Open Journal of Discrete Mathematics, 2020, 10, 74-88
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

Partitioning of Any Infinite Set with the Aid of Non-Surjective Injective Maps and the Study of a Remarkable Semigroup

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How to cite this paper: Harrafa, C. (2020) Partitioning of Any Infinite Set with the Aid of Non-Surjective Injective Maps and the Study of a Remarkable Semigroup. *Open Journal of Discrete Mathematics*, **10**, 74-88.
https://doi.org/10.4236/ojdm.2020.103008

Received: April 21, 2020
Accepted: July 6, 2020
Published: July 9, 2020

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Abstract

In this article, we will present a particularly remarkable partitioning method of any infinite set with the aid of *non-surjective injective* maps. The non-surjective injective maps from an infinite set to itself constitute a semigroup for the *law of composition* bundled with certain properties allowing us to prove the existence of remarkable elements. Not to mention a compatible equivalence relation that allows transferring the *said law* to the quotient set, which can be provided with a lattice structure. Finally, we will present the concept of *Co-injectivity* and some of its properties.

Keywords

Partitioning, Non-Surjective, Injective, Infinite Set, Fixed Points, Lattice Structure

1. Introduction

The concept of *map* in mathematics has a primordial role in understanding the links that exist between the different mathematical fields and structures. A map is binary relation over two sets that associates to every element of the first set *exactly one* element of the second set, sometimes with a specific property. For instance, a "map" is a "linear transformation" in linear algebra, a "continuous function" in topology, operators in analysis and representations in group theory, etc. In this article we will show how non-surjective injective maps allow to partitioning an infinite set in several ways.

2. Part I

Proposition 1

Let E, and F be non-empty sets. If f, g are two *non-surjective injective* maps from E to F and from F to E respectively, then:

- $\left(A, f(B), I_{f\circ g}\right)$ forms a partition of F.
- $\left(B, g(A), I_{g\circ f}\right)$ forms a partition of E.

where,

$$A=\{y\in F \mid \forall x\in E, f(x)\neq y\} \text{ and } B=\{x\in E \mid \forall y\in F, g(y)\neq x\}$$

and I_f, I_g representing the image sets under f, g respectively.

Proof

Let E and F be two non-empty sets, and let f and g be two non-surjective injective maps from E to F, and from F to E respectively. Since f and g are non-surjective, then the following sets:

$A=\{y\in F \mid \forall x\in E, f(x)\neq y\}$ and $B=\{x\in E \mid \forall y\in F, g(y)\neq x\}$ are non-empty.

Also, obviously $E=B\cup I_g$ such as $B\cap I_g=\varnothing$ as follows from the definition of I_g.

For any such map f from E to F:

$$I_f=f(E)=f\left(B\cup I_g\right)=f(B)\cup f\left(I_g\right)=f(B)\cup I_{f\circ g}$$

So $F=A\cup f(B)\cup I_{f\circ g}$

Since f is an injective map then:

$$f\left(B\cap I_g\right)=f(B)\cap I_{f\circ g}=f(\varnothing)=\varnothing$$

Therefore, $\left(A, f(B), I_{f\circ g}\right)$ forms a partition of F.

In analogy to the map g from F to E, $\left(B, g(A), I_{g\circ f}\right)$ forms a partition of E.

Note 1

This process could be applied repeatedly, and for each iteration, finer partitions of the sets E and F respectively will be obtained, e.g. after 2$^{\text{nd}}$ iteration we have:

- $E=B\cup g(A)\cup(g\circ f)(B)\cup I_{g\circ f\circ g}$
- $F=A\cup f(B)\cup(f\circ g)(A)\cup I_{f\circ g\circ f}$

In case that f and g are (two) (2) *different* non-surjective injective maps from an infinite set E to itself, we can compose either by f or by g, or by both indefinitely. Thus several *partition classes* of E can obtained, for example after a second (2$^{\text{nd}}$) iteration

- $E=A_f\cup f\left(A_f\right)\cup f^2\left(A_f\right)\cup I_{f^3}$
- $E=A_g\cup g\left(A_g\right)\cup g^2\left(A_g\right)\cup I_{g^3}$
- $E=A_f\cup f\left(A_g\right)\cup(f\circ g)\left(A_f\right)\cup I_{f\circ g\circ f}$
- $E=A_g\cup g\left(A_f\right)\cup(g\circ f)\left(A_g\right)\cup I_{g\circ f\circ g}$
- $E=A_f\cup f\left(A_g\right)\cup(f\circ g)\left(A_g\right)\cup I_{f\circ g^2}$
- $E=A_g\cup g\left(A_f\right)\cup(g\circ f)\left(A_f\right)\cup I_{g\circ f^2}$

where,

$$A_f=\{y\in F \mid \forall x\in E, f(x)\neq y\} \text{ and } A_g=\{x\in E \mid \forall y\in F, g(y)\neq x\}$$

Example 1

If $E = F = \mathbb{N}$ *i.e.* the set of natural numbers, f and g are two maps from $\mathbb{N}$ to itself (*i.e.* f and g are non-surjective injective) and defined by:

- $\forall n \in \mathbb{N}, f(n) = 2n$
- $\forall n \in \mathbb{N}, g(n) = 2n+1$

Knowing that $A_f = 2\mathbb{N}+1$ and $A_g = 2\mathbb{N}$ then:

- $f(A_g) = 4\mathbb{N}$
- $g(A_f) = 4\mathbb{N}+3$
- $(f \circ g)(A_f) = 8\mathbb{N}+6$
- $(g \circ f)(A_g) = 8\mathbb{N}+1$
- $(f \circ g \circ f)(\mathbb{N}) = 8\mathbb{N}+2$
- $(g \circ f \circ g)(\mathbb{N}) = 8\mathbb{N}+5$

Therefore, we can partition the set $\mathbb{N}$ to the second (2^{nd}) order by f and g such as:

$$\mathbb{N} = (2\mathbb{N}+1) \cup (4\mathbb{N}) \cup (8\mathbb{N}+6) \cup (8\mathbb{N}+2)$$

$$\mathbb{N} = (2\mathbb{N}) \cup (4\mathbb{N}+3) \cup (8\mathbb{N}+1) \cup (8\mathbb{N}+5)$$

2.1. Remarkable Partition

Proposition 2

Let E be an infinite set, $\mathbb{N}$ be the set of natural numbers, and f be a non-surjective injective map from E to itself, such that:

$$A_f = \{ y \in E \mid \forall x \in E, f(x) \neq y \}$$

Then,

$$\forall n \in \mathbb{N}, E = \left[\bigcup_{i=0}^{i=n} f^i(A_f) \right] \cup I_{f^{n+1}}$$

where, $f^i(A_f) = \underbrace{f \circ f \circ \cdots \circ f}_{i \text{ times}}(A_f)$

Proof

By induction:

For $n = 0$, we have, by definition $E = A_f \cup I_f$, and for $n = 1$ the proposition 1 states that, if $E = F$ and $f = g$ then, $E = A_f \cup f(A_f) \cup I_{f^2}$ Suppose that the said property is true for n. Then, by composing by f, we get:

$$I_f = f(E) = \left[\bigcup_{i=0}^{i=n} f^{i+1}(A_f) \right] \cup I_{f^{n+2}} = \left[\bigcup_{i=1}^{i=n+1} f^i(A_f) \right] \cup I_{f^{n+2}}$$

Then,

$$E = A_f \cup \left[\bigcup_{i=1}^{i=n+1} f^i(A_f) \right] \cup I_{f^{n+2}} = \left[\bigcup_{i=0}^{i=n+1} f^i(A_f) \right] \cup I_{f^{n+2}}$$

Therefore,

$$\forall n \in \mathbb{N}, E = \left[\bigcup_{i=0}^{i=n} f^i(A_f) \right] \cup I_{f^{n+1}}$$

Note 2

For all non-surjective injective maps f from an infinite set E to itself,

$$\forall n \in \mathbb{N}, A_{f^{n+1}} = \bigcup_{i=0}^{i=n} f^i\left(A_f\right)$$

Definition 1

Let f be a map from no-empty set E to itself,

- $x \in E$ is a fixed point of f if, $f(x) = x$.
- $A \subset E$ is a fixed point set of f if, $f(A) = A$.

Proposition 3

For all non-surjective injective maps f from an infinite set E to itself, $\forall n \in \mathbb{N}$, $f^n\left(A_f\right)$ contains no fixed points of f.

Proof

By induction:

For $n = 0$, let $x \in A_f$, and by definition $\forall y \in E$, $f(y) \neq x$, particularly for $y = x$ so $f(x) \neq x$ then A_f contains no fixed points of f. Suppose that the aforementioned property is true for n, let $x \in f^{n+1}\left(A_f\right) = f\left(f^n\left(A_f\right)\right)$, then $\exists y \in f^n\left(A_f\right)$ such that $x = f(y)$, we have $f(y) \neq y$ (by inductive hypothesis), since f is injective then $f\left(f(y)\right) \neq f(y)$ so $f(x) \neq x$, then $\forall x \in f^{n+1}\left(A_f\right)$, $f(x) \neq x$. Therefore, $\forall n \in \mathbb{N}$, $f^n\left(A_f\right)$ contains no fixed points of f.

Note 3

Let E be an infinite set, and f be a non-surjective injective map from E to itself, we define the following:

$$Fix_f = \left\{x \in E \mid f(x) = x\right\} \text{ and } St_f(E) = \left\{A \subset E \mid f(A) = A\right\}$$

- $\forall n \in \mathbb{N}, Fix_f \cap f^n(A_f) = \varnothing$
- $\forall n \in \mathbb{N}^*, Fix_f \subseteq Fix_{f^n}$
- $\forall n \in \mathbb{N}^*, Fix_{f^n} \subseteq Fix_{f^{2n}}$
- $\forall p \in \mathbb{N}, \forall n \in \mathbb{N}^*, f^p\left(Fix_{f^n}\right) = Fix_{f^n}$
- For all f non-surjective injective maps from an infinite set E to itself,

$$Fix_f \in St_f(E)$$

Proposition 4

For all f non-surjective injective maps from E to itself,

$$\forall A \in St_f(E), \forall n \in \mathbb{N}, f^n\left(A_f\right) \cap A = \varnothing$$

Proof

Let $A \in St_f(E)$, by induction, For $n = 0$, we have, by definition $A_f \cap I_f = \varnothing$ then $A_f \cap A = A_f \cap f(A) = \varnothing$ Suppose that the said property is true for $n \in \mathbb{N}$, let $x \in f^{n+1}\left(A_f\right) \cap A$, then, $\exists y \in f^n\left(A_f\right), x = f(y)$ and $\exists y_0 \in A, x = f\left(y_0\right)$. Since f is injective so $y = y_0 \in f^n\left(A_f\right) \cap A$, which contradicts the inductive hypothesis. Then,

$$f^{n+1}\left(A_f\right) \cap A = \varnothing .$$

Therefore, $\forall A \in St_f(E), \forall n \in \mathbb{N}, f^n(A_f) \cap A = \varnothing$

Lemma 1 (Generalization)

$$\forall p \in \mathbb{N}^*, \forall A \in St_{f^p}(E), \forall n \in \mathbb{N}, f^n(A_f) \cap A = \varnothing$$

Proof

By complete (strong) induction,

Let $p \in \mathbb{N}^*$, and $A \in St_{f^p}(E) = \{A \subset E \mid f^p(A) = A\}$: For $n = 0$, we have, by definition $A_f \cap I_f = \varnothing$, since $\exists p \in \mathbb{N}^*, I_{f^p} \subseteq I_f$, then:
$A_f \cap A = A_f \cap f^p(A) = \varnothing$

Suppose that the said property is true for all $i \in \{1, \cdots, n\}$, let $x \in f^{n+1}(A_f) \cap A$, then, $\exists y \in A_f, x = f^{n+1}(y)$ and $\exists y_0 \in A, x = f^p(y_0)$. Since f is injective then,

- $f^{n-p+1}(y) = y_0$, if $n+1 > p$, which contradicts the inductive hypothesis, because $i = n - p + 1 \in \{1, \cdots, n\}$
- $y = f^{p-n-1}(y_0)$, if $p > n+1$, which contradicts $A_f \cap I_{f^{p-n-1}} = \varnothing$
- $y = y_0$, if $n+1 = p$, which contradicts $A_f \cap A = \varnothing$

Then,

$$f^{n+1}(A_f) \cap A = \varnothing,$$

Therefore,

$$\forall p \in \mathbb{N}^*, \forall A \in St_{f^p}(E), \forall n \in \mathbb{N}, f^n(A_f) \cap A = \varnothing$$

QED

For all $n \in \mathbb{N}$, and for all f non-surjective injective maps from an infinite set E to itself, we define the following:

- $S_{f^n}(E) = \{A \subset E \mid \exists p \in \{1, \cdots, n+1\}, f^p(A) = A\}$
- $S_{f^\infty}(E) = \{A \subset E \mid \exists p \in \mathbb{N}^*, f^p(A) = A\}$
- $S_{f^n} = \{x \in A \mid A \in S_{f^n}(E)\}$
- $S_{f^\infty} = \{x \in A \mid A \in S_{f^\infty}(E)\}$

Theorem 1

Let E be an infinite set, and f be a non-surjective injective map from E to itself, then:

$$E = \left[\bigcup_{n \in \mathbb{N}} f^n(A_f)\right] \cup S_{f^\infty}$$

Proof

Note that for $n = 0$, $S_{f^0}(E) = St_f(E)$, The sequence of subsets of E, $(S_{f^n})_{n \in \mathbb{N}}$ is increasing by inclusion, so it is convergent, the limit is: $\bigcup_{n \in \mathbb{N}} S_{f^n} = S_{f^\infty}$

We have already established that,

$$\forall n \in \mathbb{N}, E = \left[\bigcup_{i=0}^{i=n} f^i(A_f)\right] \cup I_{f^{n+1}}$$

Otherly, according to the Lemma 1, $\forall n \in \mathbb{N}, E = \left[\bigcup_{i=0}^{i=n} f^i(A_f)\right] \cap S_{f^n} = \varnothing$

Let's define the following, $\forall n \in \mathbb{N}, \widehat{I_{f^{n+1}}} = I_{f^{n+1}} \setminus S_{f^n}$ the sequence of subsets of $\left(\widehat{I_{f^{n+1}}}\right)_{n\in\mathbb{N}}$ is decreasing by inclusion then, it is convergent [1], the limit is:

$$\bigcap_{n\in\mathbb{N}} \widehat{I_{f^{n+1}}} = \varnothing$$

Because, let $x \in \bigcap_{n\in\mathbb{N}} \widehat{I_{f^{n+1}}}$, then:

$$x \in \bigcap_{n\in\mathbb{N}} I_{f^{n+1}} \setminus S_{f^n} \Leftrightarrow \forall n \in \mathbb{N}, x \in I_{f^{n+1}} \setminus S_{f^n} \Leftrightarrow \forall n \in \mathbb{N}, x \in I_{f^{n+1}} \text{ and } x \in \overline{S_{f^n}}$$

$$\Leftrightarrow \forall n \in \mathbb{N}, x \in I_{f^{n+1}} \text{ and } \forall n \in \mathbb{N}, x \in \overline{S_{f^n}} \Leftrightarrow x \in \bigcap_{n\in\mathbb{N}} I_{f^{n+1}} \text{ and } x \in \bigcap_{n\in\mathbb{N}} \overline{S_{f^n}}$$

$$\Leftrightarrow x \in \bigcap_{n\in\mathbb{N}} I_{f^{n+1}} \text{ and } x \in \overline{\bigcup_{n\in\mathbb{N}} S_{f^n}} \Leftrightarrow x \in \bigcap_{n\in\mathbb{N}} I_{f^{n+1}} \text{ and } x \in \overline{S_{f^\infty}}$$

$$\Leftrightarrow x \in \left[\bigcap_{n\in\mathbb{N}} I_{f^{n+1}}\right] \setminus S_{f^\infty}$$

On the other hand, the sequence of subsets of E, $\left(I_{f^{n+1}}\right)_{n\in\mathbb{N}}$ is *strictly* decreasing by inclusion, then it is convergent and the limit is $H = \bigcap_{n\in\mathbb{N}} I_{f^{n+1}}$, and since f is injective, so $f(H) = H$ then f is bijective from H to itself, and $H \in St_f(E) \subset S_{f^\infty}(E)$.

$$\forall n \in \mathbb{N}, E = \left[\bigcup_{i=0}^{i=n} f^i(A_f)\right] \cup \widehat{I_{f^{n+1}}} \cup S_{f^n}$$

Therefore,

$$E = \left[\bigcup_{n\in\mathbb{N}} f^n(A_f)\right] \cup S_{f^\infty}$$

N.B. If $S_{f^\infty} = \varnothing$, then we get:

$$E = \bigcup_{n\in\mathbb{N}} f^n(A_f)$$

QED

Example 2

$\mathbb{R}$ is the set of real numbers, and $P(\mathbb{R})$ is the set of subsets of $\mathbb{R}$.

Let f be a map from $\mathbb{R}$ to itself, defined by:

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$$

Therefore, f is injective non-surjective from $\mathbb{R}$ to itself, because f is injective and $A_f = [0,1[$.

where,

$$\forall n \in \mathbb{N}^*, f^n(x) = \begin{cases} x+n, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$$

We have, $f(A_f) = [f(0), f(1)[= [1,2[\subset \mathbb{R}_+$ since f is increasing Therefore,

$$\forall n \in \mathbb{N}^*, f^n(A_f) = [n, n+1[$$

- $\forall n \in \mathbb{N}, S_{f^n} = S_{f^\infty} = \mathbb{R}_-^*$
- $\forall n \in \mathbb{N}, S_{f^n}(E) = S_{f^\infty}(E) = P(\mathbb{R}_-^*)$

According to the theorem 1, we can write:

$$\mathbb{R}=\mathbb{R}_{-}^{*}\cup\left[\bigcup_{n\in\mathbb{N}}[n,n+1[\right]$$

Example 3

Remark

If h is an affine map from $\mathbb{R}$ to $\mathbb{R}$, so that $h(x)=ax+b/a,b\in\mathbb{R}$ and $a\neq 0$, then:

$$\forall n\in\mathbb{N}^{*},\forall x\in\mathbb{R},h^{n}(x)=a^{n}x+b\left(1+a+\cdots+a^{n-1}\right)$$

If $a\neq 1$, then:

$$\forall n\in\mathbb{N}^{*},\forall x\in\mathbb{R},h^{n}(x)=a^{n}x+b\left(\frac{1-a^{n}}{1-a}\right)$$

Let f be a map defined from $E=[0,4]$ to itself by:

$$\forall x\in[0,4],f(x)=\begin{cases}-x+1, & \text{if } x\in[0,1]\\ x+1, & \text{if } x\in]1,2]\\ \frac{1}{2}x+2, & \text{if } x\in[2,4]\end{cases}$$

f is non-surjective injective map from E to itself, so that:

- $A_f=]1,2]$
- $Fix_f=\{4\}$
- The set $A=[0,1]\subset E$, fulfills the condition $f(A)=A$
- We have $A_f=]1,2]$, $f(A_f)=f(]1,2])=]2,3]\subset[2,4]$, $f(]2,3])=\left]3,\frac{7}{2}\right]$, …
- $S_{f^{\infty}}=\{4\}\cup[0,1]$

The restriction of f to $[2,4]$ is an affine function such as $a=\frac{1}{2}$, and $b=2$, then:

$$\forall n\in\mathbb{N}^{*},f^{n}(x)=\frac{1}{2^{n}}x+2\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right)$$

We have: $f^{n}(2)=4-\frac{1}{2^{n-1}}$, and $f^{n}(3)=4-\frac{1}{2^{n}}$,

According to Theorem 1:

$$[0,4]=\left\{]1,2]\bigcup_{n\in\mathbb{N}}\left]4-\frac{1}{2^{n-1}},4-\frac{1}{2^{n}}\right]\right\}\cup\{4\}\cup[0,1]$$

Example 4

Let f be a map defined from $E=[0,3]$ to itself by:

$$\forall x\in[0,3],f(x)=\begin{cases}x+2, & \text{if } x\in[0,1]\\ \frac{1}{2}x+1, & \text{if } x\in]1,2[\\ x-2, & \text{if } x\in[2,3]\end{cases}$$

f is non-surjective injective from E to itself, so that:

- $A_f = \left]1, \frac{3}{2}\right]$
- $Fix_f = \varnothing$
- The set $A = [0,1] \subset E$, fulfills the condition $f(A) \neq A$, and $f^2(A) = A$
- The set $B = [2,3] \subset E$, fulfills the condition $f(B) \neq B$, and $f^2(B) = B$
- $S_{f^\infty} = [0,1] \cup [2,3]$

We have,

- $\forall x \in]1,2[, \forall n \in \mathbb{N}^*, f^n(x) = \frac{1}{2^n}x + 2\left(1 - \frac{1}{2^n}\right)$
- $\lim_{x \to 1^-} f^n(x) = 2 - \frac{1}{2^n}$ and $f^n\left(\frac{3}{2}\right) = 2 - \frac{1}{2^{n+1}}$

Therefore, according to Theorem 1:

$$[0,3] = \left\{\bigcup_{n\in\mathbb{N}} \left]2 - \frac{1}{2^n}, 2 - \frac{1}{2^{n+1}}\right]\right\} \cup [0,1] \cup [2,3]$$

Corollary 1

Let f, g be non-surjective injective maps from an infinite set E to itself such as $S_{f^\infty} = \overline{S_{g^\infty}}$, then:

$$E = \bigcup_{n\in\mathbb{N}} B_n$$

where, $\forall n \in \mathbb{N}, B_n = f^n(A_f) \cup g^n(A_g)$

Therefore, $(B_n)_{n\in\mathbb{N}}$ forms a partition of *E*.

Definition 2

- Let E be an infinite set, we write: $Injns(E)$ as being the set of non-surjective injective maps from E to itself.
- $Injns(E,F)$ as the set of non-surjective injective maps from a set E to a set *F*, that being said, E and F are supposed to be non-empty and $|E| \leq |F|$ ($|E|$ is the cardinal of *E*).

Properties

1) $(Injns(E), \circ)$ is a *semigroup*, because the composite of 2 (two) injective maps f and g is injective and $I_{f\circ g} \subset I_f$, so $\forall f, g \in Injns(E)$, $f \circ g \in Injns(E)$.

2) $\forall f, g \in Injns(E)$, $A_{f\circ g} = A_f \cup f(A_g)$, $(A_f, f(A_g), I_{f\circ g})$ is, indeed, a partition of E and $E = A_{f\circ g} \cup I_{f\circ g}$, because $f \circ g \in Injns(E)$.

3) There is *no idempotent* element for the law of composition in $Injns(E)$ and if such an element exists, then $f^2 = f$ and since f is injective, then $f = I_d$ which is contradictory, because, $f \in Injns(E)$.

4)Let $f \in Injns(E)$, and assuming that f is a map from E to itself such as:

$$\tilde{f}(x) = \begin{cases} x, & \text{if } x \in A_f \\ f(x), & \text{if } x \in I_f \end{cases}$$

5) We have, $\forall f \in Injns(E), \tilde{f} \in Injns(E)$ knowing that $I_{\tilde{f}} = A_f \cup I_{f^2}$ and $A_{\tilde{f}} = f(A_f)$

6) Let $f \in Injns(E)$, we have, $\tilde{f}(A_{\tilde{f}}) = \tilde{f}(f(A_f)) = f^2(A_f)$, and

$$I_{\tilde{f}^2} = \tilde{f}\left(I_{\tilde{f}}\right) = \tilde{f}\left(A_f \cup I_{f^2}\right) = \tilde{f}\left(A_f\right) \cup \tilde{f}\left(I_{f^2}\right) = A_f \cup I_{f^3}$$

Then,

$$E = A_{\tilde{f}} \cup \tilde{f}\left(A_{\tilde{f}}\right) \cup I_{\tilde{f}^2} = A_f \cup f\left(A_f\right) \cup f^2\left(A_f\right) \cup I_{f^3}$$

Therefore, $\tilde{f}$ allow us to reduce order of iteration.

7) $\forall f \in Injns(E)$,

- $A_{\tilde{f} \circ f} = A_{\tilde{f}} \cup \tilde{f}\left(A_f\right) = A_f \cup f\left(A_f\right) = A_{f^2}$, then $I_{\tilde{f} \circ f} = I_{f^2}$
- $A_{f \circ \tilde{f}} = A_f \cup f\left(A_{\tilde{f}}\right) = A_f \cup f^2\left(A_f\right)$, then, $I_{f \circ \tilde{f}} = f\left(A_f\right) \cup I_{f^3}$
- $A_{\tilde{\tilde{f}}} = \tilde{f}\left(A_{\tilde{f}}\right) = f^2\left(A_f\right)$, then $I_{\tilde{\tilde{f}}} = A_f \cup f\left(A_f\right) \cup I_{f^3}$

8) Let $f \in Injns(E, F)$ and $g \in Injns(F, E)$, so:

$$f \circ g \in Injns(F) \quad \text{and} \quad g \circ f \in Injns(E)$$

2.2. Equivalence Relations

Example 5

Let $f, g \in Injns(E)$, we define the binary relation R on $Injns(E)$, by:

$$fRg \Leftrightarrow A_f = A_f \Leftrightarrow I_f = I_g$$

R is, indeed, an equivalence relation, because:

- R is *reflexive*, $fRf, \forall f \in Injns(E)$
- R is *symmetric*, $fRg \Leftrightarrow A_f = A_g \Leftrightarrow gRf, \forall f, g \in Injns(E)$
- R is *transitive*, $\forall f, g$ and $h \in Injns(E)$, fRg and $gRh \Leftrightarrow A_f = A_g$ and $A_g = A_h \Leftrightarrow A_f = A_h \Leftrightarrow fRh$

Example 6

Let $f, g \in Injns(E)$, we define the binary relation R on $Injns(E)$, by:

$$\forall f, g \in Injns(E): fRg \Leftrightarrow \forall n \in \mathbb{N}^*, I_{f^n} = I_{g^n}$$

R is, indeed, an **equivalence relation**, because:

- R is reflexive, $fRf, \forall f \in Injns(E)$
- R is symmetric, $fRg \Leftrightarrow \forall n \in \mathbb{N}^*, I_{f^n} = I_{g^n} \Leftrightarrow gRf, \forall f, g \in Injns(E)$
- R is transitive, f, g and $h \in Injns(E)$, fRg and $gRh \Leftrightarrow \forall n \in \mathbb{N}^*, I_{f^n} = I_{g^n}$ and $\forall n \in \mathbb{N}^*, I_{g^n} = I_{h^n} \Leftrightarrow \forall n \in \mathbb{N}^*, I_{f^n} = I_{h^n} \Leftrightarrow fRh$

Example 7

$$\forall f, g \in Injns(E), fRg \Leftrightarrow f\left(A_f\right) = g\left(A_g\right)$$

Example 8

$$\forall f, g \in Injns(E), fRg \Leftrightarrow S_{f^\infty} = S_{g^\infty} \Leftrightarrow \bigcup_{n \in \mathbb{N}} f^n\left(A_f\right) = \bigcup_{n \in \mathbb{N}} g^n\left(A_g\right)$$

$\bigcup_{n \in \mathbb{N}} f^n\left(A_f\right)$: as being the f-semicoverage of a set *E*.

3. Part II

Let E be an infinite set, $f \in Injns(E)$, and $\overline{f} = \left\{g \in Injns(E) \mid I_g \cap I_f \neq \varnothing\right\}$

We get the following:

- $\forall f \in Injns(E), \forall n \in \mathbb{N}^*, f^n \in \overline{f}$
- $\forall f, g \in Injns(E): I_f \subset I_g \Rightarrow \overline{f} \subset \overline{g}$

Let $h \in \overline{f}$ *i.e.* $I_f \cap I_h \neq \varnothing$ let $x \in I_h \cap I_f$, so $x \in I_h$ and $x \in I_f \subset I_g$, so $x \in I_h$ and $x \in I_g$, so $x \in I_h \cap I_g$ so $I_h \cap I_g \neq \varnothing$, then $h \in \overline{g}$ therefore $\overline{f} \subset \overline{g}$

Theorem 2

Let E be an infinite set, then there exists a non-surjective injective map ψ from E to itself, so that for any such non-surjective injective map ϕ from E to itself, we have:

$$I_\psi \cap I_\phi \neq \varnothing$$

Proof

By contradiction, let's suppose that for all ψ non-surjective injective maps from E to itself, there exists a map $\phi_\psi \in Injns(E)$ such as $I_\psi \cap I_{\phi_\psi} = \varnothing$ so that $I_{\phi_\psi} \subseteq A_\psi$. Additionally E is an infinite set, so E is equipotential [2] to $E \setminus \{a\}, \forall a \in E$. Considering this bijection f as a map from E to itself, then $f \in Injns(E)$, and $A_f = \{a\}$, according to all of the above, there exists a map $f_\phi \in Injns(E)$ such as $I_{f_\phi} \subseteq Af = \{a\}$, which contradicts the fact that f_ϕ injective.

QED

Note 4

- $\exists \psi \in Injns(E)$, so that: $\forall \varphi \in Injns(E)$, $\psi^{-1}(I_\varphi) \neq \varnothing$
- If ψ applies to the former theorem then: $\overline{\psi} = Injns(E)$.

Proposition 5

$\forall f \in Injns(E)$, $\exists g \in Injns(E) \setminus \{f\}$, so that $A_f \cap A_g \neq \varnothing$

Proof

By contradiction, assuming that exists a map $f \in Injns(E)$, so that $\forall g \in Injns(E) \setminus \{f\}$, $A_f \cap A_g = \varnothing$, then $\forall g \in Injns(E) \setminus \{f\}$, $I_f \cup I_g = E$.

Let $g = f^2$, so $I_f \cup I_g = I_f \subsetneq E$ because $f \in Injns(E)$ which is contradictory.

Proposition 6

$\forall f \in Injns(E)$, $\exists g \in Injns(E) \setminus \{f\}$, such as $A_f \cap I_g \neq \varnothing$

Proof

By contradiction, assuming that exists a map $f \in Injns(E)$, so that, $\forall g \in Injns(E) \setminus \{f\}$, $A_f \cap I_g = \varnothing$ then $\forall g \in Injns(E) \setminus \{f\}, I_f \cup A_g = E$. On the other hand, if $g = \tilde{f}, I_f \cup A_g = E$, additionally $A_g = f(A_f) \subset I_f$ so $I_f \cup A_g = I_f = E$ which is contradictory.

Note 5

We can define another composition law T for $Injns(E)$ so that,

$$\forall f, g \in Injns(E), (fTg)(x) = \begin{cases} g(x), & \text{if } x \in f^{-1}(I_g) \\ f(x), & \text{if } x \in f^{-1}(A_g) \end{cases}$$

- $fTf = f, \forall f \in Injns(E)$ then every element is *idempotent* under the law T.

- $\forall f, g \in Injns(E)$, if $I_f \cap I_g = \varnothing$, then $fTg = f$.
- $\forall f, g \in Injns(E)$, if $I_f \subseteq I_g$, then $fTg = g$.
- $\forall f \in Injns(E)$, $fT\tilde{f} = f$ and $\tilde{f}Tf = \tilde{f}$.
- Generally, $\forall f, g \in Injns(E)$, $fTg \neq gTf$.

4. Part III

4.1. Study of the Quotient Set

Let E be an infinite set, and $f, g \in Injns(E)$. We define the binary relation R on $Injns(E)$, by:

$$fRg \Leftrightarrow A_f \text{ and } A_g \text{ have the same cardinal}$$

The binary relation R is *reflexive, symmetric and transitive*, so R is an equivalence relation on $Injns(E)$.

We define $Cl(f) = \{g \in Injns(E) \mid |A_g| = |A_f|\}$ the equivalence class of a map f.

Note 6

- Let $f \in Injns(E)$, as $A_{\tilde{f}} = f(A_f)$ so A_f and $A_{\tilde{f}}$ have the same cardinal, because f is injective then $\tilde{f} \in Cl(f)$, therefore,

$$\forall f \in Injns(E), \tilde{f} \in Cl(f)$$

- Let $f, g \in Injns(E)$, assuming that the cardinals of A_f and A_f are finite, and thus, $|A_{f \circ g}| = |A_f \cup f(A_g)| = |A_f| + |A_g| - |A_f \cap f(A_g)|$, and since $A_f \cap f(A_g) = \varnothing$, then: $|A_{f \circ g}| = |A_f \cup f(A_g)| = |A_f| + |A_g|$.Since the composition of two (2) maps f and g on $Injns(E)$ yields a disjoint union, *i.e.* $A_{f \circ g} = A_f \cup f(A_g)$, with $A_f \cap f(A_g) = \varnothing$,then we can extend the sum of the cardinals even for infinite sets, such as:

$$|A_{f \circ g}| = |A_f \cup f(A_g)| = |A_f| + |A_g| = \sup\{|A_f|, |A_g|\}$$

- For all maps f, g, h and $t \in Injns(E)$ so that fRg and hRt *i.e.* $|A_f| = |A_g|$ and $|A_h| = |A_t|$. Since:

$$|A_{f \circ h}| = \sup\{|A_f|, |A_g|\} = \sup\{|A_g|, |A_t|\} = |A_{g \circ t}|, \text{ therefore,}$$

$$f \circ gRh \circ t$$

- The map's composition law is compatible under the equivalence relation R, then we can provide the quotient set $(Injns(E)/R, *)$ with a structure of a *semigroup*.
- $\forall Cl(f)$ and $Cl(g) \in Injns(E)/R$: $Cl(f) * Cl(g) = Cl(f \circ g)$

4.2. Partial Order

Let $Cl(f), Cl(g) \in Injns(E)/R$, define a binary relation on $Injns(E)/R$ by:

$$Cl(f) \leq Cl(g) \Leftrightarrow |A_f| \leq |A_g|$$

- $\forall Cl(f) \in Injns(E)/R, Cl(f) \leq Cl(f)$

  So $\leq$ is reflexive.

- $\forall Cl(f), Cl(g) \in Injns(E) \setminus R$, $Cl(f) \le Cl(g)$ and $Cl(g) \le Cl(f)$ $\Leftrightarrow$ $|A_f| \le |A_g|$ and $|A_g| \le |A_f|$ $\Leftrightarrow$ $|A_f| = |A_g|$ so $Cl(f) = Cl(g)$

  So $\le$ is asymmetric.
- $\forall Cl(f), Cl(g)$ and $Cl(h) \in Injns(E)/R$: $Cl(f) \le Cl(g)$ and $Cl(g) \le Cl(h)$ $\Leftrightarrow$ $|A_f| \le |A_g|$ and $|A_g| \le |A_h|$ $\Rightarrow$ $|A_f| \le |A_h|$ $\Rightarrow$ $Cl(f) \le Cl(h)$

  So $\le$ is transitive.

  Therefore, the relation $\le$ is a partial order on $Injns(E)/R$.

  Note 7
- Since $\forall Cl(f), Cl(g) \in Injns(E)/R$, $Cl(f) \le Cl(g)$ or $Cl(g) \le Cl(f)$ then $\le$ is a *total partial* order on $Injns(E)/R$.
- The partial order on the semigroup $(Injns(E)/R, *)$ is, indeed, compatible [3] with the equivalence class's composition law of composition *, then:

$$\forall Cl(f), Cl(g) \text{ and } Cl(h) \in Injns(E)/R,$$

If $Cl(f) \le Cl(g)$ then

$Cl(f) * Cl(h) = Cl(f \circ h) \le Cl(g \circ h) = Cl(g) * Cl(h)$ and

$Cl(h) * Cl(f) = Cl(h \circ f) \le Cl(h \circ g) = Cl(h) * Cl(g)$

- $Injns(E)/R$ is *well-ordered*, because any non-empty subset has a *smallest* element.
- $Injns(E)/R$ is a *lattice*, because it is ordered and every pair of elements has both *upper* bound and *lower* bound [4].
- $Injns(E)/R$ provided with the order relation has a *minimal* element $Cl(f)$, so that $|A_f| = 1$, and also *maximal* element $Cl(g)$, so that A_g has the same cardinality as *E*.
- If *E* is an *infinite countable* set, the map φ defined by:

$$\varphi : Injns(E) \to \mathbb{N}^* \cup \{\aleph 0\} = M, \forall f \in Injns(E), \varphi(f) = |A_f|,$$

where, $\aleph 0$ represents the cardinal of $\mathbb{N}$.

Considering the convention: $\forall n \in \mathbb{N}^*$, $n + \aleph 0 = \aleph 0$, φ is a *morphism* of semigroups of $(Injns(E), \circ)$ on $(M, +)$.

$$\forall f, g \in Injns(E), \varphi(f \circ g) = |A_{f \circ g}| = |A_f| + |A_g| = \varphi(f) + \varphi(g)$$

Complement

Let $f, g \in Injns(E)$ so that $|A_f| < |A_g|$, with the assumption that A_f and A_g are considered finite, I_f and I_g are equipotential because, the map ψ defined from I_f to I_g by:

$\forall x \in I_f$, $\psi(x) = (g \circ f^{-1})(x)$ is bijective, where f^{-1} is defined from I_f to *E*, and for all $x \in I_f$, we associated its inputs by *f*, we have:

- $\forall x, y \in I_f$, $\psi(x) = \psi(y)$ $\Rightarrow$ $(g \circ f^{-1})(x) = (g \circ f^{-1})(y)$ $\Rightarrow$ $g(f^{-1}(x)) = g(f^{-1}(y))$ $\Rightarrow$ $f^{-1}(x) = f^{-1}(y) \Rightarrow x = y$ (because both *g* and f^{-1} are injectives). Therefore ψ is injective.
- On the other hand $\psi(I_f) = (g \circ f^{-1})(I_f) = g(f^{-1}(I_f)) = g(E) = I_g$ so ψ is surjective.

Let $\varphi \in Injns(A_f, A_g)$ so that the map θ is defined by:

$$\theta(x) = \begin{cases} \varphi(x), & \text{if } x \in A_f \\ \psi(x), & \text{if } x \in I_f \end{cases}$$

Belong to $Injns(E)$ and $A_\theta = A_\varphi$.

Note 8

- If $g = f^2$ so $\forall x \in I_f, \psi(x) = f(x)$ so if $\varphi = f/A_f$ then we have, $\theta = f$, and if $\varphi = I_d/A_f$ so $\theta = \tilde{f}$
- We considered, previously, that, A_f and A_g are finite. That said, we can build the φ map even in case that A_f and A_g are infinite and that $|A_f| = |A_g|$.

5. Part IV

E Permutations Group Action

Let $\sigma(E)$ be the permutations group of *E*, $f \in Injns(E)$ and $\sigma \in \sigma(E)$, $f \circ \sigma \in Injns(E)$, because, since *f* and σ are injective then $f \circ \sigma$ is injective and $\forall \sigma \in \sigma(E)$, $(f \circ \sigma)(E) = f(\sigma(E)) = f(E) = I_f \subsetneq E$ then $f \circ \sigma$ is non-surjective.

where $\forall \sigma \in \sigma(E), I_{f\circ\sigma} = I_f$ then $A_{f\circ\sigma} = A_f$.

We have,

- $\forall \sigma_1, \sigma_2 \in \sigma(E)$ and $f \in Injns(E)$: $(f \circ \sigma_1) \circ \sigma_2 = f \circ (\sigma_1 \circ \sigma_2)$
- $\forall f \in Injns(E)$, $f \circ I_d = f$

Let $\theta : \sigma(E) \times Injns(E) \to Injns(E)$

so that $\forall \sigma \in \sigma(E)$, $\forall f \in Injns(E)$, $\theta(\sigma, f) = f \circ \sigma$

Therefore, the E permutations group operates on the rightmost on $Injns(E)$.

Note 9

The relation R defined on $Injns(E)$ by: $fRg \Leftrightarrow \exists \sigma \in \sigma(E)$ such as $g = f \circ \sigma$ is an equivalence relation, that is called Intransitivity relation [5].

Proposition 7

Let be, $g \in Injns(E)$: $\exists \sigma \in \sigma(E)$ so that $g = f \circ \sigma \Leftrightarrow I_f = I_g$.

Proof

$\Rightarrow$) If there exists $\sigma \in \sigma(E)$ so that $g = f \circ \sigma \Rightarrow I_g = I_{f\circ\sigma} = I_f$.

$\Leftarrow$) If $I_f = I_g$, then the map $\sigma : E \to E$, so that $x \mapsto \sigma(x) = f^{-1}(g(x))$ is bijective, because:

- $\sigma(E) = f^{-1}(g(E)) = E$ so σ is surjective.
- $\forall x, y \in E$: $\sigma(x) = \sigma(y) \Rightarrow f^{-1}(g(x)) = f^{-1}(g(y)) \Rightarrow g(x) = g(y)$ $\Rightarrow x = y$ so σ is injective.
- $\forall x \in E$, $f \circ \sigma(x) = f(\sigma(x)) = g(x)$

Note 10

- The equivalence class (Intransitivity relation) of the element *f* is called the orbit of *f*, $Cl(f) = \{f \circ \sigma \mid \sigma \in \sigma(E)\} = \{g \in Injns(E) \mid I_g = I_f\}$.
- The stabilizer of *f* is: $\Delta f = \{\sigma \in \sigma(E) \mid f \circ \sigma = f\} = \{I_d\}$, *i.e.* the morphism associated with the said action is injective.

6. Part V

Let $f, g \in Injns(E)$.

Definition 3

f and *g* are said to be Co-injectives, if,

$$I_f \cap I_g \neq \varnothing \text{ and } \forall x, y \in E, f(x) = g(y) \Rightarrow x = y$$

Let $\hat{f} = \{g \in Injns(E) \mid g \text{ is Co-injective with } f\}$

We have $\hat{f} \neq \varnothing$ because $\forall f \in Injns(E)$, $I_f \cap I_f \neq \varnothing$ and $\forall x, y \in E$, $f(x) = f(y) \Rightarrow x = y$ so *f* is Co-injective with itself.

Therefore $\forall f \in Injns(E)$, $f \in \hat{f}$, then, $\forall f \in Injns(E)$: $\hat{f} \neq \phi$

Proposition 8

Let $h \in Injns(E)$, $\forall f, g \in Injns(E)$:

f, g are Co-injectives $\Rightarrow$ $h \circ f$ and $h \circ g$ are Co-injectives.

Proof:

Let $h \in Injns(E)$, and $f, g \in Injns(E)$ such as *f* and *g* are Co-injective. $I_f \cap I_g \neq \varnothing$, additionally $h(I_f \cap I_g) = I_{h\circ f} \cap I_{h\circ g} \neq \varnothing$ (*h* is injective).

Let $x, y \in E$, $h \circ f(x) = h \circ g(y) \Rightarrow f(x) = g(y) \Rightarrow x = y$ (because *h* is injective and f, g are Co-injectives).

Therefore $h \circ f$ and $h \circ g$ are Co-injectives.

Note 11

For all $f, g \in Injns(E)$, so that f, g are Co-injectives, so:

- f^2 and $f \circ g$ are Co-injectives
- $\forall z \in I_f \cap I_g$, $\exists! x \in E$, so that, $z = f(x) = g(x)$
- $f^{-1}(I_g) = g^{-1}(I_f)$
- If, $I_f = I_g$, then $f = g$
- Let $h \in Injns(E)$, if $f(I_h) \cap g(I_h) \neq \varnothing$, then $f \circ h$ and $g \circ h$ are Co-injectives.

Proposition 9

$\forall f, g \in Injns(E)$, if $I_f \subsetneq I_g$, then f, g are not Co-injective.

Proof

$\forall f, g \in Injns(E)$, suppose that $I_f \subsetneq I_g$, then we have $I_g = I_f \cup (I_g \setminus I_f)$. If f, g are Co-injectives, then $f^{-1}(I_g) = g^{-1}(I_f) = E$, which is contradictory because:

$$E = g^{-1}(I_f) \cup g^{-1}(I_g \setminus I_f), \text{ and } g^{-1}(I_g \setminus I_f) \neq \varnothing, \text{ so } g^{-1}(I_f) \subsetneq E.$$

Proposition 10

Let $f, g \in Injns(E)$, so that *f* and *g* are Co-injectives, $\forall h \in Injns(E)$, if *g* and *h* are Co-injectives and $I_f \cap I_h = I_f \cap I_g$ then *f* and *h* are Co-injectives.

Proof

$I_f \cap I_h \neq \varnothing$, let $x, y \in E$, so that $f(x) = h(y)$. We have $f(x) = h(y) \in I_f \cap I_h = I_f \cap I_g$, then $f(x) = g(x) = h(y)$ (because f, g are Co-injectives), and since *g* and *h* are Co-injectives then $x = y$. So $\forall x, y \in E$, $f(x) = h(y) \Rightarrow x = y$. Therefore f, h are Co-injectives.

Acknowledgements

I would like to thank my dear friends, especially Chafik Bouazzaoui for us discussing the Cantor-Bernstein theorem and Mai Mohammed mainly who for his help with translating this document.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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