

# The Generalization of Signed Domination Number of Two Classes of Graphs

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## Abstract

Let  $G = (V, E)$  be a graph. A function  $f : V \rightarrow \{-1, 1\}$  is said to be a Signed Dominating Function (SDF) if  $f[v] = \sum_{v \in N[u]} f(v) \geq 1$  holds for all  $u \in V$ . The signed domination number  $\gamma_s(G) = \min \left\{ \sum_{v \in V} f(v) \mid f \text{ is an SDF of } G \right\}$ . In this paper, we determine the exact value of the Signed Domination Number of graphs  $C_n^k$  and  $P_n^k$  for  $k \geq 1$ , which is generalized the known results, respectively, where  $C_n^k$  and  $P_n^k$  are denotes the  $k$ -th power graphs of cycle  $C_n$  and path  $P_n$ .

## Keywords

Signed Domination Function, Signed Domination Numbers, Graphs  $C_n^k$ , Graphs  $P_n^k$

## 1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). For notation and graph theory terminology, please refer to the reference [1]. Let  $G$  be a simple undirected graph. The vertex set and the edge set of  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. For a vertex  $v \in V(G)$ , the *neighbor set*  $N_G(v)$  is the set of vertices adjacent to  $v$ , the degree of  $v$  is the number of adjacent vertices of  $v$  and denoted by  $d_G(v)$ .  $\delta(G) = \min \{d_G(v) : v \in V(G)\}$  and  $\Delta(G) = \max \{d_G(v) : v \in V(G)\}$  is the minimum degree and maximum degree of  $G$ . When no confusion can occur, we shall write  $N(v), N[v], \Delta, \delta$ , instead of  $N_G(v), N_G[v], \Delta(G), \delta(G)$ . For a subset  $U \subseteq V(G)$ , the subgraph induced by  $U$  is denoted by  $G[U]$ , which is the graph on  $U$  whose edges are precisely the edges of  $G$  with both ends in  $U$ .

For  $u \in V(G)$  and  $U \subseteq V(G)$ , the distance between  $u$  and  $U$ , denoted by  $d_G(u, U)$ , is the length of a shortest path from  $u$  to a vertex in  $U$ . When  $U$  consists of a single vertex  $v$ , we write  $d_G(u, v)$ , instead of  $d_G(u, \{v\})$ . For a positive integer  $k$ , the  $k$ -th power  $G^k$  of a graph  $G$  is the graph  $G^k$  whose vertex set is  $V(G)$ , two distinct vertices being adjacent in  $G^k$  if and only if their distance in  $G$  is at most  $k$ . If  $k=1$ ,  $G^1 = G$ . In particular, the graph  $G^2$  and the graph  $G^3$  are as the square of  $G$  and the cube of  $G$ . A cycle on  $n$  ( $n \geq 3$ ) vertices is a graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise, denoted by  $C_n$ . A path  $P_n$  is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.

In recent years, several kinds of signed domination problems in graphs have been investigated [2] [3] [4] [5]. Most of those belong to the vertex domination (or edge domination) of graphs, such as signed (edge) domination [6] [7], minus domination [8], cycle domination [9], signed roman (total) domination [10], weak roman domination [11], inverse signed total domination [12], etc. The signed domination number of cycles  $C_n$ , paths  $P_n$ , the square  $C_n^2$  of  $C_n$  and the square  $P_n^2$  of  $P_n$  were given in [13] [14] [15], respectively. In the present paper, we compute the exact values of signed domination number of  $C_n^k$  and  $P_n^k$  for  $k \geq 1$ .

Let  $G = (V, E)$  be a graph and  $\mathbb{R}$  be a real set. For a real function  $f: V \rightarrow \mathbb{R}$  and a nonempty subset  $S \subseteq V(G)$ , we may assume that  $f(S) = \sum_{v \in S} f(v)$ . In the following, for the sake of simplicity, simply write  $f(N[v])$  as  $f[v]$ , denotes  $f(V) = f(V(G))$ . In addition,  $\lceil x \rceil$  and  $\lfloor x \rfloor$  denotes the smallest integer not less than  $x$  and the largest integer not greater than  $x$ .

## 2. Preliminaries

**Definition 1.** [4] Let  $G = (V, E)$  be a graph, a function  $f: V \rightarrow \{-1, 1\}$  is said to be a Signed Dominating Function (SDF) if  $f[v] = \sum_{v \in N[u]} f(v) \geq 1$  holds for all  $u \in V$ , the signed domination number:

$$\gamma_s(G) = \min \left\{ \sum_{v \in V} f(v) \mid f \text{ is an SED of } G \right\}.$$

By Definition 1, it is easy to see the following a conclusion.

**Lemma 1.** For any vertex  $v \in V(G)$  and an SED  $f$  of  $G$ , if  $d_G(v)$  is even, then  $f[v] \geq 1$ . If  $d_G(v)$  is odd, then  $f[v] \geq 2$ .

For some vertex  $v_i$ , it is called a *optimal* if  $f[v_i]$  is reach it's lower bound.

**Lemma 2.** [8] Let  $P_n$  be a path. For  $n \geq 2$ , we have  $\gamma_s(P_n) = n - 2 \left\lfloor \frac{n-2}{3} \right\rfloor$ .

**Lemma 3.** [13] Let  $C_n$  be a cycle. For  $n \geq 3$ , we have  $\gamma_s(C_n) = n - 2 \left\lfloor \frac{n-2}{3} \right\rfloor$ .

**Lemma 4.** [13] If  $K_n$  is a complete graph, then:

$$\gamma_s(K_n) = \begin{cases} 1, & n \equiv 1 \pmod{2} \\ 2, & n \equiv 0 \pmod{2} \end{cases}$$

**Lemma 5.** [14] If  $C_n^2$  is a square of  $C_n$ , then:

$$\gamma_s(C_n^2) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil, & n \equiv 0,1,3 \pmod{5} \\ \left\lceil \frac{n}{5} \right\rceil + 1, & n \equiv 2,4 \pmod{5} \end{cases}$$

**Lemma 6.** [15] For a square  $P_n^2$  of  $P_n$ , if  $n = 3$ , then  $\gamma_s(P_3^2) = 1$ . If  $n \geq 4$ , then:

$$\gamma_s(P_n^2) = \begin{cases} 0.2(n+12), & n = 5k+3 \\ 0.2(n+4), & n = 5k+1 \\ 0.2(n+6), & n = 5k+4 \quad k \in \mathbb{N} \\ 0.2n, & n = 5k \\ 0.2(n+8), & n = 5k+2 \end{cases}$$

**Lemma 7.** [13] If  $G$  is a  $r$ -regular graph on  $n$  order, then  $\gamma_s(G) \geq \frac{n}{r+1}$ .

In this paper, we give an exact value of signed domination number of  $C_n^k$  and  $P_n^k$  for  $k \geq 1$  as follows.

**Theorem 1.** Let  $C_n^k$  be a  $k$ -th power graph of cycle  $C_n$ . For  $n \geq 3, k \geq 1$ , we have:

$$\gamma_s(C_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 0,1,3, \dots, 2k-1 \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & \text{if } n \equiv 2,4, \dots, 2k \pmod{2k+1} \end{cases}$$

If  $k = 1$  in Theorem 1, then it is equal to Lemma 3. If  $k = 2$  in theorem 1, then it is equal to Lemma 5.

**Theorem 2.** Let  $P_n^k$  be a  $k$ -th power graph of path  $P_n$ . For  $n \geq 2, 1 \leq k \leq n-1$ , we have the following results.

If  $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$  and  $k$  is odd, then:

$$\gamma_s(P_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil + 2, & \text{if } n \equiv 0,1, \dots, 2k-1 \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & \text{if } n \equiv 2,4, \dots, 2k \pmod{2k+1} \end{cases}$$

If  $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$  and  $k$  is even, then:

$$\gamma_s(P_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 0,1 \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & \text{if } n \equiv 2,4, \dots, 2k \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 2, & \text{if } n \equiv 3,5, \dots, 2k-1 \pmod{2k+1} \end{cases}$$

If  $\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n-1 (n \neq 3, k \neq 1)$ , then:

$$\gamma_s(P_n^k) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{2} \\ 2, & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

If  $n = 3, k = 1$ , then  $\gamma_s(P_3) = 3$ .

In fact, if  $k = 1$  in Theorem 2, then it is equal to Lemma 2. If  $k = 2$  in theorem 2, then it is equal to Lemma 6.

### 3. Proof of Theorems

#### 3.1. Proof of Theorem 1

Proof. Let  $C_n^k$  be a  $k$ -th power graph of cycle  $C_n$ . By definition of  $k$ -th power graph, it is easy to see that  $C_n^k$  is a  $2k$ -regular graph. By symmetry of cycle, it is enough to show that  $k \leq \frac{n}{2}$ . It remains to show that it is true for  $1 \leq k \leq \frac{n}{2}$ . By

Lemma 1, we have  $\gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil$ . We may assume that  $G$  has  $t$  vertices with  $-1$  label and  $s$  vertices with  $+1$  label, then  $s + t = n$ ,  $\gamma_s(G) = n - 2t$ . Let  $n = (2k + 1)q + r$ ,  $0 \leq r \leq 2k$ .

If  $r = 0$ , then  $\gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil$ .

If  $r = 1, 3, \dots, 2k - 1$ , then  $\gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil$ .

If  $r = 2, 4, \dots, 2k$ , then  $\gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil = q + 1$ . If  $n$  is odd, then  $q$  is odd, and  $q + 1$  is even. But  $\gamma_s(C_n^k) = n - 2t$  is odd and therefore:

$$\gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1 = q + 2$$

If  $n$  is even, then  $q$  is even, and  $q + 1$  is odd. But

$$\gamma_s(C_n^k) = n - 2t \text{ is even and therefore } \gamma_s(C_n^k) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1 = q + 2$$

In summary, we have:

$$\gamma_s(C_n^k) \geq \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & n \equiv 0, 1, \dots, 2k - 1 \pmod{2k + 1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & n \equiv 2, 4, \dots, 2k \pmod{2k + 1} \end{cases}$$

By definition of signed domination number, we only need to give a signed domination function  $f$  of  $C_n^k$ . Let  $C_n$  be a cycle on  $n$  vertices  $v_1, v_2, \dots, v_n$ .

If  $n \equiv 0 \pmod{2k + 1}$ , then let:

$$f(v_i) = \begin{cases} +1, & \text{if } i \equiv 1, 3, \dots, 2k + 1 \pmod{2k + 1} \\ -1, & \text{if } i \equiv 2, 4, \dots, 2k \pmod{2k + 1} \end{cases}$$

It is easy to see that  $C_n^k$  has a  $t = k \cdot \frac{n}{2k + 1}$  vertices with  $-1$  label and  $n - t$  vertices with  $+1$  label. It is follows that:

$$f(V) = n - 2 \left( k \cdot \frac{n}{2k + 1} \right) = n - 2k \cdot \frac{n}{2k + 1} = \left\lceil \frac{n}{2k + 1} \right\rceil$$

So, we have  $\gamma_s(C_n^k) = \left\lfloor \frac{n}{2k+1} \right\rfloor$ .

If  $n \equiv 1, 3, \dots, 2k-1 \pmod{2k+1}$ , then let:

$$f(v_i) = \begin{cases} +1, & \text{if } i \equiv 1, 3, 4, 6, \dots, 2k \pmod{2k+1} \\ -1, & \text{if } i \equiv 2, 5, 7, \dots, 2k+1 \pmod{2k+1} \end{cases}$$

By the definition of  $f$ , for every  $2k+1$  consecutive vertices on the  $C_n^k$ , there are  $k$  vertices that can be signed as  $-1$ , and  $\left\lfloor \frac{r}{2} \right\rfloor$  of the remaining  $n-r$  vertices can be signed as  $-1$ . So, it is easy to see that  $C_n^k$  has a

$$t = k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor + \left\lfloor \frac{n-(2k+1)\left\lfloor \frac{n}{2k+1} \right\rfloor}{2} \right\rfloor \text{ vertices with } -1 \text{ label and } n-t \text{ vertices}$$

with  $+1$  label. It follows that:

$$\begin{aligned} f(V) &= n - 2 \left( k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor + \left\lfloor \frac{n-(2k+1)\left\lfloor \frac{n}{2k+1} \right\rfloor}{2} \right\rfloor \right) \\ &= n - 2k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor - \left( n - (2k+1) \left\lfloor \frac{n}{2k+1} \right\rfloor - 1 \right) \\ &= \left\lfloor \frac{n}{2k+1} \right\rfloor + 1 = \left\lceil \frac{n}{2k+1} \right\rceil \end{aligned}$$

So, we have  $\gamma_s(C_n^k) = \left\lceil \frac{n}{2k+1} \right\rceil$ .

If  $n \equiv 2, 4, \dots, 2k \pmod{2k+1}$ , then let:

$$f(v_i) = \begin{cases} +1, & \text{if } i \equiv 1, 2, 4, 6, \dots, 2k \pmod{2k+1} \\ -1, & \text{if } i \equiv 3, 5, 7, \dots, 2k+1 \pmod{2k+1} \end{cases}$$

Same as above, it is easy to see that  $C_n^k$  has a

$$t = k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor + \frac{n-(2k+1)\left\lfloor \frac{n}{2k+1} \right\rfloor - 2}{2} \text{ vertices with } -1 \text{ label and } n-t \text{ vertices}$$

with  $+1$  label. It follows that:

$$\begin{aligned} f(V) &= n - 2 \left( k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor + \frac{n-(2k+1)\left\lfloor \frac{n}{2k+1} \right\rfloor - 2}{2} \right) \\ &= n - 2k \cdot \left\lfloor \frac{n}{2k+1} \right\rfloor - n + (2k+1) \left\lfloor \frac{n}{2k+1} \right\rfloor + 2 \\ &= \left\lfloor \frac{n}{2k+1} \right\rfloor + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1 \end{aligned}$$

So, we have  $\gamma_s(C_n^k) = \left\lceil \frac{n}{2k+1} \right\rceil + 1$ .

As mentioned above, we have:

$$\gamma_s(C_n^k) = \begin{cases} \left\lfloor \frac{n}{2k+1} \right\rfloor, & n \equiv 0, 1, \dots, 2k-1 \pmod{2k+1} \\ \left\lfloor \frac{n}{2k+1} \right\rfloor + 1, & n \equiv 2, 4, \dots, 2k \pmod{2k+1} \end{cases} \quad \square$$

### 3.2. Proof of Theorem 2

Proof. Let  $P_n^k$  be a  $k$ -th power graph of  $P_n$ , denoted by  $G$ . By definition of  $k$ -th power graph, we may assume that the vertex set of  $G$  are  $\{v_1, v_2, \dots, v_n\}$ , its edge set are  $\{E_1, E_2, \dots, E_{n-1}\}$ , where:

$$E_j = \{v_i v_{i+j} \mid 1 \leq i \leq n-j\}, 1 \leq j \leq n-1.$$

It is follows that:

$$d_G(v_j) = d_G(v_{n-j+1}) = k + j - 1, j = 1, \dots, k$$

$$d_G(v_{k+1}) = \dots = d_G(v_{n-k}) = 2k.$$

If  $n = 2$ , then  $k = 1$ . So,  $\gamma_s(P_2) = 2$ .

If  $n = 3$ , then  $k = 1$  (or  $k = 2$ ). So,  $\gamma_s(P_3) = 3$  (or 1).

It remains to show that it is true for  $n \geq 4$ .

**Case 1.**  $\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n-1$ .

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ . If  $k \geq \left\lfloor \frac{n}{2} \right\rfloor$ , then  $d_G\left(v_{\left\lfloor \frac{n}{2} \right\rfloor}\right) = n-1$ . If  $n$  is odd, then:

$$f(V) = f(v_1) + f(v_2) + \dots + f(v_n) = f\left[v_{\left\lfloor \frac{n}{2} \right\rfloor}\right] \geq 1. \text{ So, } \gamma_s(G) \geq 1. \text{ If } n \text{ is even,}$$

$$\text{then } f(V) = f(v_1) + f(v_2) + \dots + f(v_n) = f\left[v_{\frac{n}{2}}\right] \geq 1. \text{ In such a case, since}$$

$$d_G\left(v_{\frac{n}{2}}\right) = n-1 \text{ is odd and } \left|f\left[v_{\frac{n}{2}}\right]\right| \text{ is even. By Lemma 1, we have } f(V) \geq 2.$$

So,  $\gamma_s(G) \geq 2$ .

We only need to give a signed domination function  $f$  and prove that it is true for the upper bound of signed domination number of  $P_n^k$ .

**Subcase 1.1.**  $n \equiv 1 \pmod{4}$ .

Due to  $\delta = d_G(v_1) = k \geq \left\lfloor \frac{n}{2} \right\rfloor$ ,  $|f[v_1]| = k + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$  and  $\left\lfloor \frac{n}{2} \right\rfloor$  is even. It follows that we are labeled  $-1$  at  $t = \left\lfloor \frac{n}{4} \right\rfloor$  vertices in  $f[v_1]$  and in  $f[v_n]$ , and other vertices is label to  $+1$ . Namely, let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, t, n-t+1, n-t+2, \dots, n \\ +1, & \text{otherwise} \end{cases}$$

It is easy to show that  $f[v_j] \geq 1, f[v_{n-j+1}] \geq 1$  for  $j = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor$  and

$f\left[\left\lfloor v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\rfloor\right] = 1$ . In fact, under this function  $f$  we have:

$$f(V) = f\left[\left\lfloor v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\rfloor\right] = f(v_1) + f(v_2) + \dots + f(v_n) = n - 4t = n - 4\left(\left\lfloor \frac{n}{4} \right\rfloor\right) = 1. \text{ So,}$$

$$\gamma_s(G) \leq 1.$$

In summary, we have  $\gamma_s(G) = 1$ .

**Subcase 1.2.**  $n \equiv 2 \pmod{4}$ .

Because of  $\delta = d_G(v_1) = k \geq \left\lfloor \frac{n}{2} \right\rfloor$ ,  $|f[v_1]| = k + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$  and  $\left\lfloor \frac{n}{2} \right\rfloor$  is odd. It follows that we are labeled  $-1$  at  $t = \left\lfloor \frac{n}{4} \right\rfloor$  vertices in  $f[v_1]$  and in  $f[v_n]$ , and other vertices is label to  $+1$ . Namely, let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, t, n-t+1, n-t+2, \dots, n \\ +1, & \text{otherwise} \end{cases}$$

It is easy to show that  $f[v_j] \geq 1, f[v_{n-j+1}] \geq 1$  for  $j = 1, \dots, \frac{n}{2} - 1$  and

$$f\left[\left\lfloor v_{\frac{n}{2}} \right\rfloor\right] = f\left[\left\lfloor v_{\frac{n}{2}+1} \right\rfloor\right] = 2. \text{ In fact, under this function } f \text{ we have:}$$

$$\begin{aligned} f(V) &= f\left[\left\lfloor v_{\frac{n}{2}} \right\rfloor\right] = f(v_1) + f(v_2) + \dots + f(v_n) \\ &= n - 4t = n - 4\left(\left\lfloor \frac{n}{4} \right\rfloor\right) = n - 4\left(\frac{n-2}{4}\right) = 2. \end{aligned} \text{ So } \gamma_s(G) \leq 2.$$

In summary, we have  $\gamma_s(G) = 2$ .

**Subcase 1.3.**  $n \equiv 3 \pmod{4}$ .

Thanks to  $\delta = d_G(v_1) = k \geq \left\lfloor \frac{n}{2} \right\rfloor$ ,  $|f[v_1]| = k + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$  and  $\left\lfloor \frac{n}{2} \right\rfloor$  is odd. It follows that we are labeled  $-1$  at  $t_1 = \left\lfloor \frac{n}{4} \right\rfloor - 1$  vertices in  $f[v_1]$  and labeled  $-1$  at  $t_2 = \left\lfloor \frac{n}{4} \right\rfloor - 1$  vertices in  $f[v_n]$ , and other vertices is label to  $+1$ . Namely, let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, t_1, n-t_2+1, n-t_2+2, \dots, n \\ +1, & \text{otherwise} \end{cases}$$

It is easy to show that  $f[v_j] \geq 1, f[v_{n-j+1}] \geq 1$  for  $j = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor$  and

$$f\left[\left\lfloor v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\rfloor\right] = 1. \text{ In fact, under this function } f \text{ we have:}$$

$$\begin{aligned} f(V) &= f\left[\left\lfloor v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\rfloor\right] = f(v_1) + f(v_2) + \dots + f(v_n) \\ &= n - 2(t_1 + t_2) = n - 2\left[\frac{n}{4}\right] - 2\left[\frac{n}{4}\right] = n - \frac{n+1}{2} - \frac{n-3}{2} = 1. \end{aligned} \text{ So, } \gamma_s(G) \leq 1.$$

In summary, we have  $\gamma_s(G) = 1$ .

**Subcase 1.4.**  $n \equiv 0 \pmod{4}$ .

Since  $\delta = d_G(v_1) = k \geq \lfloor \frac{n}{2} \rfloor$ ,  $|f[v_1]| = k + 1 \geq \lfloor \frac{n}{2} \rfloor + 1$  and  $\lfloor \frac{n}{2} \rfloor$  is even. It follows that we are labeled  $-1$  at  $t_1 = \frac{n}{4}$  vertices in  $f[v_1]$  and labeled  $-1$  at  $t_2 = \frac{n}{4} - 1$  vertices in  $f[v_n]$ , and other vertices is label to  $+1$ . Namely, let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, t_1, n - t_2 + 1, n - t_2 + 2, \dots, n \\ +1, & \text{otherwise} \end{cases}$$

It is easy to show that  $f[v_j] \geq 1, f[v_{n-j+1}] \geq 1$  for  $j = 1, \dots, \frac{n}{2} - 1$  and

$$f\left[\frac{v_n}{2}\right] = f\left[\frac{v_{\frac{n}{2}+1}}{2}\right] = 2. \text{ In fact, under this function } f, \text{ we have:}$$

$$\begin{aligned} f(V) &= f\left[\frac{v_n}{2}\right] = f(v_1) + f(v_2) + \dots + f(v_n) \\ &= n - 2(t_1 + t_2) = n - 2\left(2 \cdot \frac{n}{4} - 1\right) = n - 4 \cdot \frac{n}{4} + 2 = 2. \end{aligned} \quad . \text{ So, } \gamma_s(G) \leq 2.$$

In summary, we have  $\gamma_s(G) = 2$ .

**Case 2.**  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ .

**Subcase 2.1.**  $n \equiv 1 \pmod{2k + 1}$ .

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ .

If  $k$  is odd, then  $|f[v_1]| = k + 1$  is even. By Lemma 1, we have:

$$f[v_1] = \sum_{i=1}^{k+1} f(v_i) \geq 2, \quad f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 2. \text{ It is follows that:}$$

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+1} f(v_i) + \sum_{i=1}^{\frac{n-(2k+2)}{2k+1}} f[v_{(2k+1)i+1}] + \sum_{i=n-k}^n f(v_i) \\ &\geq 2 + \frac{n-(2k+2)}{2k+1} + 2 = 4 + \frac{n-(2k+2)}{2k+1} \quad . \text{ So, we have:} \\ &= \frac{n-1}{2k+1} + \frac{6k+3}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil + 2 \end{aligned}$$

$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

If  $k$  is even, then  $|f[v_1]| = k + 1$  is odd. By Lemma 1, we have:

$$f[v_1] = \sum_{i=1}^{k+1} f(v_i) \geq 1, \quad f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 1. \text{ It is follows that:}$$

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+1} f(v_i) + \sum_{i=1}^{\frac{n-(2k+2)}{2k+1}} f[v_{(2k+1)i+1}] + \sum_{i=n-k}^n f(v_i) \\ &\geq 1 + \frac{n-(2k+2)}{2k+1} + 1 = 2 + \frac{n-(2k+2)}{2k+1} = \frac{n-1}{2k+1} + 1 = \left\lceil \frac{n}{2k+1} \right\rceil. \end{aligned} \quad . \text{ So, we have:}$$

$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil.$$



On the other hand, we only need to give a signed domination function  $f$ .

If  $k$  is odd, then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \lfloor \frac{k}{2} \rfloor, k+2+q(2k+1), k+4+q(2k+1), \dots, \\ & 3k+q(2k+1), n-\lfloor \frac{k}{2} \rfloor+1, \dots, n \\ +1, & \text{if } i = \lfloor \frac{k}{2} \rfloor+1, \dots, k+1, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 3k+1+q(2k+1), 3k+2+q(2k+1), n-k, \dots, n-\lfloor \frac{k}{2} \rfloor \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+1)}{2k+1} - 1$ . It is easy to see that  $G$  has a

$t = 2 \cdot \lfloor \frac{k}{2} \rfloor + k \cdot \frac{n-2(k+1)}{2k+1}$  vertices with  $-1$  label and other vertex is label to  $+1$ .

$$\gamma_s(G) \leq f(V) = n - 2 \left[ 2 \cdot \lfloor \frac{k}{2} \rfloor + k \cdot \frac{n-2(k+1)}{2k+1} \right]$$

It is follows that

$$= n - 2k + 2 - 2k \cdot \frac{n-2k-2}{2k+1} = \frac{n+2k}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 2$$

So, we have  $\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil + 2$ .

If  $k$  is even, then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \frac{k}{2}, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 3k+1+q(2k+1), n-\frac{k}{2}+1, \dots, n \\ +1, & \text{if } i = \frac{k}{2}+1, \dots, k+1, k+2+q(2k+1), k+4+q(2k+1), \dots, \\ & 3k+2+q(2k+1), n-k, \dots, n-\frac{k}{2} \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+1)}{2k+1} - 1$ . It is easy to see that  $G$  has a

$t = 2 \cdot \frac{k}{2} + k \cdot \frac{n-2(k+1)}{2k+1}$  vertices with  $-1$  label and other vertex is label to  $+1$ . It

$$\gamma_s(G) \leq f(V) = n - 2 \left[ 2 \cdot \frac{k}{2} + k \cdot \frac{n-2(k+1)}{2k+1} \right]$$

is follows that

$$= n - 2k - 2k \cdot \frac{n-2k-2}{2k+1} = \frac{n+2k}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$$

So, we have:

$$\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil$$

**Subcase 2.2.**  $n \equiv 0 \pmod{2k+1}$ .

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ .

If  $k$  is odd, then  $|f[v_1]| = k+1$  is even. By Lemma 1, we have:

$f[v_1] = \sum_{i=1}^{k+1} f(v_i) \geq 2$ ,  $f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 2$ ,  $\sum_{i=n-k-2k}^{n-k-1} f(v_i) \geq 0$  (otherwise, we have  $f[v_{n-2k}] \leq 0$ , this is a contradiction to Definition 1). It follows that

$$f(V) = \sum_{i=1}^{k+1} f(v_i) + \sum_{i=1}^{\frac{n-(2k+2+2k)}{2k+1}} f[v_{(2k+1)i+1}] + \sum_{i=n-k-2k}^{n-k-1} f(v_i) + \sum_{i=n-k}^n f(v_i) \geq 2 + \frac{n-(2k+2+2k)}{2k+1} + 0 + 2 = 4 + \frac{n-4k-2}{2k+1} = \frac{n}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 2$$

So, we have  $\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 2$ .

If  $k$  is even, then  $\lfloor f[v_1] \rfloor = k+1$  is odd. By Lemma 1, we have  $f[v_1] = \sum_{i=1}^{k+1} f(v_i) \geq 1$ ,  $f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 1$ ,  $\sum_{i=n-k-2k}^{n-k-1} f(v_i) \geq 0$  (otherwise, we have  $f[v_{n-2k}] \leq 0$ , this is a contradiction to Definition 1). It follows that

$$f(V) = \sum_{i=1}^{k+1} f(v_i) + \sum_{i=1}^{\frac{n-(2k+2+2k)}{2k+1}} f[v_{(2k+1)i+1}] + \sum_{i=n-k-2k}^{n-k-1} f(v_i) + \sum_{i=n-k}^n f(v_i) \geq 1 + \frac{n-(2k+2+2k)}{2k+1} + 0 + 1 = 2 + \frac{n-4k-2}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$$

. So,

we have  $\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil$ .

On the other hand, we only need to give a signed domination function  $f$ .

If  $k$  is odd, then let

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor, k+2+q(2k+1), k+4+q(2k+1), \dots, \\ & 3k+q(2k+1), n - \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, n \\ +1, & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, k+1, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 3k+1+q(2k+1), 3k+2+q(2k+1), n-k, \dots, n - \left\lfloor \frac{k}{2} \right\rfloor \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+1)-2k}{2k+1}$ . Specially, if  $q = \frac{n-2(k+1)-2k}{2k+1}$ , then

$f(v_q) = f(v_{n-k})$  and all such vertices are label to +1. Then  $G$  has  $t = 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + k + k \cdot \frac{n-2(k+1)-2k}{2k+1}$  vertices with -1 label and other vertex is label to +1. It follows that:

$$\gamma_s(G) \leq f(V) = n - 2 \left[ 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + k + k \cdot \frac{n-2(k+1)-2k}{2k+1} \right] = n - 2(k-1) - 2k - 2k \cdot \frac{n-4k-2}{2k+1}$$

. So, we have:

$$= n - 4k + 2 - 2k \cdot \frac{n-4k-2}{2k+1} = \frac{n}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 2$$

$$\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

If  $k$  is even, then:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \frac{k}{2}, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 2k+1+q(2k+1), 2k+2+q(2k+1), \dots, \\ & 2k+2+\frac{k}{2}-1+q(2k+1), n-\frac{k}{2}+1, \dots, n \\ +1, & \text{if } i = \frac{k}{2}+1, \dots, k+1, k+2+q(2k+1), k+4+q(2k+1), \dots, \\ & 2k+q(2k+1), 2k+2+\frac{k}{2}+q(2k+1), \dots, 3k+1+q(2k+1), \\ & 3k+2+q(2k+1), n-k, \dots, n-\frac{k}{2} \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+1)-2k}{2k+1}$ . Specially, if  $q = \frac{n-2(k+1)-2k}{2k+1}$ , then

$f(v_q) = f(v_{n-k})$  and all such vertices are label to +1. Then  $G$  has a

$t = 2 \cdot \frac{k}{2} + k + k \cdot \frac{n-2(k+1)-2k}{2k+1}$  vertices with  $-1$  and other vertex is label to +1.

It is follows that 
$$\gamma_s(G) \leq f(V) = n - 2 \left[ 2 \cdot \frac{k}{2} + k + k \cdot \frac{n-2(k+1)-2k}{2k+1} \right]$$
. So, we 
$$= n - 2k - 2k - 2k \cdot \frac{n-4k-2}{2k+1} = \frac{n}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$$

have 
$$\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil$$
.

**Subcase 2.3.**  $n \equiv 2 \pmod{2k+1}$ .

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ .

If  $k$  is odd, then  $\lfloor f[v_1] \rfloor = k+1$  is even. By Lemma 1, we have:

$f[v_1] = \sum_{i=1}^{k+1} f(v_i) \geq 2, f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 2, f(v_{n-k-1}) \geq -1$ . It is follows

that 
$$f(V) = \sum_{i=1}^{k+1} f(v_i) + \sum_{i=1}^{\frac{n-(2k+2+1)}{2k+1}} f[v_{(2k+1)i+1}] + f(v_{n-k-1}) + \sum_{i=n-k}^n f(v_i)$$
 
$$\geq 2 + \frac{n-(2k+2+1)}{2k+1} \cdot (-1) + 2 = 3 + \frac{n-2k-3}{2k+1} = \frac{n-2}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1$$

So, we have 
$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1$$
.

If  $k$  is even, then  $\lfloor f[v_2] \rfloor$  is even. By Lemma 1, we have:

$f[v_2] = \sum_{i=1}^{k+2} f(v_i) \geq 2, f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 1, \sum_{i=n-k-2k-1}^{n-k-1} f(v_i) \geq 1$ . It is

follows that 
$$f(V) = \sum_{i=1}^{k+2} f(v_i) + \sum_{i=1}^{\frac{n-(2k+4+2k)}{2k+1}} f[v_{(2k+1)i+2}]$$
 
$$+ \sum_{i=n-k-2k-1}^{n-k-2} f(v_i) + \sum_{i=n-k-1}^n f(v_i)$$
 
$$\geq 2 + \frac{n-(2k+4+2k)}{2k+1} + 1 + 1 = 4 + \frac{n-4k-4}{2k+1}$$
 
$$= \frac{n-2}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1$$
. So, we have:

$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1.$$

On the other hand, we only need to give a signed domination function  $f$ .

If  $k$  is odd, then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor, k+2+q(2k+1), k+4+q(2k+1), \dots, \\ & 2k-1+q(2k+1), 2k+1+q(2k+1), 2k+2+q(2k+1), \dots, \\ & 3k-1+q(2k+1), n-k-1, n-\left\lfloor \frac{k}{2} \right\rfloor+1, \dots, n \\ +1, & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor+1, \dots, k+1, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 2k+q(2k+1), 2k+3+q(2k+1), \dots, 3k+q(2k+1), \\ & 3k+1+q(2k+1), 3k+2+q(2k+1), n-k, \dots, n-\left\lfloor \frac{k}{2} \right\rfloor \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+1)-1}{2k+1} - 1$ . It is easy to see that  $G$  has a

$$t = 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + 1 + k \cdot \frac{n-2(k+1)-1}{2k+1} \text{ vertices with } -1 \text{ label and other vertex is label}$$

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + 1 + k \cdot \frac{n-2(k+1)-1}{2k+1} \right] \\ \text{to } +1. \text{ It is follows that} & \\ &= n - 2k - 2k \cdot \frac{n-2k-3}{2k+1} = \frac{n+4k}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil + 1 \end{aligned}$$

So, we have  $\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil + 1.$

If  $k$  is even, then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \frac{k}{2}, k+3+q(2k+1), k+5+q(2k+1), \dots, \\ & 2k+1+q(2k+1), 2k+3+q(2k+1), \dots, \\ & 2k+\frac{k}{2}+2+q(2k+1), n-\frac{k}{2}+1, \dots, n \\ +1, & \text{if } i = \frac{k}{2}+1, \dots, k+2, k+4+q(2k+1), k+6+q(2k+1), \dots, \\ & 2k+2+q(2k+1), 2k+\frac{k}{2}+3+q(2k+1), \dots, 3k+2+q(2k+1), \\ & 3k+3+q(2k+1), n-k-1, \dots, n-\frac{k}{2} \end{cases}$$

where  $q = 0, \dots, \frac{n-2(k+2)-2k}{2k+1}$ . Specially, if  $q = \frac{n-2(k+2)-2k}{2k+1}$ , then

$f(v_q) = f(v_{n-k-1})$  and all such vertices are label to +1. It is easy to see that  $G$

has a  $t = 2 \cdot \frac{k}{2} + k + k \cdot \frac{n-2(k+2)-2k}{2k+1}$  vertices with  $-1$  label and other vertex is label to +1. It is follows that:

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \frac{k}{2} + k + k \cdot \frac{n - 2(k + 2) - 2k}{2k + 1} \right] \\ &= n - 2k - 2k - 2k \cdot \frac{n - 4k - 4}{2k + 1} = \frac{n + 4k}{2k + 1} = \left\lceil \frac{n}{2k + 1} \right\rceil + 1 \end{aligned}$$

. So, we have:

$$\gamma_s(G) = \left\lceil \frac{n}{2k + 1} \right\rceil + 1.$$

**Subcase 2.4.**  $n \equiv r \pmod{2k + 1}, r = 3, 5, 7, \dots, 2k - 1.$

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ .

**Claim.** Let  $P_n = v_1 v_2 \dots v_n$  be a path and  $f$  be an SED of  $P_n^k$ . If every vertex  $v_i (1 \leq i \leq n - r)$  is optimal, then  $f(v_{(2k+1)q}) = -1$ , where  $q = 1, \dots, \frac{n-r}{2k+1}$ .

**Proof:** If  $k$  is odd, then  $|f[v_1]| = k + 1$  is even. Since every vertex  $v_i$  is optimal, by Lemma 1, we have  $f[v_1] = 2$ . It follows that  $f[v_1]$  has  $\left\lfloor \frac{k}{2} \right\rfloor$  vertices with  $-1$  label,  $f[v_3]$  has  $\left\lfloor \frac{k}{2} \right\rfloor + 1$  vertices with  $-1$  label,  $\dots$ ,  $f[v_k]$  has  $\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$  vertices with  $-1$  label. Because of  $f[v_{k+1}] = f[v_k] + f(v_{2k+1})$ , and  $v_{k+1}$  is optimal and therefore  $f[v_{k+1}]$  has  $k$  vertices with  $-1$  label. So, we have  $f(v_{2k+1}) = -1$ .

If  $k$  is even, then  $|f[v_1]| = k + 2$  is even. Since every vertex  $v_i$  is optimal, by Lemma 1, we have  $f[v_2] = 2$ . It follows that  $f[v_2]$  has  $\frac{k}{2}$  vertices with  $-1$  label,  $f[v_4]$  has  $\frac{k}{2} + 1$  vertices with  $-1$  label,  $\dots$ ,  $f[v_k]$  has  $\frac{k}{2} + \frac{k}{2} - 1$  vertices with  $-1$  label. Due to  $f[v_{k+1}] = f[v_k] + f(v_{2k+1})$ , and  $v_{k+1}$  is optimal and therefore  $f[v_{k+1}]$  has  $k$  vertices with  $-1$  label. So, we have  $f(v_{2k+1}) = -1$ .

We consider that every vertex  $v_i$  is optimal and the following facts.

$$\begin{aligned} f[v_{k+1}] &= \sum_{i=1}^{2k+1} f(v_i) \\ f[v_{k+2+(2k+1) \cdot j}] &= f[v_{k+1+(2k+1) \cdot j}] - f(v_{1+(2k+1) \cdot j}) + f(v_{2k+2+(2k+1) \cdot j}) \\ f[v_{k+3+(2k+1) \cdot j}] &= f[v_{k+2+(2k+1) \cdot j}] - f(v_{2+(2k+1) \cdot j}) + f(v_{2k+3+(2k+1) \cdot j}) \\ &\dots \\ f[v_{3k+2+(2k+1) \cdot j}] &= f[v_{3k+1+(2k+1) \cdot j}] - f(v_{2k+1+(2k+1) \cdot j}) + f(v_{4k+2+(2k+1) \cdot j}) \end{aligned}$$

where  $j = 0, \dots, \frac{n-r-2k-1}{2k+1} - 1$

It follows that:

$$f(v_1) = f(v_{2k+2}) = f(v_{4k+3}) = \dots = f(v_{1+(2k+1)(q-1)})$$

$$f(v_2) = f(v_{2k+3}) = f(v_{4k+4}) = \dots = f(v_{2+(2k+1)(q-1)})$$

...

$$f(v_{2k+1}) = f(v_{4k+2}) = f(v_{6k+3}) = \dots = f(v_{2k+1+(2k+1)(q-1)})$$

where  $q = 1, \dots, \frac{n-r}{2k+1}$

So, we have:

$$f(v_{(2k+1)q}) = -1.$$

**Subsubcase 2.4.1.** If  $r \leq k + 1$ , then  $k + r \leq 2k + 1$ .

If  $k$  is odd, then  $|f[v_n]| = k + 1$  is even and  $f[v_n] = \sum_{i=n-k}^n f(v_i) \geq 2$ . So,  $\sum_{i=1}^{k+r} f(v_i) = f[v_r] \geq 2$ . It follows that:

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+r} f(v_i) + \sum_{i=1}^{\frac{n-(k+1+k+r)}{2k+1}} f[v_{(2k+1)i+r}] + \sum_{i=n-k}^n f(v_i) \\ &\geq 2 + \frac{n-(k+1+k+r)}{2k+1} + 2 = 4 + \frac{n-2k-r-1}{2k+1} \quad . \text{Hence,} \\ &= \frac{n-r}{2k+1} + 3 = \left\lceil \frac{n}{2k+1} \right\rceil + 2 \end{aligned}$$

$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

If  $k$  is even, then  $|f[v_{n-1}]| = k + 2$  is even and  $f[v_{n-1}] = \sum_{i=n-k-1}^n f(v_i) \geq 2$ . So,  $\sum_{i=1}^{k+r-1} f(v_i) \geq 2$ . It follows that:

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+r-1} f(v_i) + \sum_{i=1}^{\frac{n-(k+2+k+r-1)}{2k+1}} f[v_{(2k+1)i+r-1}] + \sum_{i=n-k-1}^n f(v_i) \\ &\geq 2 + \frac{n-(k+1+k+r)}{2k+1} + 2 = 4 + \frac{n-2k-r-1}{2k+1} \quad . \text{Hence,} \\ &= \frac{n-r}{2k+1} + 3 = \left\lceil \frac{n}{2k+1} \right\rceil + 2 \end{aligned}$$

$$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

**Subsubcase 2.4.2.** If  $r > k + 1$ , then  $k + r > 2k + 1$ .

It follows that  $f(V) = \sum_{i=1}^{\frac{n-r}{2k+1}} f[v_{(2k+1)i-k}] + \sum_{i=n-r+1}^n f(v_i)$ . Owing to  $\sum_{i=n-r+1}^n f(v_i) \geq 3$  (Otherwise, since  $r$  is odd and therefore  $\sum_{i=n-r+1}^n f(v_i) \leq 1$ ). In order to obtain the  $\gamma_s(P_n^k)$ , we have to consider that all vertices are as optimal as possible. By Claim, we have  $f(v_{(2k+1)q}) = -1$ , namely,  $f(v_{n-r}) = -1$ . So,  $f[v_{n-k+1}] = f(v_{n-r}) + \sum_{i=n-r+1}^n f(v_i) \leq -1 + 1 = 0$ , this is a contradiction to Definition 1).

$$\text{Hence, } f(V) = \sum_{i=1}^{\frac{n-r}{2k+1}} f[v_{(2k+1)i-k}] + \sum_{i=n-r+1}^n f(v_i) \geq \frac{n-r}{2k+1} + 3 = \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

On the other hand, we only need to give a signed domination function  $f$ .

If  $k$  is odd, then we give a signed domination function  $f$  by the following rules.

If  $1 \leq i \leq k + 1$  and  $n - k \leq i \leq n$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \lfloor \frac{k}{2} \rfloor, n - \lfloor \frac{k}{2} \rfloor + 1, \dots, n \\ +1, & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k + 1, n - k, \dots, n - \lfloor \frac{k}{2} \rfloor \end{cases}$$

If  $k + 2 \leq i \leq n - k - 1$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = k + 2 + q(2k + 1), k + 4 + q(2k + 1), \dots, 3k + q(2k + 1) \\ +1, & \text{if } i = k + 3 + q(2k + 1), k + 5 + q(2k + 1), \dots, 3k + 1 + q(2k + 1), \\ & 3k + 2 + q(2k + 1) \end{cases}$$

where  $q = 0, \dots, \frac{n - 2(k + 1) - (r - 1)}{2k + 1}$ . It is trivial to see that  $G$  has

$t = 2 \cdot \lfloor \frac{k}{2} \rfloor + \lfloor \frac{r}{2} \rfloor + k \cdot \frac{n - 2(k + 1) - (r - 1)}{2k + 1}$  vertices with  $-1$  label and other vertex is label to  $+1$ . It is follows that:

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \lfloor \frac{k}{2} \rfloor + \frac{r - 1}{2} + k \cdot \frac{n - 2(k + 1) - (r - 1)}{2k + 1} \right] \\ &= n - 2k + 2 - 2k \cdot \frac{n - 2k - (r - 1)}{2k + 1} - (r - 1) \quad . \text{ So, we have:} \\ &= \frac{n - r}{2k + 1} + 3 = \left\lceil \frac{n}{2k + 1} \right\rceil + 2 \end{aligned}$$

$$\gamma_s(G) = \left\lceil \frac{n}{2k + 1} \right\rceil + 2.$$

If  $k$  is even, then we give a signed domination function  $f$  by the following rules.

If  $1 \leq i \leq k + 2$  and  $n - k - 1 \leq i \leq n$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \frac{k}{2}, n - \frac{k}{2} + 1, \dots, n \\ +1, & \text{if } i = \frac{k}{2} + 1, \dots, k + 2, n - k - 1, \dots, n - \frac{k}{2} \end{cases}$$

If  $k + 3 \leq i \leq n - k - 2$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = k + 3 + q(2k + 1), k + 5 + q(2k + 1), \dots, 3k + 1 + q(2k + 1) \\ +1, & \text{if } i = k + 4 + q(2k + 1), k + 6 + q(2k + 1), \dots, 3k + 2 + q(2k + 1), \\ & 3k + 3 + q(2k + 1) \end{cases}$$

where  $q = 0, \dots, \frac{n - 2(k + 2) - (r - 3)}{2k + 1}$ . It is trivial to see that  $G$  has

$t = 2 \cdot \frac{k}{2} + \frac{r - 3}{2} + k \cdot \frac{n - 2(k + 2) - (r - 3)}{2k + 1}$  vertices with  $-1$  label and other vertex is label to  $+1$ . It is follows that:

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \frac{k}{2} + \frac{r - 3}{2} + k \cdot \frac{n - 2(k + 2) - (r - 3)}{2k + 1} \right] \quad . \text{ So, we have:} \\ &= n - 2k - (r - 3) - 2k \cdot \frac{n - 2k - r - 1}{2k + 1} = \frac{n - r}{2k + 1} + 3 = \left\lceil \frac{n}{2k + 1} \right\rceil + 2 \end{aligned}$$

$$\gamma_s(G) = \left\lceil \frac{n}{2k+1} \right\rceil + 2.$$

**Subcase 2.5.**  $n \equiv r \pmod{2k+1}, r = 2, 4, 6, \dots, 2k$ .

We first prove that it is true for the lower bound of signed domination number of  $P_n^k$ .

**Subsubcase 2.5.1** If  $r \leq k+1$ , then  $k+r \leq 2k+1$ .

If  $k$  is odd, then  $|f[v_{n-2}]| = k+3$  is even and  $f[v_{n-2}] = \sum_{i=n-k-2}^n f(v_i) \geq 2$ . Hence,  $\sum_{i=1}^{k+r-2} f(v_i) \geq 1$ . It follows that:

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+r-2} f(v_i) + \sum_{i=1}^{\frac{n-(k+3+k+r-2)}{2k+1}} f[v_{(2k+1)i+r-2}] + \sum_{i=n-k-2}^n f(v_i) \\ &\geq 1 + \frac{n-(2k+r+1)}{2k+1} + 2 = 3 + \frac{n-2k-r-1}{2k+1} = \frac{n-r}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1. \end{aligned} \quad \text{. So,}$$

we have  $\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1$ .

If  $k$  is even, then  $|f[v_{n-1}]| = k+2$  is even and  $f(v_{n-k-2}) \geq -1$ ,  $f[v_{n-1}] = \sum_{i=n-k-1}^n f(v_i) \geq 2$ . Hence,  $\sum_{i=1}^{k+r-2} f(v_i) \geq 2$ . It follows that:

$$\begin{aligned} f(V) &= \sum_{i=1}^{k+r-2} f(v_i) + \sum_{i=1}^{\frac{n-(k+2+k+r-2+1)}{2k+1}} f[v_{(2k+1)i+r-2}] \\ &\quad + f(v_{n-k-2}) + \sum_{i=n-k-1}^n f(v_i) \\ &\geq 2 + \frac{n-(2k+1+r)}{2k+1} - 1 + 2 \quad \text{. So, we have:} \\ &= 3 + \frac{n-2k-r-1}{2k+1} = \frac{n-r}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1 \end{aligned}$$

$\gamma_s(G) \geq \left\lceil \frac{n}{2k+1} \right\rceil + 1$ .

**Subsubcase 2.5.2.** If  $r > k+1$ , then  $k+r > 2k+1$ .

It follows that  $f(V) = \sum_{i=1}^{\frac{n-r}{2k+1}} f[v_{(2k+1)i-k}] + \sum_{i=n-r+1}^n f(v_i)$ . Because of  $\sum_{i=n-r+1}^n f(v_i) \geq 2$  (Otherwise, we have  $\sum_{i=n-r+1}^n f(v_i) \leq 1$ . In order to obtain the  $\gamma_s(P_n^k)$ , we have to consider that all vertices are as optimal as possible. By Claim, we have  $f(v_{(2k+1)q}) = -1$ , namely,  $f(v_{n-r}) = -1$ . It follows that:  $f[v_{n-k+1}] = f(v_{n-r}) + \sum_{i=n-r+1}^n f(v_i) \leq -1 + 1 = 0$ , this is a contradiction to Definition 1).

Hence,  $f(V) = \sum_{i=1}^{\frac{n-r}{2k+1}} f[v_{(2k+1)i-k}] + \sum_{i=n-r+1}^n f(v_i) \geq \frac{n-r}{2k+1} + 2 = \left\lceil \frac{n}{2k+1} \right\rceil + 1$ .

On the other hand, we only need to give a signed domination function  $f$ .

If  $k$  is odd, then we give a signed domination function  $f$  by the following rules.

If  $1 \leq i \leq k+1$  and  $n-k \leq i \leq n$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor, n - \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, n \\ +1, & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, k+1, n-k, \dots, n - \left\lfloor \frac{k}{2} \right\rfloor \end{cases}$$



If  $k + 2 \leq i \leq n - k - 1$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = k + 2 + q(2k + 1), k + 4 + q(2k + 1), \dots, 3k + q(2k + 1) \\ +1, & \text{if } i = k + 3 + q(2k + 1), k + 5 + q(2k + 1), \dots, 3k + 1 + q(2k + 1), \\ & 3k + 2 + q(2k + 1) \end{cases}$$

where  $q = 0, \dots, \frac{n - 2(k + 1) - (r - 1)}{2k + 1}$ . It is trivial to see that  $G$  has a

$t = 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{r - 1}{2} \right\rceil + k \cdot \frac{n - 2(k + 1) - (r - 1)}{2k + 1}$  vertices with  $-1$  label and other vertex is label to  $+1$ . It is follows that:

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + \frac{r}{2} + k \cdot \frac{n - 2(k + 1) - (r - 1)}{2k + 1} \right] \\ &= n - 2k + 2 - r - 2k \cdot \frac{n - 2k - r - 1}{2k + 1} = \frac{n - r}{2k + 1} + 1 = \left\lceil \frac{n}{2k + 1} \right\rceil + 1 \end{aligned}$$

. So, we have

$$\gamma_s(G) = \left\lceil \frac{n}{2k + 1} \right\rceil + 1.$$

If  $k$  is even, then we give a a signed domination function  $f$  by the following rules.

If  $1 \leq i \leq k + 2$  and  $n - k - 1 \leq i \leq n$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = 1, \dots, \frac{k}{2}, n - \frac{k}{2} + 1, \dots, n \\ +1, & \text{if } i = \frac{k}{2} + 1, \dots, k + 2, n - k - 1, \dots, n - \frac{k}{2} \end{cases}$$

If  $k + 3 \leq i \leq n - k - 2$ , then let:

$$f(v_i) = \begin{cases} -1, & \text{if } i = k + 3 + q(2k + 1), k + 5 + q(2k + 1), \dots, 3k + 1 + q(2k + 1) \\ +1, & \text{if } i = k + 4 + q(2k + 1), k + 6 + q(2k + 1), \dots, 3k + 2 + q(2k + 1), \\ & 3k + 3 + q(2k + 1) \end{cases}$$

where  $q = 0, \dots, \frac{n - 2(k + 2) - (r - 3)}{2k + 1}$ . It is trivial to see that  $G$  has a

$t = 2 \cdot \frac{k}{2} + \left\lceil \frac{r - 3}{2} \right\rceil + k \cdot \frac{n - 2(k + 2) - (r - 3)}{2k + 1}$  vertices with  $-1$  label and other vertex is label to  $+1$ . It is follows that:

$$\begin{aligned} \gamma_s(G) \leq f(V) &= n - 2 \left[ 2 \cdot \frac{k}{2} + \frac{r - 2}{2} + k \cdot \frac{n - 2k - r - 1}{2k + 1} \right] \\ &= n - 2k - r + 2 - 2k \cdot \frac{n - 2k - r - 1}{2k + 1} = \frac{n - r}{2k + 1} + 2 = \left\lceil \frac{n}{2k + 1} \right\rceil + 1 \end{aligned}$$

. So, we have

$$\gamma_s(G) = \left\lceil \frac{n}{2k + 1} \right\rceil + 1.$$

As mentioned above, we have:

If  $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$  and  $k$  is odd, then:

$$\gamma_s(P_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil + 2, & \text{if } n \equiv 0, 1, \dots, 2k-1 \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & \text{if } n \equiv 2, 4, \dots, 2k \pmod{2k+1} \end{cases}$$

If  $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$  and  $k$  is even, then:

$$\gamma_s(P_n^k) = \begin{cases} \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } n \equiv 0, 1 \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 1, & \text{if } n \equiv 2, 4, \dots, 2k \pmod{2k+1} \\ \left\lceil \frac{n}{2k+1} \right\rceil + 2, & \text{if } n \equiv 3, 5, \dots, 2k-1 \pmod{2k+1} \end{cases}$$

If  $\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n-1$  ( $n \neq 3, k \neq 1$ ), then:

$$\gamma_s(P_n^k) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{2} \\ 2, & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

If  $n = 3, k = 1$ , then  $\gamma_s(P_3) = 3$ . □

#### 4. Concluding Remarks

As the dominance theory of graphs becomes more and more diversified, the dominance problem of graphs is NP-complete. Therefore, studying the upper and lower bounds of domination numbers of graphs and even accurate estimation and mutual relations are the main aspects of current research. In this paper, we determine the exact value of the Signed Domination Number of  $k$ -th power graphs  $C_n^k$  and  $P_n^k$  for  $k \geq 1$ , these conclusions are the basis for the study of the bounds of the signed domination number of the  $k$ -th power graphs of generally connected graphs, which have important meanings in the structural theory of graph theory. From the signed domination number of the  $k$ -th power graph, other types of domination numbers can be further studied, which will be the key research object in the future.

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#### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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