Open Journal of Discrete Mathematics, 2021, 11, 55-60
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

A Note on *n*-Set Distance-Labelings of Graphs

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How to cite this paper: Yeh, R.K. (2021) A Note on *n*-Set Distance-Labelings of Graphs. *Open Journal of Discrete Mathematics*, 11, 55-60.
https://doi.org/10.4236/ojdm.2021.113005

Received: March 2, 2021
Accepted: June 5, 2021
Published: June 8, 2021

Abstract

This note is considered as a sequel of Yeh [1]. Here, we present a generalized (vertex) distance labeling (labeling vertices under constraints depending the on distance between vertices) of a graph. Instead of assigning a number (label) to each vertex, we assign a set of numbers to each vertex under given conditions. Some basic results are given in the first part of the note. Then we study a particular class of this type of labelings on several classes of graphs.

Keywords

Graph Distance Labeling

1. Introduction

Inspired by a channel assignment problem proposed by Roberts [2] in 1988, Griggs and Yeh [3] formulated the $L(2,1)$-labeling problem for graphs. There are considerable amounts of articles studying this labeling and its generalizations or related problems. Readers can see [4] or [5] for a survey. In this note, we like to consider a generalization of the $L(2,1)$-labeling. Let A and B be two subsets of natural numbers. Define $\|A-B\| = \min\{|a-b| : a \in A, b \in B\}$. Denote the set $[k] = \{0,1,\cdots,k\}$ and $\binom{[k]}{n}$ the collection of all n-subsets of $[k]$.

Motivated by the article [6], we propose the following labeling on a graph.

Let $G=(V,E)$ be a graph and n be a positive integer. Given non-negative integers $\delta_1 \ge \delta_2$ an $L^{(n)}(\delta_1,\delta_2)$-*labeling* is a function $f: V(G) \to \binom{[k]}{n}$ for some $k \ge 1$ such that $\|f(u) - f(v)\| \ge \delta_i$ whenever the distance between u and v in G is i, for $i = 1,2$. (The minimum value and the maximum value of $\bigcup_{v\in V(G)} f(v)$ is 0 and k, respectively.) The value k is called the span of f. The smallest k so that there is an $L^{(n)}(\delta_1,\delta_2)$-labeling f with span k, is denoted by $\lambda^{(n)}(G;\delta_1,\delta_2)$ and called the $L^{(n)}(\delta_1,\delta_2)$-labeling number of G. An $L^{(n)}(\delta_1,\delta_2)$

-labeling with span $\lambda^{(n)}(G;\delta_1,\delta_2)$ is called an optimal $L^{(n)}(\delta_1,\delta_2)$-labeling. If $n=1$ then notations $L^{(1)}$ and $\lambda^{(1)}$ will be simplified as L and λ, respectively.

Note: 1) The elements in $[k]$ are called "numbers" and $f(u)$ is called the "label" of *u*. So, a label is a set in this problem. 2) Using our notation, the labeling in [6] is the $L(\delta_1,0)$-labeling for $\delta_1 \geq 1$.

Previously, we have studied the $L^{(2)}(2,1)$-labeling problem (cf. [1]). In this note, we will first investigate properties of the $L^{(n)}(\delta_1,\delta_2)$ for $n \geq 1$. Then, we study the case of $(\delta_1,\delta_2)=(1,1)$.

2. Preliminarily

Let G be a graph and n an positive integer. Now, we construct a new graph $G^{(n)}$ by replacing each vertex v in G by n vertices v_i, $1 \leq i \leq n$ and u_i is adjacent to v_j for all i, j, in $G^{(n)}$, whenever u and v is adjacent in *G*. That is, u_i and v_j, for all i, j, induces a complete bipartite graph $K_{n,n}$. Note that $G^{(1)}=G$.

It is easy to verify that $\lambda^{(n)}(G;\delta_1,1)=\lambda\left(G^{(n)};\delta_1,1\right)$. Thus, for example, $\lambda^{(n)}(K_m;2,1)=\lambda\left(K_{n,n,\cdots,n};2,1\right)=nm+m-2$, where $m \geq 2$, by previous result on complete m-partite graph $K_{n,n,\cdots,n}$ (cf. [3]).

Next, we consider the relation between the labeling numbers for $n=1$ and $n \geq 1$. In the following, $\lambda(G;1,1)$ and $\lambda^{(n)}(G;1,1)$ are denoted by $\lambda_1(G)$ and $\lambda_1^{(n)}(G)$, respectively, for short.

Proposition 2.1. Let $n \geq 1$, $\delta_1 \geq \delta_2$ be nonnegative integers and Δ be the maximum degree of *G*. Then

1) $(n-1)(\Delta+1)+\delta_1+(\Delta-1)\delta_2 \leq \lambda^{(n)}(G;\delta_1,\delta_1)$.
2) $\lambda^{(n)}(G;\delta_1,\delta_1) \leq \lambda(G;n+\delta_1-1,n+\delta_2-1)+n-1$.

Proof.

1) A vertex u with the maximum degree Δ in a graph G is called a major vertex of *G*. By counting the numbers for the labels of a major vertex and its neighbors and numbers need to separate each label (the (δ_1,δ_2) condition), we shall have the trivial lower bound.

2) Let $\lambda(G;n+\delta_1-1,n+\delta_2-1)=k$ and f an optimal $L(n+\delta_1-1,n+\delta_2-1)$-labeling. Define sets $L_i=\{i,i+1,\cdots,i+n-1\}$, $i=0,1,\cdots,k$ and function $g_f:V(G)\to\binom{[k+n-1]}{n}$ by $g_f(u)=L_i$ whenever $f(u)=i$ for $i=0,1,\cdots,k$.

Let u and v be distinct vertices with $d_G(u,v)=j$ for $j=1,2$ in *G*. Suppose $f(u)=i$ and $f(v)=i+n+\delta_j'-1$ for $\delta_j' \geq \delta_j$ for $j=1,2$. Then
$g_f(u)=\{i,i+1,\cdots,i+n-1\}$ and
$g_f(v)=\left\{i+n+\delta_j'-1,i+n+\delta_j',\cdots,i+n+\delta_j'-1+n-1\right\}$. Hence
$\left\|g_f(u)-g_f(v)\right\|=\left(i+n+\delta_j'\right)-(i+n-1)=\delta_j' \geq \delta_j$ for $j=1,2$. Thus g_f is an $L^{(n)}(\delta_1,\delta_2)$-labeling with span $k+n-1$. Therefore
$\lambda^{(n)}(G;\delta_1,\delta_1) \leq \lambda(G;n+\delta_1-1,n+\delta_2-1)+n-1$. ■

The following is the direct consequence of Proposition 2.1 when $(\delta_1,\delta_1)=(1,1)$.

Also notice that $\lambda(G;d\delta_1,d\delta_2)=d\lambda(G;\delta_1,\delta_2)$.

Corollary 2.2. Let Δ be the maximum degree of G. Then

$$(\Delta+1)n-1\le\lambda_1^{(n)}(G)\le n\lambda_1(G)+n-1. \quad ■$$

By Corollary 2.2, we know that whenever $\lambda_1(G)=\Delta$, the lower bound and the upper bound are equal and hence $\lambda_1^{(n)}(G)=(\Delta+1)n-1$. There are several well-known classes of graphs whose λ_1 values are all Δ (see [7]). For example, tree *T*, wheel W_m (with *m* rims), the square lattice Γ_S (4-regular infinite plane graph), the hexagonal lattice Γ_H (3-regular infinite plane graph), and the triangular lattice Γ_Δ (6-regular infinite plane graph) are all with $\lambda_1=\Delta$. We summarize as follows.

Theorem 2.3.

1) $\lambda_1^{(n)}(T)=(\Delta(T)+1)n-1$.
2) $\lambda_1^{(n)}(W_m)=(m+1)n-1$.
3) $\lambda_1^{(n)}(\Gamma_S)=5n-1$.
4) $\lambda_1^{(n)}(\Gamma_H)=4n-1$.
5) $\lambda_1^{(n)}(\Gamma_\Delta)=7n-1$. ■

3. Cycles

We know that the maximum degree of a cycle C_m of order $m\ge3$ is 2. However, $\lambda_1(C_m)$ is not necessary 2. It depends on *m*. In this section, we will consider $L^{(n)}(1,1)$-labelings on cycles.

Proposition 3.1. Let C_m be a cycle of order $m\ge3$. Then $\lambda_1^{(n)}(C_m)=3n-1$ if $m\equiv0(\bmod 3)$.

Proof. Since the maximum degree of C_m is 2, the trivial lower bound is $3n-1$ by Corollary 2.2. On the other hand, we use $\{0,1,\cdots,n-1\}$, $\{n,n+1,\cdots,2n-1\}$ and $\{2n,2n+1,\cdots,3n-1\}$ consecutively to label vertices of C_m where $m\equiv0(\bmod 3)$, to obtain an $L^{(n)}(1,1)$-labeling of C_m with span $3n-1$. Thus, we have the exact value of $\lambda_1^{(n)}(C_m)$ in this case. ■

Lemma 3.2. Let C_m be a cycle of order *m* where $m\not\equiv0(\bmod 3)$. Then $\lambda_1^{(n)}(C_m)\ge3n$.

Proof. Let $V(C_m)=\{v_1,v_2,\cdots,v_m\}$ where v_i is adjacent to v_{i+1} for $i=1,2,\cdots,m$ where $v_{m+1}=v_1$. Suppose $\lambda_1^{(n)}(C_m)\le3n-1$. Let *f* be an $L^{(n)}(1,1)$-labeling with span $3n-1$. Let $f(v_1)=A, f(v_2)=B$ and $f(v_3)=C$. Since, by definition, $f(v_1), f(v_2)$ and $f(v_3)$ are distinct, that is, $|A\cup B\cup C|=3n$ and $A\cup B\cup C=[3n-1]$. Now, $f(v_4)\cap(B\cup C)=\varnothing$ and $f(v_4)\subseteq[3n-1]$. Hence $f(v_4)=A$. Consider $f(v_5)$. Again, we have $f(v_5)\cap(A\cup C)=\varnothing$ and $f(v_5)\subseteq[3n-1]$. Hence $f(v_5)=B$. In general, we have 1) $f(v_i)=A$ if $i\equiv1(\bmod 3)$, 2) $f(v_i)=B$ if $i\equiv2(\bmod 3)$ and 3) $f(v_i)=C$ if $i\equiv0(\bmod 3)$, for $i=1,2,\cdots,m$.

If $m\equiv1(\bmod 3)$ then $f(v_m)=A$. But v_m is adjacent to v_1, where $f(v_1)=A$. This violates the condition on adjacent vertices. If $m\equiv2(\bmod 3)$ then $f(v_m)=B=f(v_2)$ while the distance between v_m and v_2 is 2. Again, this violates the condition on distance 2 vertices. We have a contradiction on each case. Therefore, $\lambda_1^{(n)}(C_m) \ge 3n$ for $m \not\equiv 0(\bmod 3)$. ■

Proposition 3.3. If 1) $m \equiv 1(\bmod 3)$ and $m \ge 3n+1$ or 2) $m \equiv 2(\bmod 3)$ and $m \ge 6n+2$ then $\lambda_1^{(n)}(C_m) = 3n$.

Proof. Let $V(C_m) = \{v_1, v_2, \cdots, v_m\}$.

1) Suppose $m = 3n+1$ and Define $A_0 = \{0,1,\cdots,n-1\}$, $A_1 = \{n, n+1, \cdots, 2n-1\}$, $A_2 = \{2n, 2n+1, \cdots, 3n-1\}$ and $A_3 = \{3n, 0, \cdots, n-2\}$. Denote $X - i(\bmod k)$ to be that set $\{x - i(\bmod k) : x \in X\}$. Then we use $A_1, A_2, A_3, A_1 - 1, A_2 - 1, A_3 - 1$, $A_1 - 2, A_2 - 2, A_3 - 2, \cdots, A_1 - (n-1), A_2 - (n-1), A_3 - (n-1)$ to label $v_1, v_2, \cdots, v_{3n}$. The last vertex v_{3n+1} is labeled by A_0. We see that this is an $L^{(n)}(1,1)$ -labeling with span 3*n* of C_m.

Suppose $m > 3n+1$. Then we label first $3n+1$ vertices as we did above. And then we repeatedly use A_0, A_1 and A_2 to label remaining vertices.

2) First consider $m = 6n+2$. We use the sequence presented in (1) for $m = 3n+1$ twice to label vertices of C_{6n+2}. Obviously, it is still an $L^{(n)}(1,1)$ -labeling for C_{6n+2} with span 3*n*.

For $m > 6n+2$, we label the first $6n+1$ vertices (namely, $v_1, v_2, \cdots, v_{6n+1}$) using the same sequence as above and then repeat using A_0, A_1 and A_2 to label remaining vertices. Thus $\lambda_1^{(n)}(C_m) \le 3n$ in each case. On the other hand, by Lemma 3.2, we have the equality. ■

Lemma 3.4. Let *G* be a diameter two graph with order *p*. Then $\lambda_1^{(n)}(G) = np - 1$.

Proof. Since *G* is a diameter two graph, every vertex must receive distinct label. Thus, we need at least *np* numbers, *i.e.*, $\lambda_1^{(n)}(G) = np - 1$. On the other hand, we can use $\{in, in+1, \cdots, in+n-1\}$ for $i = 0,1,\cdots,p-1$ to label vertices of *G* in any order. Hence $\lambda_1^{(n)}(G) \le np - 1$. ■

Corollary 3.5.

$$\lambda_1^{(n)}(C_m) = \begin{cases} 5 & m \equiv 0(\bmod 3), \\ 6 & m \equiv 1(\bmod 3), m \ge 7 \text{ or } m \equiv 2(\bmod 3), m \ge 14, \\ 7 & m = 4, 8, 11, \\ 9 & m = 5. \end{cases}$$

Proof. Let $V(C_m) = \{v_1, v_2, \cdots, v_m\}$ where v_i is adjacent to v_{i+1} for $i = 1, 2, \cdots, m$ where $v_{m+1} = v_1$.

Claim 1. $\lambda_1^{(2)}(C_8) = 7$.

Suppose $\lambda_1^{(2)}(C_8) \le 6$. Let *f* be an $L^{(2)}(1,1)$ -labeling with span 6. Since $m = 8$, there must have three consecutive vertices, say v_1, v_2 and v_3, be labeled without using 6; and let $6 \in f(v_4)$. Also let $f(v_1) = \{a_1, a_2\}$, $f(v_2) = \{b_1, b_2\}$ and $f(v_3) = \{c_1, c_2\}$. Then $f(v_4) = \{a, 6\}$ where $a = a_1$ or a_2. Suppose $a = a_1$. Hence $f(v_5) \subseteq \{b_1, b_2, a_2\}$ and $f(v_8) \subset \{c_1, c_2, 6\}$. Since $(f(v_6) \cup f(v_7)) \cap (f(v_5) \cup f(v_8)) = \varnothing$, we left only 3 numbers for $f(v_6) \cup f(v_7)$, (that is two from $\{b_1, b_2, a_2, c_1, c_2, 6\} \setminus (f(v_5) \cup f(v_8))$ plus a_1). It is not enough. The case for $a = a_2$ is similar.

Thus, $\lambda_1^{(2)}(C_8) \ge 7$. On the other hand, we can use $\{0,1\},\{2,3\},\{4,5\},\{6,7\}$ consecutively to label $v_1, v_2, \cdots, v_8$ to obtain an $L^{(2)}(1,1)$-labeling with sapn 7. Hence the claim holds.

Claim 2. $\lambda_1^{(2)}(C_{11}) = 7$.

Let *f* be an $L^{(2)}(1,1)$-labeling with span 6. Similar to Claim 1, we may assume that $f(v_1) = \{a_1, a_2\}$, $f(v_2) = \{b_1, b_2\}$, $f(v_3) = \{c_1, c_2\}$ and $f(v_4) = \{6, a\}$ where $a \in \{a_1, a_2\}$ and $6 \notin \{a_1, a_2, b_1, b_2, c_1, c_2\}$.

Again, we have $f(v_{11}) \subset \{c_1, c_2, 6\}$. Consider the following cases:

1) $f(v_8) \subset \{c_1, c_2, 6\}$. Since $f(v_4) = \{6, a\}$ (as indicated above), the discussion on $f(v_4), f(v_5), f(v_6), f(v_7)$ and $f(v_8)$ is the same as Claim 1.

2) $f(v_8) \subset \{a_1, a_2, b_1, b_2, 6\}$. Since $f(v_1) = \{a_1, a_2\}$, $f(v_{10}) \cap \{a_1, a_2\} = \varnothing$. Let $c \in \{c_1, c_2, 6\} \setminus f(v_{11})$. Hence $f(v_9) \subset \{a_1, a_2, c\}$. So $f(v_8) \subset \{b_1, b_2, c\}$. Thus, there is only one number left available for $f(v_{10})$. This is a contradiction.

3) Suppose $f(v_8)$ consists of one number of $f(v_1)$ and one number of $f(v_3)$. Without loss of generality, say $f(v_8) = \{a_1, c_1\}$. Then $f(v_5) \subset \{b_1, b_2, a'\}$ where $a' = a_2$ if $a = a_1$ and vice versa. Then there only three numbers available for $f(v_6) \cup f(v_7)$ and they are one from $\{b_1, b_2, a'\} \setminus f(v_5)$, c_1 and 6. That is not enough.

Therefore, $\lambda_1^{(2)}(C_{11}) \ge 7$. On the other hand, we can use $\{0,1\},\{2,3\},\{4,5\},\{6,7\}$ consecutively to label $v_1, v_2, \cdots, v_{11}$ to obtain an $L^{(2)}(1,1)$-labeling with span 7. Hence the claim holds. Finally, we have

1) $m \equiv 0 \pmod 3$.

By Proposition 3.2, $\lambda_1^{(2)}(C_m) = 5$.

2) $m \equiv 1 \pmod 3$.

By Proposition 3.3, $\lambda_1^{(2)}(C_m) = 6$ if $m \ge 7$. Since C_4 is diameter 2 graph, by Lemma 3.4, $\lambda_1^{(2)}(C_4) = 7$.

3) $m \equiv 2 \pmod 3$.

By Proposition 3.3, $\lambda_1^{(2)}(C_m) = 6$ if $m \ge 14$. Case for $m = 11$ and 8 are obtained by Claim 1 and Claim 2. Since C_5 is also a diameter 2 graph, by Lemma 3.4, $\lambda_1^{(2)}(C_5) = 9$. ■

4. Concluding Remark

We have obtained values of $\lambda_1^{(2)}(C_m)$ for all *m* and $\lambda_1^{(n)}(C_m)$ for some *m* where $n \ge 3$. Otherwise, the labeling numbers are still unknown. It is known that $\lambda_1(C_m) = 4$ if $m \ne 0 \pmod 3$ (cf. [8]). Hence an upper bound is $4n-1$ in this case. On the other hand, the lower bound we have in Lemma 3.2 is $3n$. Thus, there is still a gap between $3n$ and $4n-1$ for $n > 1$.

Acknowledgements

The author would like to thank the referee for valuable editorial suggestions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Yeh, R.K. (2019) Pair L(2,1)-Labelings of Infinite Graphs. *Discussiones Mathematicae Graph Theory*, **39**, 257-269. https://doi.org/10.7151/dmgt.2077

[2] Roberts, F.S. (1988) Private Communication with Griggs J. R.

[3] Griggs, J.R. and Yeh, R.K. (1992) Labelling Graphs with a Condition at Distance 2. *SIAM Journal on Discrete Mathematics*, **5**, 586-595. https://doi.org/10.1137/0405048

[4] Calamoneri, T. (2011) The L(h,k)-Labeling Problems: A Update Survey and Annotated Bibliography. *The Computer Journal*, **54**, 1344-1371. https://doi.org/10.1093/comjnl/bxr037

[5] Yeh, R.K. (2006) A Survey on Labeling Graphs with a Condition at Distance Two. *Discrete Mathematics*, **306**, 1217-1231. https://doi.org/10.1016/j.disc.2005.11.029

[6] Füredi, Z., Griggs, J.R. and Kleitman, D.J. (1989) Pair Labellings with Given Distance. *SIAM Journal on Discrete Mathematics*, **2**, 491-499. https://doi.org/10.1137/0402044

[7] Griggs, J.R. and Jin, X.T. (2008) Real Number Channel Assignments for Lattices. *SIAM Journal on Discrete Mathematics*, **22**, 996-1021. https://doi.org/10.1137/060650982

[8] Liu, D.D.-F. and Yeh, R.K. (1997)On Distance Two Labelings of Graphs. *Ars Combinatoria*, **47**, 13-22.