangulations was firstly proposed in 1960 by Gardner (see [1] [2] [3]). In the same year, independently, Burago and Zalgaller [4] proved the existence of acute triangulations of general two-dimensional polyhedral surfaces. However, their method could not give an estimate on the size of the existed acute triangulations. Acute triangulations of *n*-polygons (n > 3) have been considered from 1980, such as acute triangulations of square [5], quadrilaterals [6] [7], trapezoids [8], pentagons [9] and arbitrary convex polygons [10] [11] [12].

Currently, some compact convex surfaces have also been triangulated, such as acute and non-obtuse triangulations of the surfaces of all Platonic solids in [13] [14] [15] [16], the surfaces of some Archimedean solids in [17] and [18], flat Möbius strips [19], flat tori [20]. Furthermore, other surfaces homeomorphic to the sphere have also been acutely triangulated, such as the double triangles [21], the double quadrilaterals [22] and double planar convex bodies in [23]. In 2018, Bau and Gagola III discussed acute decompositions (not triangulations) of closed orientable geometric surfaces in [24].

In 2009, acute triangulations of any polyhedral surface were considered again by Saraf [25], but there was still no estimate on the size of the existed acute triangulations. In 2011, H. Maehara [26] proved that every polyhedral surface admits a proper acute triangulation and got a upper bound for the size of the proper acute triangulation.

In this paper, we will consider the surface of any cone, and start with the surface of a bounded right circular cone. For the sake of convenience, let S be the surface of a bounded circular cone. Denote by T an acute triangulation of S and T_0 a non-obtuse triangulation of S. Let |T| denote the size of T. We prove that $|T_0| \ge 8$ and $|T| \ge 20$.

2. Non-Obtuse Triangulations

Theorem 1. The surface of any right circular cone can be triangulated into 8 non-obtuse triangles, and there is no non-obtuse triangulation with fewer triangles.

Proof. Consider the unfold figure of S. Denote by *a* the vertex of S, *b* the center of the bottom of S. Draw four segments from *a* to *b* such that the four segments divide equally the angles around *a* and *b*. Let p_1, p_2, p_3, p_4 be the intersection points of the four segments with the circle of the bottom. Choose the points $c \in bp_1$, $d \in ap_2$, $e \in bp_3$, $f \in ap_4$ such that $cd \perp ab$, $cf \perp ab$ at *c*, $de \perp ab$, $ef \perp ab$ at *e*. Let C_b be the circle of the bottom, $\{q_1\} = cd \cap C_b$, $\{q_2\} = de \cap C_b$, $\{q_3\} = ef \cap C_b$, $\{q_4\} = fc \cap C_b$.

Now we choose the suitable positions of *c* and *e* such that

$$\left| \widehat{p_1 q_1} \right| = \left| \widehat{p_1 q_4} \right| = \left| \widehat{p_3 q_2} \right| = \left| \widehat{p_3 q_3} \right|, \text{ and } \left| \widehat{p_1 q_1} \right| : \left| \widehat{p_2 q_1} \right| = \alpha : 2\pi \text{ . Obviously,}$$
$$\left| \widehat{p_2 q_1} \right| = \left| \widehat{p_2 q_2} \right| = \left| \widehat{p_4 q_3} \right| = \left| \widehat{p_4 q_4} \right|.$$

We get a triangulation of S with 8 triangles:

acd, ade, aef, afc, bcd, bde, bef, bfc.

It is not difficult to see that the four triangles *acd*, *ade*, *aef*, *afc* are congruent, and the four triangles *bcd*, *bde*, *bef*, *bfc* are also congruent. Therefore, we only need to consider the two triangles *acd* and *bcd*. Since

 $\begin{aligned} \left| \widehat{p_1 q_1} \right| &: \left| \widehat{p_2 q_1} \right| = \alpha : 2\pi, \text{ the sum of the interior angles of the triangle } acd \text{ is } \\ \pi + \frac{\alpha}{2\pi + \alpha} \cdot \frac{2\pi + \alpha}{4} = \pi + \frac{\alpha}{4}, \text{ and the sum of the interior angles of the triangle} \\ bcd \text{ is } \pi + \frac{2\pi}{2\pi + \alpha} \cdot \frac{2\pi + \alpha}{4} = \pi + \frac{\pi}{2}. \text{ By } \angle cad = \frac{\alpha}{4}, \angle acd = \frac{\pi}{2}, \text{ we have} \\ \angle adc = \left(\pi + \frac{\alpha}{4}\right) - \frac{\alpha}{4} - \frac{\pi}{2} = \frac{\pi}{2}. \text{ Then, } \angle bdc = \frac{\pi}{2}. \text{ Therefore, the above triangulation of } \mathcal{S}. \end{aligned}$

Suppose that there is a non-obtuse triangulation T_0 of S with $|T_0| < 8$, then $|T_0| = 4$ or 6. It implies that T_0 has at least two vertices with degree 3. However, the total angle at any point except a on S is equal to 2π . Therefore, T_0 has only one vertex possibly with degree 3, which is a contradiction.

The proof is complete. \Box

3. Acute Triangulations

Let *o* denote the vertex of the circular cone, and α denote the total angle around *o* on the surface *S*. Denote by *o'* the center of the bottom of *S*.

Theorem 2. The surface of any right circular cone cannot be triangulated into less than 20 acute triangles.

Proof. Let *T* denote an acute triangulation of *S*. By the definition of triangulation, the degree of any vertex of *T* is at least 3. Notice that $0 < \alpha < 2\pi$, and the total angle at any point except *o* on *S* is equal to 2π , so in *T*, the vertex *o* can have degree less than 5, but any other vertex except *o* must have degree at least 5.

If |T| = 18, then *T* has $(18 \times 3)/2 = 27$ edges and, by Euler's formula, 27-18+2=11 vertices. Clearly, the sum of the degrees of all the vertices of *T* is 2×27 = 54, so *o* must be a vertex of *T*. Furthermore, there are two cases about *T*: the vertex *o* with degree 4, any other ten vertices have degree 5; the vertex *o* with degree 3, one vertex with degree 6, any other nine vertices have degree 5. However, there is no triangulation with 18 triangles that can have these constructions.

If |T| = 16, then we can immediately get that *T* has 24 edges and 10 vertices, and then the sum of the degrees of all the vertices of *T* is 48. Then there is only one case about *T*: the vertex *o* with degree 3, any other nine vertices have degree 5. However, there is no triangulation with 16 triangles can have this construction.

If |T| = 14, then *T* has 21 edges and 9 vertices, and thus, the sum of the degrees of all the vertices of *T* is 42. Then the degree of the vertex *o* must be 2, a contradiction.

If $|T| \le 12$, it is not difficult to see that there are at least two vertices of *T* with degree at most 4, which implies impossible.

Theorem 3. The surface of any right circular cone can be triangulated into 20 acute triangles.

Proof. Let a, b, c, d, e be five points in the relative interior of the flank of the circular cone such that |oa| = |ob| = |oc| = |od| = |oe|, and

 $\angle aob = \angle boc = \angle cod = \angle doe = \angle eoa = \frac{\alpha}{5}$. Clearly, these five points a, b, c, d, eon S form a regular pentagon with center o. Draw two segments from a, b, c, d, e on the outside of the triangle abc such that the three angles at

a,b,c,d,e are equal to
$$\frac{\pi + \frac{\alpha}{5}}{3} = \frac{\pi}{3} + \frac{\alpha}{15}$$
 respectively

Now we respectively choose the suitable distances from a,b,c,d,e to o such that the intersection points of the above ten segments, denoted by a',b',c',d',e', are located in the bottom of S. Hence the pentagon a'b'c'd'e'a' is regular, and with center o'.

The ten segments ae', aa', ba', bb', cb', cc', dc', dd', ed', ee'divide the bottom of S into ten arcs, denote by S_a , $S_{a'}$, S_b , $S_{b'}$, S_c , $S_{c'}$, S_d , $S_{d'}$, S_e , $S_{e'}$. Furthermore, $|S_a| = |S_b| = |S_c| = |S_d| = |S_e|$, and

 $\frac{|S_{a'}| = |S_{b'}| = |S_{c'}| = |S_{d'}| = |S_{e'}|.$ As we respectively slide a, b, c, d, e in direction \overrightarrow{oa} , \overrightarrow{ob} , \overrightarrow{oc} , \overrightarrow{od} and \overrightarrow{oe} , the distances from a', b', c', d', e' to o' are monotone decreasing. Choose the suitable positions of a, b, c, d, e such that $|S_a| : |S_{a'}| = 7:5.$

We obtain a triangulation T of S with 20 triangles:

oab, obc, ocd, ode, oae, o'a'b', o'b'c', o'c'd', o'd'e', o'a'e', aba', aa'e', bcb', ba'b', cdc', cb'c', ded', dc'd', eae', ed'e'.

It is clear that the triangles *oab*, *obc*, *ocd*, *ode*, *oae* are planar congruent isosceles ones with vertex angle $\alpha/5 < \pi/2$ and the triangles o'a'b', o'b'c', o'c'd', o'd'e', o'a'e' are also planar congruent isosceles ones with vertex angle $2\pi/5$.

The remaining ten triangles are isosceles. Furthermore, the five triangles aba', bcb', cdc', ded', eae' are congruent, and the other five triangles aa'e', ba'b', cb'c', dc'd', ed'e' are also congruent. Hence we only need to consider the triangles aba' and ba'b'.

It is not difficult to see that the sum of curvature on the arc $S_{a'} \cup S_b$ is $\frac{2\pi + \alpha}{5}$, then the sum of curvature of S_b is $\frac{7}{12} \cdot \frac{2\pi + \alpha}{5}$, and the sum of curvature of S_b is $\frac{7}{12} \cdot \frac{2\pi + \alpha}{5}$.

vature of $S_{a'}$ is $\frac{5}{12} \cdot \frac{2\pi + \alpha}{5}$. By the Gauß-Bonnet formula, the sum of the inte-

rior angles of the triangle ba'b' is $\pi + \frac{7}{12} \cdot \frac{2\pi + \alpha}{5}$. Since $\angle a'bb' = \frac{\pi}{3} + \frac{\alpha}{15} < \frac{\pi}{2}$,

we have
$$\angle ba'b' = \angle bb'a' = \frac{\pi + \frac{7}{12} \cdot \frac{2\pi + \alpha}{5} - \left(\frac{\pi}{3} + \frac{\alpha}{15}\right)}{2} = \frac{9\pi}{20} + \frac{\alpha}{40} < \frac{9\pi}{20} + \frac{2\pi}{40} = \frac{\pi}{2}.$$

Similarly, the sum of the interior angles of the triangle aba' is $\pi + \frac{5}{12} \cdot \frac{2\pi + \alpha}{5}$.

Because of
$$\angle aba' = \angle baa' = \frac{\pi}{3} + \frac{\alpha}{15} < \frac{\pi}{2}$$
, then
 $\angle aa'b = \pi + \frac{5}{12} \cdot \frac{2\pi + \alpha}{5} - 2 \cdot \left(\frac{\pi}{3} + \frac{\alpha}{15}\right) = \frac{\pi}{2} - \frac{\alpha}{20} < \frac{\pi}{2}$.

4. Acute Triangulations with Degree of o Less Than 5

The total angle at each point of S except for o is 2π . According to the total angle of the vertex o, can we find a triangulation T of S with the number of triangles as few as possible such that the degree of o in T is less than 5?

If $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$, it is not difficult to see that the curvature of o is $2\pi - \alpha < \pi/2$, hence we can get a triangle on S with all the three interior angles being equal to $\pi - \alpha/3 < \pi/2$ such that o is located in the interior of this triangle. If $\alpha = 3\pi/2$, we can find an triangulation of S such that the degree of o is 4. If $\alpha \in \left(0, \frac{3\pi}{2}\right)$, we prove that there is an triangulation of S such that the degree of o is 3.

Theorem 4. If $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$, then the surface of the right circular cone ad-

mits an acute triangulation with 20 triangles such that the degree of o is 0.

Proof. Let a',b',c' denote three points in the relative interior of the flank of S such that |oa'| = |ob'| = |oc'| and $\angle a'ob' = \angle b'oc' = \angle c'oa' = \frac{\alpha}{3}$. Let a,b,c

denote the midpoint of b'c', c'a', a'b' respectively. By the Gauß-Bonnet formula, the sums of the interior angles of the two triangle abc and a'b'c' are both $\pi + 2\pi - \alpha = 3\pi - \alpha$. Outside of the triangle a'b'c', draw three segments from a',b',c' respectively such that the four angles at a',b',c' are equal

to
$$\frac{2\pi - \pi + \frac{\alpha}{3}}{4} = \frac{\pi}{4} + \frac{\alpha}{12} < \frac{\pi}{2}$$
.

Let $\{d'\} = b'd' \cap c'd'$, $\{e'\} = a'e' \cap c'e'$, $\{f'\} = a'f' \cap b'f'$. Consider the segments d'e', e'f', d'f', ad', be', cf'. Clearly, $d'a \perp b'c'$, $e'b \perp a'c'$,

 $f'c \perp a'b'$. Let d, e, f denote the midpoint of e'f', d'f', d'e' respectively. It is not difficult to see that d, e, f belongs to the middle segment starting from a', b', c' respectively, and $a'd \perp e'f'$, $b'e \perp d'f'$, $c'f \perp d'e'$.

Therefore, we obtain a triangulation T_1 of S with 20 triangles:

abc, abc', acb', ab'd', ac'd', bca', ba'e', bc'e', ca'f', cb'f', def, def', dfe', da'e', da'f', efd', eb'd', eb'f', fc'd', fc'e'.

Now choose the suitable positions of a',b',c' such that the twelve segments ad', be', cf', a'd, a'e', a'f', b'd', b'e, b'f', c'd', c'e' and c'f equally divide the circle of the bottom.

It is not difficult to see that the four triangles def, def', de'f, d'ef are congruent equilateral triangles with angle $\frac{\pi}{3}$. It is clear that both the triangle *abc* and a'b'c' are equilateral triangles. Notice that the sums of the interior angles of the two triangles *abc* and a'b'c' are both $3\pi - \alpha$, thus,

$$\angle abc = \angle bca = \angle cab = \angle a'b'c' = \angle b'c'a' = \angle c'a'b' = \frac{3\pi - \alpha}{3} = \pi - \frac{\alpha}{3} < \frac{\pi}{2} \quad , \quad \text{and}$$
$$\pi - \left(\pi - \frac{\alpha}{2}\right)$$

then
$$\angle a'bc = \angle a'cb = \angle b'ac = \angle b'ca = \angle c'ab = \angle c'ba = \frac{\pi - \left(\pi - \frac{\pi}{3}\right)}{2} = \frac{\alpha}{6} < \frac{\pi}{2}$$
.

Hence, the three triangles *abc*', *bca*' and *acb*' are both acute ones.

By the symmetry, the six triangles ab'd', ac'd', ba'e', bc'e', ca'f' and cb'f' are congruent right triangles. Hence we only need to consider the triangle ab'd'. In the triangle ab'd', the sum of the interior angles is

$$\pi + \frac{2\pi + \alpha}{12} = \frac{7\pi}{6} + \frac{\alpha}{12}. \text{ Notice that } \angle ab'd' = \frac{\pi}{4} + \frac{\alpha}{12} < \frac{\pi}{2}, \ \angle b'ad' = \frac{\pi}{2}, \text{ we can}$$

get $\angle ad'b' = \frac{7\pi}{6} + \frac{\alpha}{12} - \left(\frac{\pi}{4} + \frac{\alpha}{12}\right) - \frac{\pi}{2} = \frac{5\pi}{12} < \frac{\pi}{2}.$

Similarly, the six triangles da'e', da'f', eb'd', eb'f', fc'd' and fc'e' are also congruent right triangles, then we only consider the triangle da'e'. In the triangle da'e', the sum of the interior angles is $\frac{7\pi}{6} + \frac{\alpha}{12}$. Since

$$\angle da'e' = \frac{\pi}{4} + \frac{\alpha}{12} < \frac{\pi}{2}, \quad \angle a'de' = \frac{\pi}{2}, \text{ we have}$$
$$\angle de'a' = \frac{7\pi}{6} + \frac{\alpha}{12} - \left(\frac{\pi}{4} + \frac{\alpha}{12}\right) - \frac{\pi}{2} = \frac{5\pi}{12} < \frac{\pi}{2}.$$

Now, we will slightly change the positions of some vertices of T_1 to get an acute triangulation of S. During all the steps, the original acute angles remain acute.

Slide a,b,c,d,e,f slightly in direction $\overline{d'a}$, $\overline{e'b}$, \overline{fc} , $\overline{a'd}$, $\overline{b'e}$, $\overline{c'f}$ respectively, then the twelve right angles $\angle b'ad'$, $\angle c'ad'$, $\angle a'be'$, $\angle c'be'$, $\angle a'cf'$, $\angle b'cf'$, $\angle a'de'$, $\angle a'df'$, $\angle b'ed'$, $\angle b'ef'$, $\angle c'fd'$ and $\angle c'fe'$ become acute. Then all the triangles become acute.

The proof is complete. \Box

Theorem 5. If $\alpha \in \left(0, \frac{3\pi}{2}\right)$, then the surface of the right circular cone cannot e triangulated into less than 22 acute triangles such that the degree of α is less

be triangulated into less than 22 acute triangles such that the degree of o is less than 5.

Proof. By Theorem 2, we know that $|T| \ge 20$. If |T| = 20, then *T* has $(20 \times 3)/2 = 30$ edges and, by Euler's formula, 30 - 20 + 2 = 12 vertices. Clearly, the sum of the degrees of all the vertices of *T* is $2 \times 30 = 60$. Since $\alpha \in \left(0, \frac{3\pi}{2}\right]$, the vertex *o* of the cone cannot be in the interior of a triangle of *T*. That is to say,

if *o* has degree less than 5, then *o* must be a vertex of *T*.

Therefore, there are three cases about *T*: the vertex *o* with degree 4, one vertex with degree 6, any other ten vertices have degree 5; the vertex *o* with degree 3, one vertex with degree 7, any other ten vertices have degree 5; the vertex *o* with degree 3, two vertices with degree 6, any other nine vertices have degree 5. However, there is no triangulation with 20 triangles can have these constructions. \Box

Theorem 6. If $\alpha \in \left(0, \frac{3\pi}{2}\right]$, then the surface of the right circular cone admits an acute triangulation with 22 triangles such that the degree of o is less than 5.

Proof. Case 1.
$$\alpha \in \left(0, \frac{3\pi}{2}\right)$$

Let a,b,c denote three points in the relative interior of the flank of S such that |oa| = |ob| = |oc| and $\angle aob = \angle boc = \angle coa = \frac{\alpha}{3}$. Let d,e,f,g,h,i be the midpoint of the segment oa, ab, ob, bc, oc, ac respectively. Draw two segments from a,b,c on the outside of the triangle abc such that the three angles around a,b,c are equal to $\frac{2\pi - \left(\pi - \frac{\alpha}{3}\right)}{3} = \frac{\pi}{3} + \frac{\alpha}{9}$ respectively. Let the points j,k,l be on the above segments such that $\{j\} = aj \cap bj$, $\{k\} = bk \cap ck$, $\{l\} = al \cap cl$. Choose the suitable positions of a,b,c such that the nine segments aj, ej, bj, bk, gk, ck, al, il and cl equally divide the circle of the bottom. Obviously, $je \perp ab$, $kg \perp bc$, $li \perp ac$.

Therefore, we obtain a triangulation T_2 of S with 22 triangles:

ade, def, bef, bfg, odf, ofh, fgh, cgh, odh, dhi, chi, adi, aej, ail, ajl, bej, bjk, bgk, cgk, ckl, cil, jkl.

Clearly, the triangle *jkl* are equilateral triangle with angle $\frac{\pi}{3}$. It is clear that the twelve triangles *ade*, *def*, *bef*, *bfg*, *odf*, *ofh*, *fgh*, *cgh*, *odh*, *dhi*, *chi* and *adi* are congruent isosceles triangles with vertex angle $\frac{\alpha}{3} < \frac{\pi}{2}$, and

hence the other two angles are both $\frac{\pi - \frac{\alpha}{3}}{2} = \frac{\pi}{2} - \frac{\alpha}{6} < \frac{\pi}{2}$.

By the Gauß-Bonnet formula, the sums of the interior angles of all the nine triangles *aej*, *ajl*, *ail*, *bej*, *bjk*, *bgk*, *cgk*, *ckl* and *cil* are $\pi + \frac{2\pi + \alpha}{9} = \frac{11\pi}{9} + \frac{\alpha}{9}$. Furthermore, the six right triangles *aej*, *ail*, *bej*, *bgk*, *cgk* and *cil* are congruent and the three triangles *ajl*, *bjk* and *ckl* are also congruent. Thus, we only consider the two triangles *bej* and *bjk*.

In the triangle *bej*, $\angle bej = \frac{\pi}{2}$, $\angle ebj = \frac{\pi}{3} + \frac{\alpha}{9} < \frac{\pi}{2}$, then

 $\angle bje = \frac{11\pi}{9} + \frac{\alpha}{9} - \frac{\pi}{2} - \left(\frac{\pi}{3} + \frac{\alpha}{9}\right) = \frac{7\pi}{18} < \frac{\pi}{2}.$ In the triangle bjk, $\angle jbk = \frac{\pi}{3} + \frac{\alpha}{9} < \frac{\pi}{2}.$ It is easy to see that $\angle bjk = \angle bkj = \frac{\frac{11\pi}{9} + \frac{\alpha}{9} - \left(\frac{\pi}{3} + \frac{\alpha}{9}\right)}{2} = \frac{4\pi}{9} < \frac{\pi}{2}.$

Now slide the vertices e, g, i of T_2 slightly in direction \overline{je} , \overline{kg} , \overline{li} respectively. Then the six right angles $\angle aej$, $\angle ail$, $\angle bej$, $\angle bgk$, $\angle cgk$ and $\angle cil$ become less than $\frac{\pi}{2}$. During all the steps, the original acute angles remain acute.

Case 2. $\alpha = \frac{3\pi}{2}$

Let a,b denote two points in the relative interior of the flank of S such that $\angle aob = \frac{\alpha}{2} = \frac{3\pi}{4}$ and |oa| = |ob|. Clearly, there are two segments from a to

b. Let c, d be the midpoints of the two segments.

Let k,l denote the points in the relative interior of the bottom such that the segments ak and bk are orthogonal to the segment ab which contains c, and the segments al and bl are orthogonal to the segment ab which contains d. Let $e \in ak$, $f \in bk$, $g \in al$, $h \in bl$ such that $ef \perp ak$, $gh \perp al$. Extend the segments fe and hg, and then let i be the intersection point. Extend ef and gh, then let j be the intersection point. Choose the suitable positions of a,b,e,f,g,h such that the ten segments ai, ei, ek, fk, fj, bj, hj, hj, hl, gl, gi equally divide the circle of the bottom.

We get a triangulation T_3 of S with 22 triangles as follows:

aoc, aod, boc, bod, ace, bcf, cef, adg, bdh, dgh, aei, efk, eik, fjk, bfj, agi, gil, ghl, bhj, hjl, ikl, jkl.

The four triangles *aoc*, *aod*, *boc*, *bod* are congruent right triangles, so

we only consider the triangle aoc. It is clear that $\angle aoc = \frac{\pi}{2} = \frac{3\pi}{8} < \frac{\pi}{2}$, $\angle aco = \frac{\pi}{2}$ and $\angle oac = \frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8} < \frac{\pi}{2}$. Obviously, the four triangles ace, bcf, adg, bdh are congruent right triangles. We only consider the triangle ace. Without loss of generality, suppose the radius of the flank of S is unit. Clearly, $\angle cae = \frac{\pi}{2}$, $|ac| = \sin \frac{3\pi}{40}$, $|ae| = \cos \frac{9\pi}{40} - \sin \frac{3\pi}{40} \cdot \tan \frac{\pi}{8}$, that is, |ac| < |ae|. Hence $\angle aec < \frac{\pi}{4}$ and $\frac{\pi}{4} < \angle ace < \frac{\pi}{2}$. The two triangles cef and $|ac| = \cos \frac{\pi}{2} + \frac{\pi}{4} = \sin \frac{\pi}{4} = \cos \frac{\pi}{4} + \frac{\pi}{4} = \cos \frac{\pi}{4} = \cos \frac{\pi}{4}$.

dgh are congruent isosceles triangles, so we only consider the triangle *cef*. It is easy to see that $\angle ecf = 2 \angle aec < \frac{\pi}{2}$, and then the other two angles

$$\angle cef = \angle cfe < \frac{\pi}{2}.$$

The two triangles *efk* and *ghl* are congruent, so we only consider the triangle *efk*. Clearly, $\angle kef = \angle kfe = \frac{\pi}{2}$. By the Gauß-Bonnet formula, the sum of

the interior angles of the triangles *efk* is $\pi + \frac{2\pi + \frac{3\pi}{2}}{10} = \frac{27\pi}{20}$, then

 $\angle ekf = \frac{27\pi}{20} - 2 \cdot \frac{\pi}{2} = \frac{7\pi}{20} < \frac{\pi}{2}$. Since the four triangles *aei*, *agi*, *bfj*, *bhj* are

congruent right triangles, we only consider the triangle *aei*. Clearly, $\angle aei = \frac{\pi}{2}$,

 $\angle eai = \angle gai = \frac{2\pi - \pi - \frac{\pi}{4}}{2} = \frac{3\pi}{8}$. Notice that the sum of the interior angles of the triangle *aei* is $\frac{27\pi}{20}$, thus $\angle aie = \frac{27\pi}{20} - \frac{\pi}{2} - \frac{3\pi}{8} = \frac{19\pi}{40} < \frac{\pi}{2}$.

The four triangles *eik*, *fjk*, *gil*, *hjl* are congruent, so we only consider the triangle *gil*. Let $\angle gli = x$, $\angle gil = y$, $\angle lij = z$, then we have

$$x + y = \frac{27\pi}{20} - \frac{\pi}{2} = \frac{17\pi}{20}, \quad y + z = \frac{2\pi - 2 \cdot \frac{19\pi}{40}}{2} = \frac{21\pi}{40}.$$
 Obviously,
$$\tan z = \frac{\sin \frac{2\pi}{5} - \sin \frac{\pi}{10} \cdot \tan \frac{13\pi}{40}}{\sin \frac{3\pi}{10} - \sin \frac{\pi}{5} \cdot \tan \frac{\pi}{40}}.$$
 It is not difficult to check that
$$\tan \frac{\pi}{6} < \tan z < \tan \frac{7\pi}{40}, \text{ that is, } \frac{\pi}{6} < z < \frac{7\pi}{40}.$$
 Then $y > \frac{21\pi}{40} - \frac{7\pi}{40} = \frac{7\pi}{20},$
$$y < \frac{21\pi}{40} - \frac{\pi}{6} = \frac{43\pi}{120} < \frac{\pi}{2}, \text{ and hence } x < \frac{17\pi}{20} - \frac{7\pi}{20} = \frac{\pi}{2}.$$

The two triangles *ikl* and *jkl* are congruent isosceles triangles, so we only consider the triangle *ikl*. It is clear that $\angle kil = 2z < 2 \cdot \frac{7\pi}{40} < \frac{\pi}{2}$, and then the

other two angles $\angle ikl = \angle ilk < \frac{\pi - 2 \cdot \frac{\pi}{6}}{2} = \frac{\pi}{3} < \frac{\pi}{2}$.

Now, we will slightly change the positions of some vertices of T_3 to get an acute triangulation of S. During all the slidings of vertices, once an angle has been acute, then the next steps will be performed so gently that the angle remains acute.

Step 1. Slide c,d slightly in direction \overrightarrow{oc} , \overrightarrow{od} respectively, then the eight right triangles $\angle oca$, $\angle ocb$, $\angle cae$, $\angle cbf$, $\angle oda$, $\angle odb$, $\angle dag$, $\angle dbh$ become less than $\frac{\pi}{2}$.

Step 2. Slide a,b slightly in direction \overrightarrow{oa} , \overrightarrow{ob} respectively, then the four right angles $\angle aei$, $\angle agi$, $\angle bfj$, $\angle bhj$ become less than $\frac{\pi}{2}$.

Step 3. Slide e, f, g, h slightly in direction \overrightarrow{ke} , \overrightarrow{kf} , \overrightarrow{lg} , \overrightarrow{lh} respectively. Meanwhile, maintain that the ten segments ai, ei, ek, fk, fj, bj, hj, hl, gl, gi still equally divide the circle, then the four right angles $\angle iek$, $\angle jfk$, $\angle igl$, $\angle jhl$ become less than $\frac{\pi}{2}$. In this step, don't change the values of the four angles $\angle kef$, $\angle kfe$, $\angle lgh$ and $\angle lhg$.

Step 4. Slide k, l slightly in direction \vec{ko} , \vec{lo} respectively, then the four right angles $\angle kef$, $\angle lgh$, $\angle lgh$, $\angle lhg$ become less than $\frac{\pi}{2}$.

During all the steps, the original acute angles remain acute. Then all the triangles become acute.

The proof is completed. \Box

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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