Open Journal of Discrete Mathematics, 2021, 11, 13-30
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

Tiling a Plane with Semi-Regular Equilateral Polygons with 2m-Sides

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How to cite this paper: Stojanovic, N. (2021) Tiling a Plane with Semi-Regular Equilateral Polygons with 2m-Sides. *Open Journal of Discrete Mathematics*, **11**, 13-30.
https://doi.org/10.4236/ojdm.2021.111002

Received: September 26, 2020
Accepted: January 5, 2021
Published: January 8, 2021

Abstract

In this paper, tiling a plane with equilateral semi-regular convex polygons is considered, and, that is, tiling with equilateral polygons of the same type. Tiling a plane with semi-regular polygons depends not only on the type of a semi-regular polygon, but also on its interior angles that join at a node. In relation to the interior angles, semi-regular equilateral polygons with the same or different interior angles can be joined in the nodes. Here, we shall first consider tiling a plane with semi-regular equilateral polygons with 2m-sides. The analysis is performed by determining the set of all integer solutions of the corresponding Diophantine equation in the form of $t\cdot\alpha+s\cdot\beta=2\pi$, where *t*, *s* are the non-negative integers which are not equal to zero at the same time, and α, β are the interior angles of a semi-regular equilateral polygon from the characteristic angle. It is shown that of all semi-regular equilateral polygons with 2m-sides, a plane can be tiled only with the semi-regular equilateral quadrilaterals and semi-regular equilateral hexagons. Then, the problem of tiling a plane with semi-regular equilateral quadrilaterals is analyzed in detail, and then the one with semi-regular equilateral hexagons. For these semi-regular polygons, all possible solutions of the corresponding Diophantine equations were analyzed and all nodes were determined, and then the problem for different values of characteristic elements was observed. For some of the observed cases of tiling a plane with these semi-regular polygons, some graphical presentations of tiling constructions are also given.

Keywords

Tiling a Plane, Semi-Regular Plane Tiling, Diophantine Equations

1. Introduction

The problem of tiling a plane is an ancient one, which even the mathematicians of ancient Egypt, Greece, Persia, China and other old civilizations were familiar with. Tiling comes down to dividing a plane into polygons that would completely cover it without having any overlaps or gaps, following certain regularity, depending on the type, shape and arrangement of the polygons [1]. So, with tiling, the goal is to divide the plane into polygons that would only have common sides and vertices. Then, for the polygons that have one common side it is said that they are the adjacent polygons, and the point of the plane in which the vertices of the adjacent polygons meet is named the node of that partition of the plane. The node is called regular if all angles of the polygons meeting at it are equal. The two nodes are considered to be equal if the number of angles that meet at it is the same. The problem of tiling comes down to determining all possible divisions of a plane with the polygons:

1) The division of a plane with regular polygons, or when all the polygons and all the nodes are mutually equal. Such tiling is called a regular one.

2) The division of a plane in a way that several types of regular polygons meet at a node. Such tiling a plane is called an Archimedes one, or a semi-regular one.

3) Special cases of tiling a plane.

The first two cases have been largely researched, and there can be found more about them in [1] and a catalog of tiling can be seen in [2] [3] [4].

We are interested in special cases, and that is tiling a plane with semi-regular equilateral polygons, *i.e.* when equilateral semi-regular polygons meet at a node. Prior to the analysis of the problem of tiling a plane with semi-regular polygons, let us mention some basic points of view and theorems that are valid for tiling a plane with regular polygons.

Theorem 1. *The only proper tiling is possible with equilateral triangles, squares and regular hexagons, and in such a way that six, four and three of them meet at a single node.*

Proof: Since the sum of angles at each node is 2π and the value of the interior angles of regular polygon $\frac{(n-2)\pi}{n}$, if $k \in \mathbb{N}$ regular polygons meet at the vertex, then it follows that: $k \cdot \frac{(n-2)\pi}{n} = 2\pi$.

From here, after rearranging and solving the equation, we get the following equation $k(n-2) = 2n$ from which, after solving by k, we find that $k = 2 + \frac{4}{n-2}$.

From the condition $(n-2) \mid 4$ it follows that $n \in \{3, 4, 6\}$. On the basis of this, it is found that for $n = 3$, the value is $k = 6$, and for $n = 4$ it is $k = 3$, while for $n = 6$, it is $k = 3$ [1].

2. Semi-Regular Equilateral Polygons and Formulation of a Problem

1) Polygon $\mathcal{P}_n \equiv A_1 A_2 \cdots A_n$ or closed polygonal line is the union along

$A_1A_2, A_2A_3, \cdots, A_nA_{n+1}$ and is write shortly $\mathcal{P}_n \equiv \bigcup_{j=1}^{n} A_jA_{j+1}$, $(n+1 \equiv 1(\bmod n))$. Points A_j are vertices, and lines A_jA_{j+1} are sides of polygon $\mathcal{P}_n$. The angles on the inside of a polygon formed by each pair of adjacent sides are angles of the polygon.

2) Given polygon $\mathcal{P}_n$ with vertices A_j $j = 1, 2, \cdots, n$, $(n+1 \equiv 1(\bmod n))$, lines of which A_jA_i polygonal diagonals if indices are not consecutive natural numbers, that is $j \neq i$. We can draw $n-3$ diagonals from each vertex of the polygon with n number of vertices.

3) Exterior angle of the polygon $\mathcal{P}_n$ with vertex A_j is the angle $\angle A_{v,j}$ with one side A_jA_{j+1}, and vertex A_j, and the other one is extension of the side $A_{j-1}A_j$ through vertex A_j.

4) The interior angle of the polygon $\mathcal{P}_n$ with vertex A_j is the angle $\angle A_{u,j}, j = 1, 2, \cdots, n$ for which $\angle A_{v,j} + \angle A_{u,j} = \pi$. That is the angle with one $A_{j-1}A_j$, side, and the other side A_jA_{j+1}. Sum of all interior angles of the polygon is defined by equation

$\sum_{j=1}^{n} \angle A_{u,j} = (n-2k) \cdot \pi; n \in \mathbb{N}, k \in \mathbb{Z}$. In which k is number of turning around the polygon in certain direction.

5) A regular polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). Regular polygon with n sides of b length is marked as $\mathcal{P}_n^b$. The formula for interior angles γ of the regular polygon $\mathcal{P}_n^b$ with n sides is $\gamma = \frac{(n-2)\pi}{n}$. A non-convex regular polygon is a regular star polygon. For more about polygons in [4] [5] [6].

6) Polygon that is either equiangular or equilateral is called *semi-regular* polygon. Equilateral polygon with different angles within those sides are called *equilateral semi regular* polygons, whereas polygons that are *equiangular* and with sides different in length are called *equiangular sem regular* polygons. For more about polygons in [1] [2] [3].

7) If we construct a polygon $\mathcal{P}_k$ with $m = k-1$ sides, $k \geq 3$, $k \in \mathbb{N}$ with vertices $B_i, i = 1, 2, \cdots, k$ over each side of the convex polygon $\mathcal{P}_n$, $n \geq 2$, $n \in \mathbb{N}$ with vertices $A_j, j = 1, 2, \cdots, n; n+1 \equiv 1(\bmod n)$, that is $A_j = B_1, A_{j+1} = B_k$, we get new polygon with $N = m \cdot n$ (**Figure 1**) marked as $\mathcal{P}_N$.

Here are the most important elements and terms related to constructed polygons:

- Polygon $\mathcal{P}_k$ with vertices $B_1B_2, \cdots, B_{k-1}B_k$, $B_1 = A_j, B_k = A_{j+1}$ constructed over each side $A_jA_{j+1}, j = 1, 2, \cdots, n$ of polygon $\mathcal{P}_n$ with which it has one side in common is called edge polygon for polygon $\mathcal{P}_N$.
- $A_jB_2, B_2B_3, \cdots, B_{k-1}A_{j+1}$ are the sides of polygon $\mathcal{P}_k$.
- $A_jB_3, A_jB_4, \cdots, A_jB_{k-1}$ are diagonals $d_i, i = 1, 2, \cdots, k-1$ of the polygon $\mathcal{P}_k^a$ drawn from the top A_j and that implies $d_{i-2} = A_jA_{j+1} = b$.
- Angles $\angle B_{u,i}$ are interior angles of vertices B_i of the polygon $\mathcal{P}_N$. and are denoted as β_i. Interior angles $\angle A_{u,j}$ of the polygon of the vertices A_j are denoted as α.

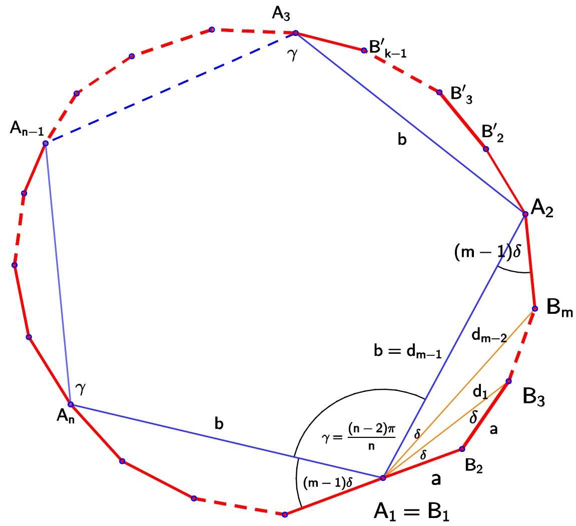

Figure 1. Convex semi regular polygon $\mathcal{P}_N$ with $N=(k-1)n=mn$ sides constructed above the regular polygon $\mathcal{P}_n^b$.

- Polygon $\mathcal{P}_k$ of the side *a* constructed over the side *b* of the polygon $\mathcal{P}_n$ is isosceles, with $m=k-1$ equal sides, is denoted as $\mathcal{P}_k^a$.
- $\delta=\angle(d_{i-1},d_i), i=1,2,\cdots,m-1$ denotes the angle between its two consecutive diagonals drawn from the vertices $A_j, j=1,2,\cdots,n$ for which it is true $d_0=a, d_{m-1}=b$.
- Regular polygon $\mathcal{P}_n^b$ is called corresponding regular polygons of the semi-regular polygon $\mathcal{P}_N^{a,\delta}$.
- If the isosceles polygon $\mathcal{P}_k^a$ is constructed over each side of the *b* regular polygon $\mathcal{P}_n^b$ with *n* sides, then the constructed polygon with $N=mn$ of equal sides is called equilateral *semi-regular polygon* which is denoted as $\mathcal{P}_N^{a,\delta}$.
- Interior angles of a semi-regular polygon at odd vertices are marked with α, and those at even vertices are marked with β (**Figure 1**).
- To a semi-regular equilateral polygon $\mathcal{P}_N \equiv A_1A_2\cdots A_N$ with $N=2\cdot n, n\in\mathbb{N}$ with equal sides, there can be "inscribed" regular *n*-side polygons: by joining odd vertices, $\mathcal{P}_n^1 \equiv A_1A_3A_5\cdots A_{2n-3}A_{2n-1}$ or even vertices $\mathcal{P}_n^2 \equiv A_2A_3A_4\cdots A_{2n-2}A_{2n}$. [5].

To analyze the metric properties of regular polygons, it is sufficient for us to know one basic element, the length of a side, while for the semi-regular polygons this is not sufficient [5] Therefore, in addition to side a of a semi-regular polygon, for the analysis of the metric properties we will use another element of it, and that is the angle between side *a* of the semi-regular polygon and side *b* of its "inscribed" regular polygon, which we mark with *δ*, *i.e.* $\delta=\angle(a,b)$ (**Figure 2**) [5] [6] [7].

To show that a semi-regular equilateral 2*n*-side polygon is given by side *a* and angle δ we write: $\mathcal{P}_{2n}^{a,\delta}$.

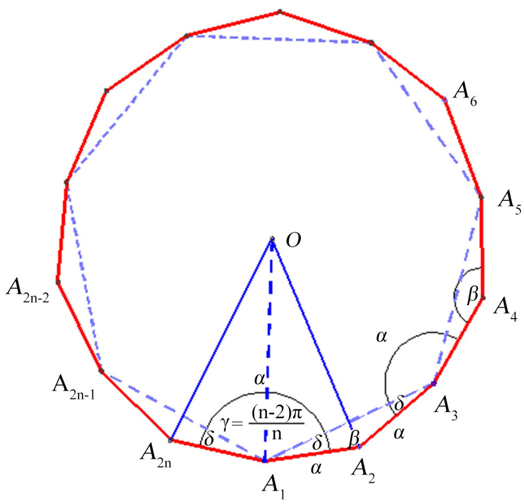

Figure 2. Basic elements of equilateral semi-regular polygon $\mathcal{P}_{2n}^{(a,\delta)}, n \ge 2 \in \mathbb{N}$ of side α and angle δ.

If $\gamma = \frac{(n-2)\pi}{n}, n \ge 2$ is the interior angle of the "inscribed" regular polygon $\mathcal{P}_n^b$, then $\alpha = \gamma + 2\delta = \frac{(n-2)\pi}{n} + 2\delta$ gives interior angles at odd vertices, and $\beta = \pi - 2\delta$ gives the ones at even vertices of the semi-regular polygon $\mathcal{P}_{2n}^a$ of a side *a*, where $\delta = \angle(b,a)$ marks the angle between the sides of polygons $\mathcal{P}_n^1$ and $\mathcal{P}_n^1$ (Figure 2). Here, we consider that a regular polygon with $n = 2$ sides (segment) is "inscribed" to a semi-regular equilateral quadrilateral (rhombus) [5] [6] [7] [8].

Let us mark semi-regular polygon $\mathcal{P}_N$ with $N = n \cdot m$ sides as determined with the characteristic elements n, m, δ and interior angles α, β, with $\mathcal{P}_N \vdots \begin{pmatrix} n & m \\ & \delta \end{pmatrix}_\beta^\alpha$.

Here are some more results on the interior angles of semi-regular polygon $\mathcal{P}_N$, with $N = n \cdot m; n, m \ge 2 \in \mathbb{N}$ sides, which we need, and the proof of which can be seen in papers [5].

Theorem 2. A semi-regular equilateral convex polygon $\mathcal{P}_N$, with $N = n \cdot m; n, m \ge 2 \in \mathbb{N}$ sides and characteristic angle $\delta = \angle(a,b)$ has the following:

$-n$ interior angles along the vertices of the "inscribed" regular polygon $\mathcal{P}_n$, with side *b*, all equal to angle α and the following applies

$$\alpha = \gamma + 2\delta = \frac{(n-2)\pi}{n} + 2(m-1)\delta \quad (1)$$

$(m-1) \cdot n$ interior angles, along the vertices of isosceles polygons $\mathcal{P}_k$, with *m* equal arms constructed over each side of a regular polygon $\mathcal{P}_n$, as a common side. These interior angles are equal to angle β and the following applies:

$$\beta = \pi - 2\delta, m, n, k \in \mathbb{N}, m = k - 1 \quad (2)$$

The dependence of convexity of a semi-regular equilateral polygon $\mathcal{P}_N$, with $N = n \cdot m; n, m \geq 2 \in \mathbb{N}$ sides from the value of angle δ is expressed by the following theorem:

Theorem 3. Semi-regular, equilateral polygon $\mathcal{P}_N$ with $N = n \cdot m$ sides is convex if the following is valid for angle δ:

$$\delta \in \left\langle 0, \frac{\pi}{N-n} \right\rangle; \quad \delta \neq \frac{\pi}{N}, \tag{3}$$

Note that for $\delta = \frac{\pi}{N}$, a convex semi-regular polygon $\mathcal{P}_N$ becomes regular, and since this is not the subject of our research here, these values of angle δ are excluded from further consideration. The proof of this theorem can be seen in [6] [7] [8].

3. Main Results

3.1. Tiling a Plane with Semi-Regular Polygons of the Same Type

Tiling a plane with equilateral convex semi-regular polygons differs from tiling a plane with regular ones, and it belongs to a special group of tiling. Based on the characteristics of the semi-regular equilateral polygons, the following types of tiling a plane with semi-regular polygons can be differentiated:

A) Tiling a plane with semi-regular polygons when the equal number of semi-regular polygons of the same type meet at each node;

B) Tiling a plane with semi-regular polygons when semi-regular equilateral polygons of different types and equal sides meet at one node;

C) Tiling a plane with semi-regular polygons when semi-regular equilateral polygons of different types and different sides meet at one node.

In this paper we shall consider the cases of tiling a plane as stated under section (A).

Let us consider tiling a plane if tiling is performed with semi-regular equilateral polygons.

If it is possible to perform tiling with one type of semi-regular polygons, than due to the existence of the two types of the interior angles: the angles along the vertices of the inscribed regular polygon, equal to angle

$$\alpha = \frac{(n-2)\pi}{n} + 2(m-1)\delta,$$

and angles along the vertices of the edge polygons, equal to angle $\beta = \pi - 2\delta$, the following types of nodes can be differentiated [9]:

1) Nodes at which the vertices meet, to which the interior angles equal to the angle α correspond;

2) Nodes at which the vertices meet, to which the interior angles equal to angle β correspond;

3) Nodes at which the vertices meet, to which the interior angles equal to angles α and β correspond.

Let us assume that it is possible to perform tiling with semi-regular equilateral convex polygons of the same type, constructed in the manner as described above, with the characteristic elements n, m, δ and interior angles α, β.

Then in that case there is any non-negative integers *t*, *s* which are not simultaneously equal to zero, such that at one node, there are *t* vertices meeting, to which the interior angles equal to angle α correspond, and/or *s* vertices, to which the interior angles equal to angle β correspond.

Based on the value of interior angles α and β, and the fact that in that case, the sum of angles at each node is equal to 2π, the stated conditions may be written down in the form of the following equation

$$t \cdot \alpha + s \cdot \beta = 2\pi \Leftrightarrow t \cdot \left[\frac{(n-2)\pi}{n} + 2(m-1)\delta \right] + s \cdot (\pi - 2\delta) = 2\pi \tag{4}$$

In this way, the problem of tiling a plane with one type of semi-regular convex equilateral polygons can be described as the solution of Equations (3.1). A semi-regular equilateral polygon is convex for $\delta \in \left\langle 0; \frac{\pi}{N-n} \right\rangle, \delta \neq \frac{\pi}{N}$ when the polygon is regular.

Based on this, for different values of $n, m \geq 2 \in \mathbb{N}$ and δ we also have different Diophantine equations. Here we analyze the case when characteristic angle $\delta = \frac{\pi}{l \cdot (N-n)}$ and l is any natural number greater than one, $l > 1 \in \mathbb{N}$. (A case when $l = 2^m$ was dealt with in paper [9]). For this value of angle δ, the Equation (4) has the following form:

$$t \cdot \left[\frac{(n-2)\pi}{n} + \frac{2(m-1)\pi}{l \cdot (N-n)} \right] + s \cdot \left(\pi - \frac{2\pi}{l \cdot (N-n)} \right) = 2\pi$$

$$\Leftrightarrow t \cdot \left[\frac{n-2}{n} + \frac{2}{l \cdot n} \right] + s \cdot \left(1 - \frac{2}{l \cdot n(m-1)} \right) = 2$$

$$\Leftrightarrow t \cdot \left(\frac{l \cdot (n-2) + 2}{l \cdot n} \right) + s \cdot \left(\frac{l \cdot n(m-1) - 2}{l \cdot n \cdot (m-1)} \right) = 2.$$

After rearranging and solving, the last equation can be written in the following form:

$$(m-1)(l \cdot n - 2l + 2) \cdot t + (l \cdot n(m-1) - 2) \cdot s = 2l \cdot n(m-1) \tag{5}$$

representing the Diophantine equation in which the unknown parameters are t, s.

Should we denote the coefficients respectively with:

$$A = (m-1)(l \cdot n - 2l + 2), B = l \cdot n(m-1) - 2, C = 2l \cdot n \cdot (m-1) \tag{6}$$

the equation can be written in a simpler form $At + Bs = C$.

For different values of n, m, l Equation (5) has different forms. Let us analyze the possibilities of tiling a plane with semi-regular polygons with $N = 2 \cdot m, m \geq 2$, *i.e.* with those with which the inscribed regular polygon has $n = 2$ sides closed segment.

The Diophantine equation for tiling a plane with semi-regular equilateral polygons with $2\cdot m$ sides reads

$$(m-1)\cdot t+\left(l\cdot(m-1)-1\right)\cdot s=2l\cdot(m-1) \tag{7}$$

where $l>1, m\geq 2; m,l\in\mathbb{N}$ and unknowns t,s are the non-negative integers, *i.e.* $t,s\in\mathbb{N}_0$ The following theorem applies:

Theorem 4. Let $\mathcal{P}_N$ be an equilateral semi-regular polygon with $N=2m, m\geq 2\in\mathbb{N}$ sides and let $\delta=\dfrac{\pi}{2\cdot l\cdot(m-1)}, l>1, l\in\mathbb{N}$ be its characteristic angle. Tiling a plane with semi-regular equilateral polygon $\mathcal{P}_{2m}$ can be performed only if $m=2$ or $m=3$, that is, the Diophantine Equation (7) has a non-negative integer solutions only for $m=2$ or $m=3$.

Proof: To show the theorem's claim, let us note that the problem of tiling a plane with a semi-regular equilateral polygon with $2m$ sides is equivalent to determining the set of all the solutions of the corresponding Diophantine Equation (7). Any pair (t,s) of the non-negative integers that are not simultaneously equal to zero and that meet the given equation represent its solution.

Let us show that the Diophantine Equation (7) has solutions only when $m=2$ or $m=3$. Further, note that for a given value of angle

$\delta=\dfrac{\pi}{2\cdot l\cdot(m-1)}, l>1, m\geq 2; m,l\in\mathbb{N}$ for the interior angles of a semi-regular equilateral polygon equal to angle α the following relation applies:

$\alpha=2(m-1)\cdot\dfrac{\pi}{2\cdot l\cdot(m-1)}=\dfrac{\pi}{l}$, and for angles equal to angle β the following applies: $\beta=\pi-2\delta=\left(1-\dfrac{1}{l\cdot(m-1)}\right)$.

Note that the solution of Equation (7) represents a pair of non-negative integers (t, s) that depend on the value of natural number $m\geq 2\in\mathbb{N}$ and that the following cases can be distinguished:

Case 1: If $t\neq 0, s=0$, Equation (7) has the following form:

$$(m-1)\cdot t=2l\cdot(m-1) \tag{8}$$

From here we find that one solution, for $m\neq 1$ of Equation (7) is a pair $(t,s)=(2l,0)$, which does not depend on the choice of natural numbers $m\geq 2, l>1\in\mathbb{N}$.

Case 2: If $t=0, s\neq 0$ the Equation (7) reads: $\left(l\cdot(m-1)-1\right)\cdot s=2l\cdot(m-1)$

From here we find that

$$s=\frac{2l\cdot(m-1)}{l\cdot(m-1)-1}=2+\frac{2}{l\cdot(m-1)-1}.$$

And from here, we find that s is a natural number, if and only if $\dfrac{2}{l\cdot(m-1)-1}$ is a natural number, or if $\left[l\cdot(m-1)-1\right]|2$.

If it were that: $\left[l\cdot(m-1)-1\right]|2$, then one of these cases could arise:

1) $l\cdot(m-1)-1=\pm 1$.

2) $l\cdot(m-1)-1=2$ because $s\neq 0$.

Let us consider both cases.

1) If it were that: $l\cdot(m-1)-1=-1$, then it would be that $l\cdot(m-1)=0$, which is impossible, because by assumption $m\geq 2$ and $l>1$.

If it where that: $l\cdot(m-1)-1=1$ then it would be that $l\cdot(m-1)=2$ were from it would follow that $l=\frac{2}{m-1}$. Since $l\in\mathbb{N}$ then it means that $m-1=1$ or $m-1=2$, whence we get that $l\in\mathbb{N}$ if $m=2$ or $m=3$.

- If $m=2$, then $l=2$, and then it is that $s=4$, $t=0$. A pair of (0, 4) is another solution of Diophantine Equation (7).
- if $m=3$, then $l=1$, which is contrary to the assumption that $l>1$, and, in this case, the Equation (7) has no solution.

2) If the case is that $l\cdot(m-1)-1=2$, then $l=\frac{3}{m-1}$. Hence, it follows that *l* is a natural number greater than 1, if $m-1=1$ or $m-1=3$, that is, if $m=2$ or $m=4$.

- If it were that $m=2$, then it would be that $l=3$. Based on that, it would be that $s=3$, so pair $(t,s)=(0,3)$ is another solution of the corresponding Diophantine Equation (7)
- and the case when $m=4$ is not possible, because then it would be that $l=1$, which is contrary to the assumption that $l>1$.

Case 3: Let $t,s\neq 0, t,s\in\mathbb{N}$. Let us transform Equation (7) as follows:

$(m-1)\cdot t+(l\cdot(m-1)-1)\cdot s=2l\cdot(m-1)$

$$s=\frac{2l\cdot(m-1)-(m-1)\cdot t}{l\cdot(m-1)-1}=2+\frac{2-(m-1)\cdot t}{l\cdot(m-1)-1} \tag{9}$$

From the last equation $s=2+\frac{2-(m-1)\cdot t}{l\cdot(m-1)-1}$ it follows that *s* is a natural number, if one of the cases occur:

1) $2-(m-1)\cdot t=0$ or

2) $l\cdot(m-1)-1\,|\,2-(m-1)\cdot t$.

Let us consider each of the cases.

1) If $2-(m-1)\cdot t=0$, then $s=2$ and $t=\frac{2}{m-1}$ is a natural number only for $m=2$ and $m=3$.

In case when $m=2$, then $t=2$, so the pair of $(t,s)=(2,2)$ is also the solution of Equation (7).

When $m=3$ then $t=1$, and $s=2$, thus another pair of non-negative integers $(t,s)=(1,2)$ has been determined $(t,s)=(1,2)$ which is the solution of Equation (7).

2) Let us also consider case 2). Suppose that $s\in\mathbb{N}_0$ is defined by Equation (8) and that $\frac{2-(m-1)\cdot t}{l\cdot(m-1)-1}=p$. Then from the assumption that $s\in\mathbb{N}_0$, it follows that $s=p+2\geq 0$ *i.e.* then the integer is $p\geq -2\in\mathbb{Z}$.

In relation to the constraint of the value of integer *p*, let us observe the following cases:

1) $p = 0$, then $2-(m-1)\cdot t = 0$, We have already considered this case;

2) $p = -1$ then from equation $\frac{2-(m-1)\cdot t}{l\cdot(m-1)-1} = -1$ it follows that $2-(m-1)\cdot t = 1 - l\cdot(m-1)$, and from this we get that $t = \frac{1}{m-1} + l$. Since from the assumption that $l \in \mathbb{N}$, it follows that it must be that $m = 2$.

For this value of number, it is $t = l+1$ and $s = 1$. So the pair of $(t,s) = (l+1,1)$ is the solution of Diophantine Equation (7) for every natural number $l > 1$.

1) If $p \in \mathbb{N}$, then from equation $\frac{2-(m-1)\cdot t}{l\cdot(m-1)-1} = p$ we get that $t = \frac{2}{m-1} - \frac{p\cdot[l\cdot(m-1)-1]}{m-1}$, or $t = \frac{2}{m-1} + \frac{p}{m-1} - p\cdot l$. It follows from this equation that *t* is a natural number, if and only if $\frac{2}{m-1}$ and $\frac{p}{m-1}$ are natural numbers, that is, if $m-1 \mid 2$ and $(m-1) \mid p$.

From the requirement $m-1 \mid 2$, it follows that it must be that $m = 2$ or $m = 3$.

For $m = 2$ it applies that $(m-1) \mid p$ and then it is $t = 2 + p - p\cdot l = 2 + p(1-l)$ and $s = 2 + p$. Furthermore, from the requirement that $p = \frac{2-t}{l-1} > 0$, it follows that it must be that: $2 - t > 0$, because $l - 1 > 0$ by assumption. It follows then that $t = 1 \in \mathbb{N}$. Now, for $t = 1$ from $p = \frac{2-t}{l-1}$, it follows that $p \in \mathbb{N}$ if $l = 2$ and $p = 1$, then $s = 3$.

Based on that, for $m = 2$ we have determined another solution of Diophantine Equation (7), and that is $(t,s) = (1,3)$.

If $m = 3$, then from the condition that $(m-1) \mid p$ and equation $t = 2 + p - p\cdot l$, it follows that $t = 2 + \frac{p}{2} - p\cdot l$ is a natural number only if *p* is an even number, *i.e.* $p = 2q, q \in \mathbb{N}$. Furthermore, since $t \in \mathbb{N}$, by assumption, it follows that $t = 2 + q - 2q\cdot l > 0$. From this inequation we then get that $q < \frac{2}{2l-1}$. Since $q \in \mathbb{N}$ is only for $l = 1$, which is contrary to the assumption that $l > 1$. Thus, there is no $q \in \mathbb{N}$ for which $t \in \mathbb{N}$, and the consequence of this is that for $m = 3$, in this case, there is no solution for Diophantine Equation (7).

We have, thus, shown that the Diophantine Equation (7) has a solution only if $m = 2$ or $m = 3$ and that the set of solutions of Equation (7) is:

$$\mathcal{R} = \{(t,s) \mid t,s \in \mathbb{N}_0\} = \{(2l,0),(l+1,1),(0,4),(0,3),(2,2),(1,2),(1,3)\},$$

$$\forall l > 1 \in \mathbb{N}.$$

3.2. Tiling a Plane with Semi-Regular Equilateral Quadrilaterals

If in Equation (7) we put that $m = 2$, we get the following equation:

$$t + (l-1)\cdot s = 2l; l-1 > 0 \tag{10}$$

Equation (10) represents the corresponding Diophantine equation for the problem of tiling a plane with equilateral semi-regular quadrilaterals with a characteristic angle $\delta = \frac{\pi}{2\cdot l}$ and interior angles which are equal to angle $\alpha = \frac{\pi}{l}$ or angle $\beta = \left(1 - \frac{1}{l}\right)\pi$.

Note that for $l = 2$ the interior angles of the quadrilateral are equal, and they are $\alpha = \beta = \frac{\pi}{2}$, so it is regular, and it is not subject to our investigation here.

Therefore, we shall further consider the case when $l > 2, l \in \mathbb{N}$, *i.e.* when the interior angles of an equilateral quadrilateral are different. The theorem on nodes in tiling a plane with semi-regular equilateral quadrilateral holds.

Theorem 5. In relation to the selected value $l > 2, l \in \mathbb{N}$, when tiling a plane with a semi-regular equilateral quadrilateral, only the following nodes can appear: $\mathcal{R} = \{(2l,0),(0,3),(l+1,1),(2,2)\}$, and in this case node (0, 3) appears only in the case of tiling a plane with a semi-regular quadrilateral whose characteristic angle is $\delta = \frac{\pi}{6}$, and interior angles are $\alpha = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$.

Proof. The problem of determining nodes is equivalent to the problem of finding all non-negative solutions of the corresponding Diophantine Equation (10). The following cases are possible:

1) If $s = 0$, then $t = 2l$, so, obviously, pair $(t,s) = (2l,0)$ is the one solution of Equation (10) for all $l > 2, l \in \mathbb{N}$. That is, when tiling a plane with a semi-regular equilateral quadrilateral, a node $(2l,0)$ appears. That is, $2l$ vertices of semi-regular quadrilaterals which have the internal angles equal to angle $\alpha = \frac{\pi}{l}$ are joined at one node.

2) If $t = 0$, then Equation (10) has the following form $(l-1)\cdot s = 2l$. The solution of equation $s = \frac{2l}{l-1}$ can be written in the form of $s = \frac{2l}{l-1} = 2 + \frac{2}{l-1}$, from which it follows that $s \in \mathbb{N}_0$ is a non-negative integer if, and only if $l-1 = \pm 1$ or $l-1 = \pm 2$. If it were that $l-1 = \pm 1$, value $l = 2$ (when the quadrilateral is regular) and value $l = 0$ do not meet the assumption that $l > 2, l \in \mathbb{N}$. Only value $l-1 = 2$ meets the assumption, and then $l = 3$, and it follows that $s = 3$. It follows that node $(t,s) = (0,3)$ appears only when tiling a plane with a semi-regular equilateral quadrilateral with a characteristic angle $\delta = \frac{\pi}{6}$, and interior angles $\alpha = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$.

3) Let $t, s \neq 0$ still be natural numbers. If we write Equation (10) in the form of $s = \frac{2l-t}{l-1} = 2 + \frac{2-t}{l-1}$, it follows that s is a natural number, if $\frac{2-t}{l-1} \in \mathbb{Z}$. Since

$\frac{2-t}{l-1} \in \mathbb{Z}$, if:

a) $2-t=0$, that is, if $t=2$, then it still is that $s=2$. Thus, pair $(t,s)=(2,2)$ is the solution of Equation (10). Hence, it follows that when tiling a plane with a semi-regular equilateral quadrilateral, a node (2, 2) appears at which two vertices join, with an interior angle equal to angle $\alpha = \frac{\pi}{l}$ and two vertices whith corresponding interior angles equal to angle β for each choice of natural number $l > 2, l \in \mathbb{N}$.

b) If it were that $\frac{2-t}{l-1} = -1 \in \mathbb{Z}$, and then it would be that $2-t=1-l$ and then $t=l+1$, and $s=1$, so pair $(t,s)=(l+1,1)$ is another solution of Diophantine Equation (10).

Thus, when tiling a plane with semi-regular equilateral quadrilateral, there is a node $(l+1,1)$ at which the following vertices meet: vertex $l+1$ with an interior angle equal to angle $\alpha = \frac{\pi}{l}$ and another vertex with a corresponding interior angle which is equal to angle β.

c) If it were that $\frac{2-t}{l-1} = 1 \in \mathbb{Z}$, and then it would be that $2-t=l-1$, and then $t=3-l$. From the requirement that $t=3-l>0$, it follows that $l<3$. Since by assumption $l > 2, l \in \mathbb{N}$, it follows that the case is not possible.

d) If $\frac{2-t}{l-1} = p \in \mathbb{Z}$, then it would be $t = 2 - p(l-1)$ and $s = 2 + p$. Since by assumption $t,s \neq 0$ are natural numbers, the following conjunction must hold: $p > -2$ and $2 - p(l-1) > 0$. Hence we find that the conjunction is valid only when $-2 < p < 1$, *i.e.* $p \in \{-1, 0\}$ if $l=3$. Thus, for $p=-1$ it is $t=4$ and $s=1$. This pair (4, 1) is obtained from the previous pair $(t,s)=(l+1,1)$ if $l=3$. For $p=0$, we have that $t=2$ and $s=2$. We got a pair, as in case a).

We have, thus, determined the set of all solutions of Equation (10), $\mathcal{R} = \{(2l,0),(0,3),(l+1,1),(2,2)\}$, *i.e.* all nodes that can appear when tiling a plane with semi-regular equilateral quadrilaterals, with a characteristic angle $\delta = \frac{\pi}{2 \cdot l}$ and interior angles equal to angle $\alpha = \frac{\pi}{l}$ and angle $\beta = \left(1 - \frac{1}{l}\right)\pi$.

A graphical presentation of the corresponding Diophantine equation with nodes when tiling a plane with semi-regular quadrilaterals is shown in **Figure 3**.

Table 1 also lists the basic values of semi-regular quadrilaterals with the corresponding Diophantine equation, as an example of tiling a plane with semi-regular equilateral quadrilaterals for various values $l > 1, l \in \mathbb{N}$.

Let us now consider tiling a plane with some of the semi-regular equilateral quadrilaterals.

We noted that for $l=2$ the characteristic angle is $\delta = \frac{\pi}{4}$, and that then the

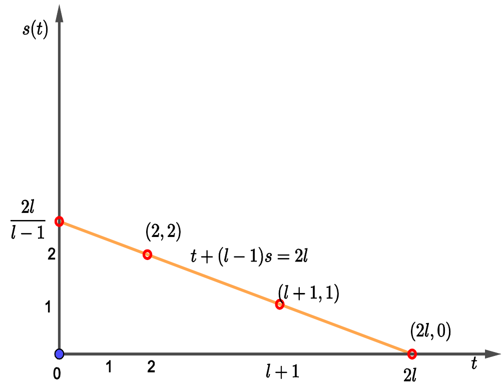

Figure 3. Position of nodes when tiling a plane with a semi-regular equilateral quadrilateral on the graph of the corresponding Diophantine equation.

Table 1. Diophantine equations with a set of solutions for tiling a plane with semi-regular equilateral quadrilaterals with characteristic values of angle δ, and interior angles, for various values of parameter $l>1, l\in\mathbb{N}$.

l	Diophantine equation	Set of Solutions	$\delta=\frac{\pi}{2l}$	$\alpha=\frac{\pi}{l}$	$\beta=\left(1-\frac{1}{l}\right)\pi$
1	$t+s=4$	$\{(4,0),(0,4),(1,3),(3,1),(2,2)\}$ [1]	π/4	π/2	π/2
2	$t+2s=6$	$\{(6,0),(0,3),(4,1),(2,2)\}$	π/6	π/3	2π/3
3	$t+3s=8$	$\{(8,0),(5,1),(2,2)\}$	π/8	π/4	3π/4
4	$t+4s=10$	$\{(10,0),(6,1),(2,2)\}$	π/10	π/5	4π/5
5	$t+5s=12$	$\{(12,0),(7,1),(2,2)\}$	π/12	π/6	5π/6
…	…	…	…	…	…
k	$t+(k-1)\cdot s=2k$	$\{(2k,0),(k+1,1),(2,2)\}$	$\pi/2k$	π/k	$\left(1-\frac{1}{k}\right)\pi$

1—quadrilateral is regular.

interior angles of the quadrilateral are equal; $\alpha=\frac{\pi}{2}$ and $\beta=\frac{\pi}{2}$ so the observed quadrilateral is regular (square). Tiling a plane with these quadrilaterals has been previously considered [2].

For $l=3$, the corresponding Diophantine equation is $t+2s=6$. The set of solutions of this equation is:

$\mathcal{R}=\{(6,0),(0,3),(4,1),(2,2)\}$, and characteristic angle $\delta=\frac{\pi}{6}$, while the values of the interior angles are $\alpha=\frac{\pi}{3}$, $\beta=\frac{2\pi}{3}$ respectively.

Let us consider the example of tiling a plane when the following nodes appear: (6, 0) and (0, 3) (**Figure 4**) and nodes (0, 3), (4, 1) and (2, 2) (**Figure 5**).

For $l=4$ when tiling a plane with a semi-regular quadrilateral with a characteristic angle $\delta=\frac{\pi}{8}$ and one node, $(t,s)=(2,2)$, at which two quadrilaterals meet, with vertices to which two interior angles equal to angle $\alpha = \frac{\pi}{4}$ correspond, and two quadrilaterals with vertices to which interior angles equal to angle $\beta = \frac{3\pi}{4}$ correspond (**Figure 6**).

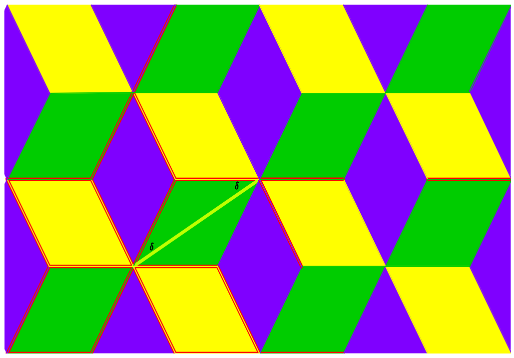

Figure 4. A fragment of tiling a plane with a semi-regular quadrangle with $\delta = \frac{\pi}{6}$ and with nodes (6, 0) and (0, 3).

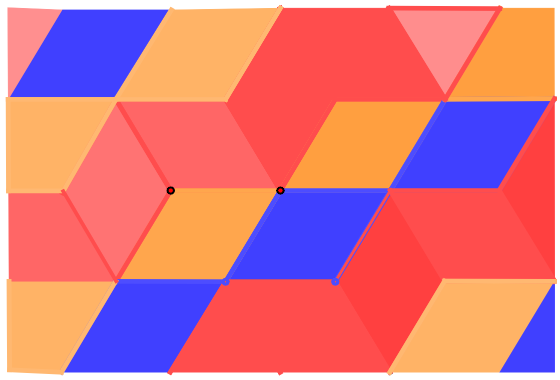

Figure 5. Tiling a plane with semi-regular equilateral quadrilaterals with $\delta = \frac{\pi}{6}$, and nodes (0, 3) and (4, 1).

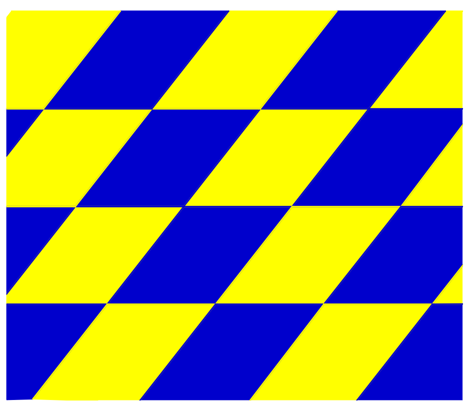

Figure 6. A fragment of tiling a plane with a semi-regular quadrilateral with $\delta = \frac{\pi}{8}$ and with node (2, 2).

A case of tiling a plane with a semi-regular quadrilateral in which there is a node (8.0) and a node (2.2) is shown in **Figure 7**, while tiling a plane with a semi-regular equilateral quadrilateral with nodes (5.1) and (2.2) is shown in **Figure 8**.

3.3. Tiling a Plane with Semi-Regular Equilateral Hexagons

If, in Equation (7) which corresponds to the problem of tiling a plane with semi-regular equilateral polygons with $2m$-sides,

$$(m-1)\cdot t+(l\cdot(m-1)-1)\cdot s=2l\cdot(m-1)$$

with $l>1, m\geq 2; m,l\in\mathbb{N}$ and unknowns t, s are non-negative integers, *i.e.* $t,s\in\mathbb{N}_0$, we insert that $m=3$, we get the following equation:

$$2t+(2l-1)\cdot s=4l;\ l-1>0. \tag{11}$$

Equation (11) represents Diophantine equation for the problem of tiling a plane with equilateral semi-regular hexagons with characteristic angle $\delta=\frac{\pi}{4\cdot l}$ and interior angles that are equal to angle $\alpha=2\frac{\pi}{l}$ or angle $\beta=\left(1-\frac{1}{2l}\right)\pi$. Each pair (t,s) of the non-negative integers that are not simultaneously equal to zero and that meet Equation (11) is the solution of the equation.

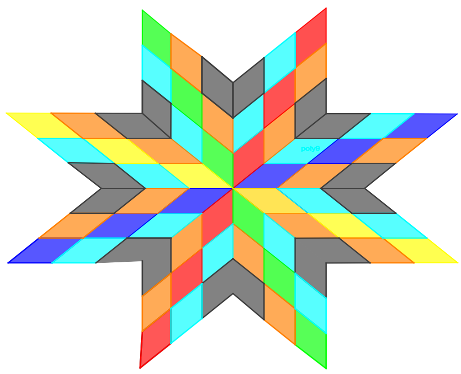

Figure 7. Tiling a plane with semi-regular quadrilaterals with node (8.0) and node (2.2).

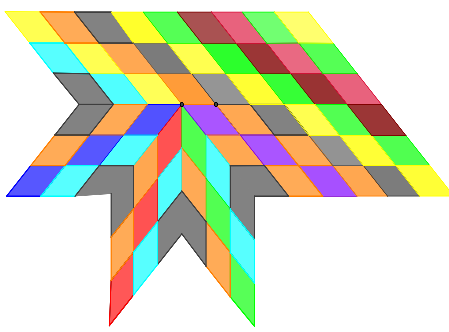

Figure 8. Tiling a plane with semi-regular quadrilateral when with $\delta=\frac{\pi}{8}$ and nodes (5, 1) and (2, 2).

The theorem holds.

Theorem 6. In relation to the selected value $l > 1, l \in \mathbb{N}$ when tiling a plane with a semi-regular equilateral hexagon, only nodes $(2l, 0), (1, 2)$ can appear for all values of the characteristic angle $\delta = \frac{\pi}{4l}$ and interior angles $\alpha = 2\frac{\pi}{l}$ and $\beta = \left(1 - \frac{1}{2l}\right)\pi$.

Proof: The problem of determining nodes is equivalent to the problem of finding all non-negative solutions of Diophantine Equation (11). In doing so, let us consider the following cases:

Case 1: If $t \neq 0, s = 0$, the equation has the following form $2t = 4l$, whence we find that there is a pair $(t, s) = (2l, 0)$, which represents the solution of Equation (11) for all $l > 1, l \in \mathbb{N}$.

From this, it follows that when tiling a plane with a semi-regular equilateral hexagon, there appears node (2*l*) at which two vertices meet with an interior angle equal to angle $\alpha = 2\frac{\pi}{l}$ and for each choice of a natural number $l > 1, l \in \mathbb{N}$.

Case 2: *If* $t = 0, s \neq 0$, then Equation (11) reads $(2l - 1) \cdot s = 4l$. Let us examine for which values of $l > 1, l \in \mathbb{N}$ it is that $s = \frac{4l}{2l - 1}$ and it is a natural number. Note that $s = \frac{4l}{2l - 1} = 2 + \frac{2}{2l - 1}$ and that $s \in \mathbb{N}$ is valid only when $2l - 1$ is divided by 2. This is possible only when $2l - 1 = \pm 1$ or $2l - 1 = \pm 2$. If it were that $2l - 1 = \pm 1$ than it would be that $l = 0$ or $l = 1$ and these values do not meet the requirement that $l > 1, l \in \mathbb{N}$. If it were that $2l - 1 = \pm 2$ then it could be that $l = \frac{3}{2}$ or $l = -\frac{1}{2}$, and also none of these values meets the requirement that $l > 1, l \in \mathbb{N}$. Thus, we conclude that there is no pair (*t*, *s*) in which $t = 0, s \neq 0$, and which is the solution of Equation (11) and which meets the requirement $l > 1, l \in \mathbb{N}$, and there is no node (0, *s*) as well, when tiling a plane with semi-regular equilateral hexagon.

Case 3: Let us assume that $t \neq 0, s \neq 0$. Then from equation $2t + (2l - 1) \cdot s = 4l$ we get that $t = 2l - l \cdot s + \frac{s}{2}$. Note that $t \in \mathbb{N}$ only if s is an even number.

Let $s = 2p, p \in \mathbb{N}$. Let us determine the values for natural number *p*.

Since $t = 2l - 2l \cdot p + p$ for $s = 2p$ then from the assumption that $t \neq 0$ and $t \in \mathbb{N}$ it follows that it must be that $t = 2l - 2l \cdot p + p > 0$, which is possible if $2l - 2l \cdot p \geq 0$, which is equivalent to the requirement that $l(1 - p) \geq 0$, or the requirement that $p \leq 1$. Hence, from the requirement that $p \in \mathbb{N}$ it follows that $t > 0$ only when $p = 1$. For this value $p \in \mathbb{N}$ is $t = 1$, and $s = 2$, for all values of $l > 1, l \in \mathbb{N}$.

We have, thus, determined another solution of Diophantine Equation (11).

That is, when tiling a plane with a semi-regular equilateral hexagon, a node (1,

2) appears at which one vertex of a semi-regular equilateral hexagon meets, with a corresponding interior angle equal to angle α, and two vertices of a semi-regular equilateral hexagon, with a corresponding interior angle equal to angle β for each selection of natural number $l > 1, l \in \mathbb{N}$.

We have, thus, shown that the set of all solutions of Diophantine Equation (11) is $\mathcal{R} = \{(2l,0),(1,2)\}$, *i.e.* that only nodes $(2l,0),(1,2)$ can occur when tiling a plane with a semi-regular equilateral hexagon, for all values of $l > 1, l \in \mathbb{N}$.

Examples of tiling a plane with semi-regular equilateral hexagons with corresponding Diophantine equations are shown in **Table 2**, and graphically in **Figure 9** and **Figure 10**.

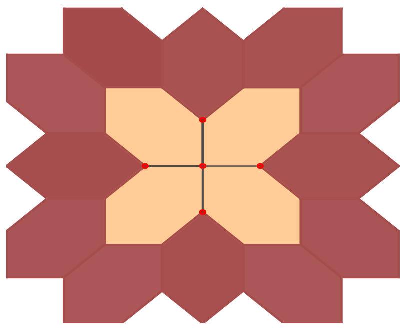

Figure 9. Tiling a plane with a semi-regular equilateral hexagon, with nodes (4, 0) and (1, 2).

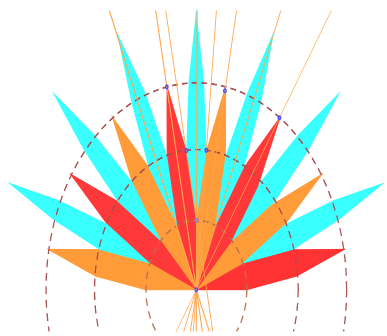

Figure 10. A case of tiling a plane with a semi-regular equilateral hexagon with nodes (16, 0) and (1, 2).

Table 2. Diophantine equations with a set of solutions for tiling a plane with a semi-regular equilateral hexagon, and with characteristic values of angle δ, and interior angles, for various values of parameter $l > 1, l \in \mathbb{N}$.

l	Diophantine equation	Set of Solutions	$\delta = \frac{\pi}{4 \cdot l}$	$\alpha = \frac{\pi}{l}$	$\beta = \left(1 - \frac{1}{2l}\right)\pi$
1	$2t + 3s = 8$	$\{(4,0),(1,2)\}$ [1]	$\pi/8$	$\pi/2$	$3\pi/4$
2	$2t + 5s = 12$	$\{(6,0),(1,2)\}$	$\pi/6$	$\pi/3$	$5\pi/6$
3	$t + 7s = 16$	$\{(8,0),(1,2)\}$	$\pi/16$	$\pi/4$	$7\pi/8$
…	…………..	…………	….	….	….
k	$2t + (2k-1) \cdot s = 4k$	$\{(2k,0),(1,2)\}$	$\pi/2k$	π/k	$\left(1 - \frac{1}{2k}\right)\pi$

4. Conclusions

The paper dealt with the possibility of tiling the Euclidean plane with convex semi-regular equilateral polygons. The research was conducted by observing a set of solutions for the corresponding Diophantine equation of the following form: $t\alpha + s\beta = 2\pi$, where t, s are the nonnegative integers that are not simultaneously equal to zero, and α, β, are the interior angles of a semi-regular equilateral polygon P_N.

It has been shown that each solution of this equation represents one node and it shows how many semi-regular equilateral polygons with the corresponding interior angles meet at that node. It has also been shown that of all semi-regular equilateral polygons with 2m-sides, a plane may be tiled only with semi-regular quadrilaterals and semi-regular hexagons. Graphically presented cases are just some of the possible ones that depend on the value of the characteristic angle.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Coxeter, M.S.H. (1948) Regular Polytopes. Methuen & Co.LTD., London.

[2] Chavey, D. (1989) Tilings by Regular Poligons—II A Catalog of Tilings. In: Hargittai, I., Ed., *Symmetry* 2 *Unifying Human Understanding*, 147-165. https://doi.org/10.1016/B978-0-08-037237-2.50019-2

[3] Chavey, D. (1984) Periodic Tilings and Tilings by Regular Poligons. Ph.D. Dissertation, University of Wisconsin, Madison.

[4] Deza, M., Grišuhin, V.P, Štogrin, M.I. (2007) Izometričeskie, poliedralnie podgrafi v hiperkubah i kubičeskih rešetkah, Moskva.

[5] Stojanović, N. (2015) Neka metrička svojstva polupravilnih poligona, PhD. Disertacija Filozofski fakultet Pale. Istočno Sarajevo.

[6] Stojanović, N. (2013) Inscribed Circle of General Semi-Regular Polygon and Some of Its Features. *International Journal of Geometry*, **2**, 5-22.

[7] Stojanović, N (2012) Some Metric Properties of General Semi-Regular Polygons. *Global Journal of Advanced Research on Classical and Modern Geometries*, **1**, 39-56.

[8] Stojanović, N. and Govedarica,V. (2013) Jedan pristup analizi konveksnosti i računanju površine jednakostranih polupravillnih poligona, II Matematička konferencija Republike Srpske, Trebinje, Zbornik radova, 87-105.

[9] Stojanović, N. and Govedarica,V. (2015) Diofantove jednačine i parketiranje ravni polupravilnim poligonima jedne vrste. *Fourth Mathematical Conference of the Republic of Srpska*, *Trebinje Proceedings*, Trebinje, June 2014, 184-194.