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On the Number of Idempotent Partial Contraction Mappings of a Finite Chain

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Abstract

Let $\mathcal{P}_n$ be the partial symmetric semigroup on $X_n = \{1, 2, \cdots, n\}$ and let $\mathcal{OCP}_n$ and $\mathcal{ODCP}_n$ be its subsemigroups of order-preserving contractions and order-preserving, order-decreasing contractions mappings of X_n, respectively. In this paper we investigate the cardinalities of $E(\mathcal{OCP}_n)$ and $E(\mathcal{ODCP}_n)$, the set idempotents of $\mathcal{OCP}_n$ and $\mathcal{ODCP}_n$, respectively. We also investigate the cardinalities of certain equivalences on $E(OCP_n)$ and $E(\mathcal{ODCP}_n)$.

Keywords

Height, Right (Left) Waist and Fix of a Transformation, Idempotents

1. Introduction

Let $X_n = \{1, 2, \cdots, n\}$. A (partial) transformation $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\,\alpha \subseteq X_n$ is said to be *full* or *total* if $\mathrm{Dom}\,\alpha = X_n$; otherwise it is called *strictly* partial. The set of partial transformations of X_n, denoted by $\mathcal{P}_n$, more commonly known as the *partial transformation semigroup* is also known as the *partial symmetric semigroup or monoid* with *composition of mappings* as the semigroup operation. Similarly, the set of full (or total) transformations of X_n, denoted by $\mathcal{T}_n$, more commonly known as the *full transformation semigroup* is also known as the (*full*) *symmetric semigroup* or *monoid*.

We shall write the image of x under α as $(x)\alpha$ (or simply $x\alpha$) instead of $\alpha(x)$. This is called the *right-hand* notation and it has the advantage that composition of maps is read from left to right, that is, $(x)\alpha\beta = ((x)\alpha)\beta$. Further, a transformation $\alpha \in \mathcal{P}_n$ is said to be *order-preserving* (*order-reversing*) if

$(\forall x, y \in \mathrm{Dom}\alpha)\quad x \le y \Rightarrow x\alpha \le y\alpha\,(x\alpha \ge y\alpha)$ and, a *contraction mapping* (or simply a *contraction*) if $(\forall x, y \in \mathrm{Dom}\alpha)\quad |x-y| \ge |x\alpha - y\alpha|$. We shall denote by $\mathcal{OCP}_n$ and $\mathcal{ODCP}_n$, the semigroups of order-preserving partial contractions and of order-preserving, order-decreasing partial contractions of X_n, respectively.

Recently, Zhao and Yang [1] initiated the algebraic study of semigroups of order-preserving partial contractions of X_n, where they referred to our contractions as *compressions*. A general systematic studied of various semigroups of contraction mappings of a finite chain was initiated in the papers [2] [3] [4] [5]. While the papers [2] [4] [5] investigated algebraic properties of various semigroups of contraction mappings of a finite chain, Adeshola and Umar [3] investigated combinatorial properties of the semigroups of order-preserving full contraction mappings, $\mathcal{OCT}_n$ and its subsemigroup of order-decreasing full contraction mappings, $\mathcal{ODCT}_n$. This paper investigates the combinatorial properties of the set of idempotents of $\mathcal{OCP}_n$ and $\mathcal{ODCP}_n$ and of certain natural equivalences on them.

An element e in a semigroup S, is said to be an *idempotent* if $e^2 = e$. Every finite semigroup contains an idempotent and in a group there is only one idempotent, namely the identity. Semigroups which contain a "sufficient" supply of idempotents (for example, *regular*, *abundant*, etc.) have been the object of study by many semigroup theorists largely due to the role idempotents play in determining the structure of such semigroups. Counting the number of idempotents in a semigroup has attracted the attention of several authors, see for example [3] [6]-[17], which is by no means a complete list. For a detailed account on combinatorial enumeration problems in transformation semigroup theory we refer the reader to Umar [17]. It is fairly obvious that the number of full and partial transformation idempotents of X_n denoted by $|E(\mathcal{T}_n)|$ and $|E(\mathcal{P}_n)|$ are, respectively,

$$|E(\mathcal{T}_n)| = \sum_{p=0}^{n} \binom{n}{p} p^{n-p} \text{ [14] and } |E(\mathcal{P}_n)| = \sum_{p=0}^{n} \binom{n}{p} (p+1)^{n-p} \text{ [18].}$$

However, at least two non-trivial cases are worth mentioning. In an elegant paper in 1971, Howie [10] showed that the number of full order-preserving idempotents denoted by $|E(\mathcal{O}_n)|$ is

$$|E(\mathcal{O}_n)| = f_{2n},$$

where f_m denotes the mth Fibonacci number, defined recursively for $m > 2$ by

$$f_1 = f_2 = 1,\ f_m = f_{m-1} + f_{m-2}.$$

More recently, Adeshola and Umar [3] showed that the number of full order-preserving contraction idempotents denoted by $|E(\mathcal{OCT}_n)|$ is

$$|E(\mathcal{OCT}_n)| = n(n+1)/2 = \binom{n+1}{2}.$$

We conclude this section with a breakdown of our investigation section by section. In Section 2 we obtain the cardinalities of various equivalence classes defined on $E(\mathcal{OCP}_n)$. In Section 3 we obtain the analogues of the results from Section 2 for $E(\mathcal{ODCP}_n)$. These cardinalities lead to formulae for the orders of $E(\mathcal{OCP}_n)$ and $E(\mathcal{ODCP}_n)$ as well as new triangles of numbers which are as at the time of submitting this paper not yet recorded in [19].

For standard concepts in semigroup and transformation semigroup theory, see for example [8] [9]. In particular, the set of idempotents of a semigroup S is denoted by $E(S)$.

2. The Semigroup $\mathcal{OCP}_n$

Our approach is essentially that of [20]. For $\alpha \in \mathcal{P}_n$ define $F(\alpha) = \{x \in [n] : x\alpha = x\}$ as the set of fixed points of α, and let

- $e_n = |E(\mathcal{OCP}_n)|$,
- $\beta_n = |\{\alpha \in E(\mathcal{OCP}_n) : F(\alpha) = \varnothing\}|$,
- $u_n = |\{\alpha \in E(\mathcal{OCP}_n) : \{1\} \subseteq F(\alpha)\}|$,
- $v_n = |\{\alpha \in E(\mathcal{OCP}_n) : F(\alpha) = \{1\}\}| = |\{\alpha \in E(\mathcal{OCP}_n) : F(\alpha) = \{n\}\}|$,
- $a_n = |\{\alpha \in E(\mathcal{OCP}_n) : \{1, n\} \subseteq F(\alpha)\}|$,
- $\gamma_n = |\{\alpha \in E(\mathcal{OCP}_n) : F(\alpha) = \{1, n\}\}|$.

Then we have the following results:

Lemma 2.1. *For* $n \geq 1$, $\gamma_n = 1$, *and* $\gamma_0 = 0$.

Lemma 2.2. *For* $n \geq 2$, $a_n = 2^{n-2}$, $a_1 = 1$ *and* $a_0 = 0$.

Proof. Let $\alpha \in E(\mathcal{OCP}_n)$ be such that $\{1, n\} \subseteq F(\alpha)$. It is clear that for all $x \in X_n \setminus \{1, n\}$ either $x \in \mathrm{Dom}\,\alpha$ or $x \notin \mathrm{Dom}\,\alpha$. In the former, we must have $x\alpha = x$ and there are $n-2$ choices for x. Thus each x has two degrees of freedom. The result is clear when $n = 1$. □

Lemma 2.3. *For* $n \geq 1$, $v_n = 2^{n-1}$, *and* $v_0 = 0$.

Proof. Let $\alpha \in E(\mathcal{OCP}_n)$ be such that $\{1\} = F(\alpha) = \mathrm{Im}\,\alpha$. Then for all $x \in \mathrm{Dom}\,\alpha$, we must have $x\alpha = 1$, and again, each with 2 degrees of freedom. □

Next, we state and prove an analogue of ([20], Lemma 4) which will be useful in what follows.

Lemma 2.4. *For* $n \geq 2$, $u_n = v_n + \gamma_2 u_{n-1} + \cdots + \gamma_n u_1$.

Proof. $u_n - v_n$ is the number of maps $\alpha \in \mathcal{OCP}_n$ with $1\alpha = 1$ and $i\alpha = i$ for some $i > 1$: if r is the smallest integer greater than 1 such that $r\alpha = r$; then clearly the number of such maps α is $\sum_{r=2}^{n} \gamma_r u_{n-r+1}$, as required. □

This leads to the following lemma:

Lemma 2.5. *For* $n \geq 1$, $u_n = 2^{n-2}(n+1)$, *and* $u_0 = 0$.

Proof. Substituting γ_n and v_n with 1 and 2^{n-1}, respectively into Lemma 2.4 we get

$$u_n = 2^{n-1} + u_{n-1} + u_{n-2} + \cdots + u_1 \tag{1}$$

Replacing n by $n-1$ we get

$$u_{n-1} = 2^{n-2} + u_{n-2} + u_{n-3} + \cdots + u_1 \tag{2}$$

Subtracting (2) from (1) we get

$$u_n = 2^{n-2} + 2u_{n-1},$$

which implies (by iteration)

$$u_n = 2^{n-2}(n+1),$$

as required. □

We also have

Lemma 2.6. *For* $n \geq 0$, $\beta_n = 1$.

Next, we state and prove an analogue of ([20], Lemma 5) which will be useful in what follows.

Lemma 2.7. *For* $n \geq 2$, $e_n = \beta_n + v_1 u_{n-1} + v_2 u_{n-2} + \cdots + v_n u_1$.

Proof. The nilpotents of $\mathcal{OCP}_n$ are precisely those that do not have fixed points (see [21]), hence $e_n - \beta_n$ is the number of maps $\alpha \in \mathcal{OCP}_n$ with at least one fixed point. If $u_{n,r}$ is the number of such maps α with r as the smallest fixed point, then $u_{n,r} = v_r u_{n-r+1}$. Hence $e_n - \beta_n = \sum_{r=1}^{n} v_r u_{n-r+1}$. The result now follows. □

Now we are ready to state and prove one of the two main results of this section.

Theorem 2.8. *For* $n \geq 0$, $e_n = 1 + 2^{n-3} n(n+3)$.

Proof. Using Lemmas 2.5, 2.6 & 2.7 we see that

$$\begin{aligned} e_n &= 1 + 2^0 \times 2^{n-2}(n+1) + 2^1 \times 2^{n-3} n + 2^2 \times 2^{n-4}(n-1) + \cdots \\ &\quad + 2^{n-2} \times 2^0 \times 3 + 2^{n-1} \times 2^{-1} \times 2 \\ &= 1 + 2^{n-2}\left[(n+1) + n + (n-1) + \cdots + 3 + 2 + 1 - 1\right] \\ &= 1 + 2^{n-2}\left[(n+1)(n+2)/2 - 1\right] \\ &= 1 + 2^{n-3} n(n+3), \end{aligned}$$

as required. □

Now we let $e(x)$ be the (ordinary) generating functions of the sequence e_n above. The proof of the next result is routine using Theorem 2.8 above.

Theorem 2.9. $e(x) = \sum_{n\geq 0} e_n x^n = \dfrac{1 - 5x + 10x^2 - 6x^3}{(1-x)(1-2x)^3} = \dfrac{(1-x)(1-4x+6x^2)}{(1-x)(1-2x)^3}$.

As in Umar [17], for natural numbers $n \geq k \geq p \geq 0$ we define

$$F_{pk}(n; p, k) = F(n; p, k) = \left|\left\{\alpha \in S : h(\alpha) = |\operatorname{Im}\alpha| = p, w^+(\alpha) = k\right\}\right|, \tag{3}$$

$$\begin{aligned} F_p(n; p) &= F(n; p) = \left|\left\{\alpha \in S : h(\alpha) = |\operatorname{Im}\alpha| = p\right\}\right|, \\ F_k(n; k) &= F(n; k) = \left|\left\{\alpha \in S : w^+(\alpha) = k\right\}\right|. \end{aligned} \tag{4}$$

Then we have

Proposition 2.10. *Let* $S = E(\mathcal{OCP}_n)$. *Then* $F_{pk}(n;1,k) = 2^{n-1}$ *and*

$$F(n; p, k) = \sum_{t=1}^{k-p+1} \binom{k-t+1}{p-2} 2^{n-(k-t+1)}, (n \geq k \geq p \geq 2).$$

Proof. Let $\alpha \in E(\mathcal{OCP}_n)$ be such that $h(\alpha) = |\operatorname{Im}\alpha| = p$ and $w^+(\alpha) = k$.

First, notice that if $p=1$ then $\{k\}=F(\alpha)=\operatorname{Im}\alpha$, and so $k\alpha=k$. Now for all $x\in[n]\setminus\{k\}$ there are two degrees of freedom: $x\in \operatorname{Dom}\alpha$ and $x\alpha=k$; or $x\notin \operatorname{Dom}\alpha$. Hence $F_{pk}(n;1,k)=2^{n-1}$.

Next, let $t=\min(\operatorname{Im}\alpha)$ then $t\in\{1,2,\cdots,k-p+1\}$. Next, note that we can choose the remaining $p-2$ elements of $F(\alpha)$ in $\binom{k-t+1}{p-2}$ ways, since $t,k\in\operatorname{Im}\alpha=F(\alpha)\subseteq\operatorname{Dom}\alpha$. Similarly, the remaining elements of $\operatorname{Dom}\alpha$ can be chosen from $\{1,2,\cdots,t-1\}\cup\{k+1,k+2,\cdots,n\}$ in $2^{t-1}\cdot 2^{n-k}=2^{n-(k-t+1)}$ ways, as each has two degrees of freedom. Now taking the sum over the range of t yields the required result. □

Immediately, we deduce the following

Corollary 2.11. *Let* $S=E(\mathcal{OCP}_n)$. *Then* $F_k(n;0)=1$ *and*

$$F(n;k)=(k+1)2^{n-2},(n\geq k\geq 1).$$

Corollary 2.12. *Let* $S=E(\mathcal{OCP}_n)$. *Then* $F_p(n;0)=1$, $F_p(n;1)=n2^{n-1}$ *and*

$$F(n;p)=\sum_{k=p}^{n}\sum_{t=1}^{k-p+1}\binom{k-t+1}{p-2}2^{n-(k-t+1)},(n\geq p\geq 2).$$

Finally, notice that we may recover Theorem 2.8 from either of the two corollories above. For some selected values of $F(n;k)$ and $F(n;p)$ in $E(\mathcal{OCP}_n)$ see Table 1 and Table 2 below:

Table 1. Some selected values of $F(n;k)$ in $E(\mathcal{OCP}_n)$.

n \ k	0	1	2	3	4	5	6	$\sum_{k=0}^{n}F(n;k)=\lvert E(\mathcal{OCP}_n)\rvert$
0	1							1
1	1	1						2
2	1	2	3					6
3	1	4	6	8				19
4	1	8	12	16	20			57
5	1	16	24	32	40	48		161
6	1	32	48	64	80	96	112	433

Table 2. Some selected values of $F(n;p)$ in $E(\mathcal{OCP}_n)$.

n \ p	0	1	2	3	4	5	6	$\sum_{p=0}^{n}F(n;p)=\lvert E(\mathcal{OCP}_n)\rvert$
0	1							1
1	1	1						2
2	1	4	1					6
3	1	12	5	1				19
4	1	32	17	6	1			57
5	1	80	49	23	7	1		161
6	1	192	129	72	30	8	1	433

3. The Semigroup $\mathcal{ODCP}_n$

Let

- $d_n = \left|E\left(\mathcal{ODCP}_n\right)\right|$,
- $z_n = \left|\left\{\alpha \in E\left(\mathcal{ODCP}_n\right) : F(\alpha) = \{n\}\right\}\right|$.

Then observe that Lemmas 2.1, 2.2, 2.4, 2.5 & 2.6 are all valid if we replace $\mathcal{OCP}_n$ with $\mathcal{ODCP}_n$. Moreover, in contrast to Lemma 2.3 we have

Lemma 3.1. *For* $n \geq 1$, $z_n = 1$.

To prove the main result of this section, the following lemma (which can be proved by induction) will be needed:

Lemma 3.2. *For* $n \geq 1$, *we have* $\sum_{i=1}^{n} i2^{n-i} = 2^{n+1} - (n+2)$.

An analogue of Lemma 2.7 will also be needed.

Lemma 3.3. *For* $n \geq 2$, $d_n = \beta_n + z_1 u_{n-1} + z_2 u_{n-2} + \cdots + z_n u_1$.

Thus we can now state and prove one of the main results of this section.

Theorem 3.4. *For* $n \geq 0$, $d_n = 1 + n2^{n-1}$.

Proof. Using Lemmas 2.5, 2.6, 3.1 & 3.3 we see that

$$\begin{aligned} d_n &= 1 + 2^{n-2}(n+1) + 2^{n-3}n + 2^{n-4}(n-1) + \cdots + 2^0\left[n-(n-3)\right] + 1 \\ &= 2 + n\left[2^{n-2} + 2^{n-3} + \cdots + 2^0\right] + 2^{n-2} \\ &\quad -\left[2^{n-4} - 2\times 2^{n-5} - 3\times 2^{n-6} - \cdots - (n-3)\times 2^0\right] \\ &= 2 + n\left(2^{n-1} - 1\right) + 2^{n-2} - \left[2^{n-2} - (n-1)\right] \quad \text{(by Lemma 3.2)} \\ &= 1 + n2^{n-1}, \end{aligned}$$

as required. □

Now we let $d(x)$ be the (ordinary) generating functions of the corresponding sequence above. Then we have the following result whose proof is routine using Theorem 3.4.

Theorem 3.5. $d(x) = \sum_{n\geq 0} d_n x^n = \dfrac{1-3x+3x^2}{(1-x)(1-2x)^2} = \dfrac{(1-x)^3 + x^3}{(1-x)(1-2x)^2}$.

As in the previous section we obtain expressions for the following combinatorial functions.

Proposition 3.6. *Let* $S = E\left(\mathcal{ODCP}_n\right)$. *Then*

$$F(n; p, k) = \binom{k-1}{p-1} 2^{n-k}, (n \geq k \geq p \geq 0).$$

Proof. Let $\alpha \in E\left(\mathcal{ODCP}_n\right)$ be such that $h(\alpha) = |\mathrm{Im}\,\alpha| = p$ and $w^+(\alpha) = k$. First, note that we can choose the remaining $p-1$ elements of $F(\alpha)$ in $\binom{k-1}{p-1}$ ways, since $k \in \mathrm{Im}\,\alpha = F(\alpha) \subseteq \mathrm{Dom}\,\alpha$. Similarly, the remaining elements of $\mathrm{Dom}\,\alpha$ can be chosen from $[n]\setminus\{1,2,\cdots,k\}$ in 2^{n-k} ways, as each has two degrees of freedom: $x\alpha = k$ or $x \notin \mathrm{Dom}\,\alpha$. □

Immediately, we deduce the following

Corollary 3.7. *Let* $S = E\left(\mathcal{ODCP}_n\right)$. *Then* $F_k(n;0) = 1$ *and* $F(n;k) = 2^{n-1}, (n \geq k \geq 1)$.

Corollary 3.8. *Let* $S=E(\mathcal{ODCP}_n)$. *Then* $F_p(n;0)=1$ *and* $F(n;p)=\sum_{k=p}^{n}\binom{k-1}{p-1}2^{n-k},(n\geq p\geq 1)$.

Finally, notice that we may recover Theorem 3.4 from either of the two corollories above. For some selected values of $F(n;k)$ and $F(n;p)$ in $E(\mathcal{ODCP}_n)$ see Table 3 and Table 4 below:

Table 3. Some selected values of $F(n;k)$ in $E(\mathcal{ODCP}_n)$.

n \ k	0	1	2	3	4	5	6	$\sum_{k=0}^{n}F(n;k)=\lvert E(\mathcal{ODCP}_n)\rvert$
0	1							1
1	1	1						2
2	1	2	2					5
3	1	4	4	4				13
4	1	8	8	8	8			33
5	1	16	16	16	16	16		81
6	1	32	32	32	32	32	32	193

Table 4. Some selected values of $F(n;p)$ in $E(\mathcal{ODCP}_n)$.

n \ p	0	1	2	3	4	5	6	$\sum_{p=0}^{n}F(n;p)=\lvert E(\mathcal{ODCP}_n)\rvert$
0	1							1
1	1	1						2
2	1	3	1					5
3	1	7	4	1				13
4	1	15	11	5	1			33
5	1	31	26	16	6	1		81
6	1	63	57	42	22	7	1	193

Remark 3.9. *The sequence* d_n, *is recorded* (*in* [19]) *as A005183.*

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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