Open Journal of Discrete Mathematics, 2025, 15(3), 55-71
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

On the Monoid of All Order Preserving Full Contractions with a Fixed Set

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How to cite this paper: Zubairu, M.M. and Lawan, Z.D. (2025) On the Monoid of All Order Preserving Full Contractions with a Fixed Set. *Open Journal of Discrete Mathematics*, **15**, 55-71.
https://doi.org/10.4236/ojdm.2025.153004

Received: April 17, 2025
Accepted: July 1, 2025
Published: July 4, 2025

Abstract

In this paper, we consider the monoid $\mathrm{Fix}\mathcal{OCT}([n],D)$ of all order-preserving full contraction mappings that fix a subset say D of a finite n-element chain $\{1,2,\cdots,n\}$. We characterize regularity, Green's relations, and starred Green's relations, and show that this monoid is left adequate. Furthermore, we determine the cardinality of $\mathrm{Fix}\mathcal{OCT}([n],D)$, the ranks of its two-sided ideals, and demonstrate that the ranks of the two-sided ideals and their corresponding Rees quotients are equal. Moreover, we deduce the rank of the monoid $\mathrm{Fix}\mathcal{OCT}([n],D)$.

Keywords

Contraction Map, Order Preserving, Fixed Set, Rank

1. Introduction and Preliminaries

Denote $[n]$ to be a finite n-chain $\{1,2,\cdots,n\}$. A map that has both its domain and range subsets of $[n]$ is said to be a *transformation* of the set $[n]$. A transformation that has its domain subset of $[n]$ is said to be *partial*. The collection of all partial transformations on $[n]$ is known as *the semigroup of partial transformations* and is usually denoted by $\mathcal{P}_n$. A partial transformation whose domain is equal to $[n]$ is said to be a *full* (or *total*) transformation. The collection of all full transformations on $[n]$ is known as *the semigroup of full transformations*, which is usually denoted by $\mathcal{T}_n$. A map $\alpha \in \mathcal{T}_n$ is said to be *order preserving* if (for all $x, y \in [n]$) $x \le y$ implies $x\alpha \le y\alpha$. The collection of all order preserving full transformations on $[n]$ is known as the *semigroup of order preserving full transformations* and is usually denoted by $\mathcal{O}_n$. The algebraic, combinatorial and the rank properties of the semigroup $\mathcal{O}_n$ have been extensively studied over the years, see for example [1]-[11].

Let D be a nonempty subset of $[n]$. Denote by $\text{Fix}\mathcal{T}([n],D)$ to be the collection of all $\alpha\in\mathcal{T}_n$ that fix elements of D that is, $\text{Fix}\mathcal{T}([n],D)=\{\alpha\in\mathcal{T}_n : a\alpha=a, \text{ for all } a\in D\}$. The set $\text{Fix}\mathcal{T}([n],D)$ is known as *the semigroup of transformations with a fixed set* under the usual composition of functions. This semigroup first appeared in [12], where its algebraic, combinatorial and rank properties were investigated. In particular, the authors in [12] investigated its Green's relations, ideals, isomorphisms and rank properties. Later, Chaiya *et al.* [13] characterized all the maximal subsemigroups of $\text{Fix}\mathcal{T}([n],D)$, while the natural partial order on $\text{Fix}\mathcal{T}([n],D)$ was investigated in [14]. There are various semigroups of transformations with fixed sets that have been studied by various authors, see for example [15] [16]. For basic concepts in semigroup theory, we refer the reader to the books [17]-[19].

A map $\alpha\in\mathcal{T}_n$ is said to be an *isometry* if for all $x,y\in[n]$, $|x\alpha-y\alpha|=|x-y|$; and is said to be a *contraction* if for all $x,y\in[n]$, $|x\alpha-y\alpha|\le|x-y|$. Let

$$\mathcal{CT}_n=\{\alpha\in\mathcal{T}_n : (\text{for all } x,y\in[n])\,|x\alpha-y\alpha|\le|x-y|\} \quad (1)$$

be *the semigroup of all of full contractions on* $[n]$ and

$$\mathcal{OCT}_n=\{\alpha\in\mathcal{CT}_n : (\text{for all } x,y\in[n])\, x\le y\Rightarrow x\alpha\le y\alpha\} \quad (2)$$

be *the semigroup of all order-preserving full contractions of* $[n]$. A general study of these semigroups was initiated in 2013 by Umar and Al-Kharousi [20], in a research proposal to the Research Council of Oman. In this proposal, notations for these semigroups and their various subesmigroups were given. We are adopting the same notations for these semigroups in this paper.

The combinatorial properties of the semigroup $\mathcal{OCT}_n$ have been investigated by Adeshola and Umar [21], where they showed that the order of $\mathcal{OCT}_n$ is $(n+1)2^{n-2}$. Ali *et al.* [22] characterized both the regular elements and Green's relations of the semigroups $\mathcal{CT}_n$ and $\mathcal{OCT}_n$. Moreover, they proved that the collection of all regular elements of $\mathcal{OCT}_n$ (denoted by $\text{Reg}\mathcal{OCT}_n$) is a subsemigroup, and also showed that the Rees quotient of the two-sided ideal of $\text{Reg}\mathcal{OCT}_n$ is an inverse semigroup. Recently, Toker [23] investigated the rank properties of $\mathcal{OCT}_n$.

For a nonempty subset D of $[n]$, let

$$\text{Fix}\mathcal{OCT}([n],D)=\{\alpha\in\mathcal{OCT}_n : d\alpha=d, \text{ for all } d\in D\} \quad (3)$$

be *the monoid of all order preserving full contraction mappings with a fixed set* D. It is straightforward to show that $\text{Fix}\mathcal{OCT}([n],D)$ is a subsemigroup of $\mathcal{OCT}_n$. Moreover, it is evident that $\text{id}_{[n]}\in\text{Fix}\mathcal{OCT}([n],D)$ (where $\text{id}_{[n]}$ denotes the identity map on $[n]$) for every nonempty subset $D\subseteq[n]$. Hence, $\text{Fix}\mathcal{OCT}([n],D)$ is a *monoid*. Notice that if $\{1,n\}\subseteq D$, then $\text{Fix}\mathcal{OCT}([n],D)=\{id_{[n]}\}\cong\mathbb{Z}_1$ (*i.e.*, $\text{Fix}\mathcal{OCT}([n],D)$ is the trivial group). Thus, for the rest of the paper, we will only consider case where D does not contain

$\{1,n\}$. Furthermore, let $a=\max D$ and $b=\min D$, and let $\bar{D}=\{x\in[n]:a\le x\le b\}$. Then, we shall refer to $\bar{D}$ as *closure* of D.

Now, for $1\le r\le p\le n-1$, let

$$L(n-r,p)=\{\alpha\in \mathrm{Fix}\mathcal{OCT}([n],D):r\le|\mathrm{Im}\,\alpha|\le p\} \tag{4}$$

be the two-sided ideal of $\mathrm{Fix}\mathcal{OCT}([n],D)$, and let

$$J_p^*=\{\alpha\in \mathrm{Fix}\mathcal{OCT}([n],D):|\mathrm{Im}\,\alpha|=p\} \tag{5}$$

be the collection of elements in $\mathrm{Fix}\mathcal{OCT}([n],D)$ of height p. Evidently, $J_r^*\cup J_{r+1}^*\cup\cdots\cup J_p^*=L(n-r,p)$. Additionally, let

$$F(n,p)=|J_p^*|. \tag{6}$$

Furthermore, for $p>1$, let

$$D_n(p)=L(n-r,p)/L(n-r,p-1) \tag{7}$$

be the Rees quotient semigroup of $L(n-r,p)$, where the product of two elements α and β in $D_n(p)$ is zero if their product has height less than p, otherwise it is $\alpha\beta$.

The monoid $\mathrm{Fix}\mathcal{OCT}([n],D)$ does not seem to have been studied in the literature. In this paper, we investigate the algebraic, combinatorial and rank properties of this monoid. Section 1 of this paper contains the definitions of basic terms. In Section 2, we give a characterization of regular elements as well as a necessary and sufficient condition for the monoid $\mathrm{Fix}\mathcal{OCT}([n],D)$ to be regular. Moreover, we characterize all the Green's relations and starred Green's relations on the monoid $\mathrm{Fix}\mathcal{OCT}([n],D)$. In Section 3, we compute the cardinality and the rank of $\mathrm{Fix}\mathcal{OCT}([n],D)$. Finally, we study certain isomorphism properties on the monoid.

In line with [17], every transformation $\alpha\in \mathrm{Fix}\mathcal{OCT}([n],D)$ can be written as

$$\alpha=\begin{pmatrix}A_1 & \cdots & A_p\\ x_1 & \cdots & x_p\end{pmatrix}(1\le p\le n), \tag{8}$$

where A_i $(1\le i\le p)$ (also referred to as *blocks*) are equivalence classes under the relation $\ker\alpha=\{(x,y)\in[n]\times[n]:x\alpha=y\alpha\}$. The collection of all the equivalence classes of the relation $\ker\alpha$, is the partition of $[n]$ usually denoted by $\mathbf{Ker}\,\alpha$, *i.e.*, $\mathbf{Ker}\,\alpha=\{A_1,\cdots,A_p\}$ ($p\le n$). A subset X of $[n]$ is said to be *convex* if $x\le y$ ($x,y\in X$) and if there exists $z\in[n]$ such that $x<z<y$ implies $z\in X$. Let P be a *partition* of $[n]$, then P is called *convex* if any element E of P is convex. Moreover, a partition P is said to be *ordered* if for all $A_i,A_j\in P$, $A_i<A_j$ if and only if $a<b$ for all $a\in A_i$ and $b\in A_j$. Notice that if P is an ordered partition then its necessarily convex. A subset T_α of $[n]$ is said to be a *transversal* of the partition $\mathbf{Ker}\,\alpha$ if $|T_\alpha|=p$, and $|A_i\cap T_\alpha|=1$ $(1\le i\le p)$.

For any transformation α, we shall denote $\mathrm{Im}\,\alpha$, $h(\alpha)$, $F(\alpha)=\{x\in[n]:x\alpha=x\}$ and $f(\alpha)$ to mean the image set of α, $|\mathrm{Im}\,\alpha|$, the set of fixed points of α and the number of the fixed points of α (*i.e.*, $|F(\alpha)|$), respectively. For $\alpha, \beta \in \mathrm{Fix}\mathcal{OCT}([n], D)$, we shall write the composition of α and β as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all $x \in [n]$.

Let S be a semigroup, a subset A of S is said to be a *generating set of* S, if every element of S can be written as a product of elements of A. If A generates S, we simply write $\langle A \rangle = S$. The *rank* of a semigroup S is defined and denoted by

$$\mathrm{rank}(S) = \min\{|A| : A \subset S, \langle A \rangle = S\}.$$

An element $a \in S$ is said to be an *idempotent* if $a^2 = a$. The collection of all idempotents of S is usually denoted by $E(S)$. It is well known that an element α (where α is expressed as in (8)) in any transformation semigroup is an idempotent if and only if A_i is stationary in the sense that $x_i \in A_i$ for all $1 \le i \le p$.

Before we begin our discussion, the following remark is worth noticing:

Remark 1.1.

1) *For* $n \ge 3$ *and any nonempty subset* D *of* $[n]$ *of order* 1, $\mathrm{Fix}\mathcal{OCT}([n], D) \ne \mathcal{OCT}_n$. *Evidently, for all* $1 \le i \le n-2$ *and* $D = \{i\}$, *the element*

$$\begin{pmatrix} \{1, \cdots, i\} & \{i+1, \cdots, n\} \\ i+1 & i+2 \end{pmatrix}$$

is in $\mathcal{OCT}_n$, *but not in* $\mathrm{Fix}\mathcal{OCT}([n], D)$.

Moreover, if $i = n-1$ *or* $i = n$, *the element* $\alpha = \begin{pmatrix} \{1, \cdots, n-1\} & n \\ n-2 & n-1 \end{pmatrix} \in \mathcal{OCT}_n$, *but does not fix* $n-1$ *and* n, *so* $\alpha \notin \mathrm{Fix}\mathcal{OCT}([n], D)$.

2) *For* $n \ge 4$ *and any nonempty subset* D *of* $[n]$ *of order* $r \ge 2$, $\mathrm{Fix}\mathcal{OCT}([n], D) \ne \mathcal{OCT}_n$.

Notice that for any $D \subseteq [n]$ *with* $\min D = i$ *and* $\max D = i + r - 1$, *the element*

$$\begin{pmatrix} \{1, \cdots, i-1\} & \{i, i+1, \cdots, i+r-1\} & \{i+r, \cdots, n\} \\ i-1 & i & i+1 \end{pmatrix} \in \mathcal{OCT}_n,$$

but not in $\mathrm{Fix}\mathcal{OCT}([n], D)$ *since the element* $i+1$ *is an element of* D *and is not a fixed point, and so,* $\mathrm{Fix}\mathcal{OCT}([n], D) \ne \mathcal{OCT}_n$.

The following Lemma from [21] is needed in our subsequent discussion.

Lemma 1.2. ([21] Lemma 1.2) *Let* $\alpha \in \mathcal{CT}_n$ *and let* $|\mathrm{Im}\,\alpha| = p$. *Then,* $\mathrm{Im}\,\alpha$ *is convex.*

We now have the following lemma.

Lemma 1.3. *If* $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$. *Then,* $a\alpha = a$ *for all* $a \in \overline{D}$. *In other words, if* $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$, *then* α *must fix* $\overline{D}$.

Proof. Let $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$ and let $a = \min D$ and $b = \max D$. Now, suppose by way of contradiction that there exists $c \in \overline{D}$ such that $c\alpha \ne c$. Notice that $a < c < b$. Then, by the order preserving property of α, we must have

$a\alpha < c\alpha < b\alpha$, *i.e.*, $a < c\alpha < b$. Now since $c\alpha \neq c$, then either $c > c\alpha$ or $c\alpha > c$. Thus, we consider the two cases separately.

Case 1. Suppose $c\alpha > c$, *i.e.*, $a < c\alpha < b$. Then, $|a\alpha - c\alpha| = |a - c\alpha| > |a - c|$, contradicting the fact that α is a contraction.

Case 2. Suppose $c\alpha < c$ *i.e.*, $a < c\alpha < c < b$. Then, $|b\alpha - c\alpha| = |b - c\alpha| > |b - c|$, contradicting the fact that α is a contraction. The result now follows. □

The elements in the monoid $\mathrm{Fix}\mathcal{OCT}([n], D)$ have a general expression as in the lemma below.

Lemma 1.4. *Let* $\min D = a + i$ *and* $\max D = a + r + i - 1$ *for some* $1 \le i \le p$ *and for some* $0 \le a \le n - p$, *so that* $\bar{D} = \{a + j : i \le j \le r + i - 1\}$, *where* $r = |\bar{D}|$. *Then, every* $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$ *of height* p *can be expressed as*

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i, \cdots, a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1, \cdots, \max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix}. \quad (9)$$

Proof. Let $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$ be of height $1 \le p \le n$. Then, as in (8), α can be expressed as

$$\begin{pmatrix} A_1 & \cdots & A_p \\ x_1 & \cdots & x_p \end{pmatrix}.$$

Now since α is a contraction, then by Lemma 1.2, $\mathrm{Im}\,\alpha$ is convex *i.e.*, $\mathrm{Im}\,\alpha = \{a+1, \cdots, a+p\}$ for some $0 \le a \le n - p$. Thus, α can be expressed as

$$\begin{pmatrix} A_1 & \cdots & A_p \\ a+1 & \cdots & a+p \end{pmatrix}.$$

Now since $\alpha \in \mathrm{Fix}\mathcal{OCT}([n], D)$, then by Lemma 1.3, α must fix $\bar{D}$. Notice that $\bar{D} = \{a + j : i \le j \le r + i - 1\}$ for some $1 \le i \le p$, and so, $(a + j)\alpha = a + j$ for all $i \le j \le r + i - 1$, where $a + i = \max A_i = \min \bar{D}$ and $a + r + i - 1 = \min A_{i+r-1} = \max \bar{D}$. Thus, α can be expressed as

$$\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i, \cdots, a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1, \cdots, \max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix},$$

as required. □

We note the following remark.

Remark 1.5. *It is worth noting that the block* A_k ($1 \le k \le i - 1$ *and* $i + r \le k \le p$) *can be empty.*

For the purpose of illustrations, take $n = 9$, *i.e.*, $[9] = \{1, \cdots, 9\}$ and $D = \{4, 6\}$. Let $\alpha \in \mathrm{Fix}\mathcal{OCT}([9], \{4, 6\})$ be

$$\begin{pmatrix} 1 & \{2,3\} & 4 & 5 & \{6,7\} & \{8,9\} \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Notice that $r = |\bar{D}| = 3$ and $p = 6$. Moreover, the position of the first element in D is in A_3, *i.e.*, $i = 3$, and so $a = 1$. The block containing the maximum element of $\bar{D}$ is at the position $A_{i+r-1} = A_{3+3-1} = A_5$. Thus, the blocks containing the elements of $\bar{D}$ are $A_3 = \{4\}$, $A_4 = \{5\}$ and $A_5 = \{6,7\}$. Other blocks are $A_1 = \{1\}$, $A_2 = \{2,3\}$ and $A_6 = \{8,9\}$.

Now, consider

$$\alpha = \begin{pmatrix} \{1,2,3,4\} & 5 & \{6,7\} & \{8,9\} \\ 4 & 5 & 6 & 7 \end{pmatrix} \in \mathrm{Fix}\mathcal{OCT}([9],\{4,6\}).$$

Clearly, A_1 is the block containing the minimum element 4 while A_3 contain the maximum element 6 in $\bar{D}$, and so the blocks that contains the elements of $\bar{D}$ are $A_1 = \{1,2,3,4\}$, $A_2 = \{5\}$ and $A_3 = \{6,7\}$. It is worth noting that there is no any block before A_1 and there is one block after A_3, *i.e.*, the block $A_4 = \{8,9\}$.

Moreover, consider

$$\alpha = \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & \{6,7,8,9\} \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix} \in \mathrm{Fix}\mathcal{OCT}([9],\{4,6\}).$$

Notice that the blocks containing the elements in $\bar{D}$ are $A_3 = \{4\}$, $A_4 = \{5\}$ and $A_5 = \{6,7,8,9\}$. While other blocks are $A_1 = \{1\}$ and $A_2 = \{2,3\}$. Obviously, there are two blocks before A_3 and there is no block after A_5.

Furthermore, consider $\alpha \in \mathrm{Fix}\mathcal{OCT}([9],\{4,6\})$ as

$$\begin{pmatrix} \{1,2,3,4\} & 5 & \{6,7,8,9\} \\ 4 & 5 & 6 \end{pmatrix}.$$

$A_1 = \{1,2,3,4\}$ is the block containing the minimum element of $\bar{D}$ while A_3 contain the maximum element. Thus, there is not any block before A_1 and after A_3.

The following is worth remarking.

Remark 1.6. *Clearly if* $\bar{D} = \{a+j : i \le j \le r+i-1\}$, *then* $A_j = \{a_j\}$ $(i+1 \le j \le r+i-2)$.

We now present the following lemma.

Lemma 1.7. *For any* $D \subseteq [n]$, $\mathrm{Fix}\mathcal{OCT}([n],D) = \mathrm{Fix}\mathcal{OCT}([n],\bar{D})$.

Proof. Let $\alpha \in \mathrm{Fix}\mathcal{OCT}([n],D)$. Then, by Lemma 1.3, α fix $\bar{D}$, and so $\alpha \in \mathrm{Fix}\mathcal{OCT}([n],\bar{D})$.

On the other hand, if $\alpha \in \mathrm{Fix}\mathcal{OCT}([n],\bar{D})$, then α fix D (since $D \subseteq \bar{D}$) and therefore $\alpha \in \mathrm{Fix}\mathcal{OCT}([n],D)$. Thus, the result follows.

2. Regularity and Green's Relations on $\mathrm{Fix}\mathcal{OCT}([n],D)$

Let S be a semigroup without identity element and S^1 be a monoid. The five equivalences classes on S known as Green's relations were first introduced by J. A. Green in 1951. The primary aim of defining these relations is to study the structure of a semigroup S. These relations are defined as follows. For $a,b \in S$, $a\mathcal{L}b$ if and only if $S^1a = S^1b$; $a\mathcal{R}b$ if and only if $aS^1 = bS^1$; $a\mathcal{J}b$ if and if $S^1aS^1 = S^1bS^1$. The relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, while the relation $\mathcal{D}$ is a join of the relations $\mathcal{L}$ and $\mathcal{R}$, *i.e.*, $\mathcal{D}=\mathcal{L}\circ\mathcal{R}$. For more details on the properties of Green's relations, we refer the reader to [3] [18] [19].

An element a in a semigroup S is *regular* if there exists $b\in S$ such that $a=aba$. If every element of S is regular, then S is called a *regular semigroup*. If a semigroup with idempotent elements is not regular, then there is need to investigate the regular elements, so as to identify the $\mathcal{L}$-classes and $\mathcal{R}$-classes that contain idempotents. The semigroup $\mathrm{Fix}\mathcal{OCT}([n],D)$ is not regular in general, but can be regular for certain $D\subseteq[n]$, as we are going to discuss below, in this section.

The Greens relations for the semigroup $\mathcal{CT}_n$ and some of its subsemigroups have been investigated in [22]. Here, we also characterize these relations on the semigroup $\mathrm{Fix}\mathcal{OCT}([n],D)$. Throughout this section, we will consider $1\le|D|<[n]$.

We begin our investigation by first noting the following well-known lemmas:

Lemma 2.1. ([22], *Corollary* 44) *Let* $\alpha,\beta\in\mathcal{OCT}_n$ *be as expressed as in* (8). *Then,* $(\alpha,\beta)\in\mathcal{L}$ *if and only* $\operatorname{Im}\alpha=\operatorname{Im}\beta$ *and* $\alpha\overset{\ker}{=}\beta$.

Lemma 2.2. ([22], *Corollary* 45) *Let* $\alpha,\beta\in\mathcal{OCT}_n$ *be expressed as in* (8). *Then,* $(\alpha,\beta)\in\mathcal{R}$ *if and only* $\ker\alpha=\ker\beta$.

Lemma 2.3. ([21], *Lemma* 1.1) *Let* $\alpha\in\mathcal{CT}_n$ *be such that* $f(\alpha)=m$. *Then,* $F(\alpha)=\{i,i+1,\cdots,i+m-1\}$. *Equivalently,* $F(\alpha)$ *is convex.*

Lemma 2.4. *If* $\alpha,\beta\in\mathrm{Fix}\mathcal{OCT}([n],D)$ *such that* $\ker\alpha=\ker\beta$, *then* $\alpha=\beta$.

Proof. Let $\alpha,\beta\in\mathrm{Fix}\mathcal{OCT}([n],D)$ such that $\ker\alpha=\ker\beta$. Suppose α is expressed as in (9), *i.e.*,

$$\alpha=\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix}$$

and let

$$\beta=\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ b+1 & \cdots & b+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & b+p \end{pmatrix}.$$

Thus, by Lemma 1.2, $\operatorname{Im}\beta$ is convex, and therefore $a+i=b+i-1+1$. This implies $a=b$, and as such $\alpha=\beta$, as required. □

From this point onward, we shall let α and β be of the form:

$$\alpha=\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix} \tag{10}$$

and

$$\beta=\begin{pmatrix} B_1 & \cdots & B_{i-1} & \{\min B_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max B_{i+r-1}\} & B_{i+r} & \cdots & B_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix}. \tag{11}$$

Next, we characterize the Green's relations on $\mathrm{Fix}\mathcal{OCT}([n],D)$.

Theorem 2.5. *Let* $\alpha,\beta\in\mathrm{Fix}\mathcal{OCT}([n],D)$ *be expressed as in and* (11), *respec-*

tively. Then,

1) $(\alpha,\beta)\in\mathcal{L}$ *if and only* $\alpha=\beta$;

2) $(\alpha,\beta)\in\mathcal{R}$ *if and only* $\alpha=\beta$.

Proof. 1) Let $\bar{D}=\{a+i,\cdots,a+i+r-1\}$, $|\bar{D}|=r$ and $\alpha,\beta\in\text{Fix}\mathcal{OCT}([n],D)$ be as expressed in (10) and (11) such that $(\alpha,\beta)\in\mathcal{L}$. Since $\alpha,\beta\in\mathcal{OCT}_n$, then by Corollary 2.1, $\text{Im}\,\alpha=\text{Im}\,\beta$ and $\alpha\overset{\ker}{=}\beta$. This means that there is an isometry from A_j to B_j for all $j\in\{1,\cdots,i,i+r-1,\cdots,p\}$. Notice that $1\in A_1$, $1\in B_1$ and since there is an isometry from A_1 to B_1 and from B_1 to A_1, then $A_1=B_1$. Inductively, we see that $A_j=B_j$, for all $j\in\{1,\cdots,i,i+r-1,\cdots,p\}$. Hence, $\alpha=\beta$.

Conversely, suppose $\alpha=\beta$. Now let $\gamma=\text{id}_n$. Clearly, $\text{id}_n\in\text{Fix}\mathcal{OCT}([n],D)$, $\alpha=\gamma\beta$ and $\beta=\gamma\alpha$.

2) The result follows directly from Lemma 2.2 and Lemma 2.4.

□

Consequently, we have the following Corollaries.

Corollary 2.6. *On the monoid* $\text{Fix}\mathcal{OCT}([n],D)$, $\mathcal{L}=\mathcal{R}=\mathcal{J}=\mathcal{D}=\mathcal{H}$.

As a consequence of the above Corollary, we deduce the following characterization of Green's relations on the semigroup $S\in\{D_n(p),L(n-r,p)\}$.

Theorem 2.7. *Let* $S\in\{D_n(p),L(n-r,p)\}$. *Then,* S *is* $\mathcal{J}$*-trivial and therefore, the semigroup* S *is non-regular.*

Next, we deduce the characterization of the regular element in $\text{Fix}\mathcal{OCT}([n],D)$ below:

Corollary 2.8. *Let* $\alpha\in\text{Fix}\mathcal{OCT}([n],D)$ *be as expressed in Equation* (10). *Then,* α *is regular if and only if* α *is an idempotent.*

Proof. The result follows from the fact that $\text{Fix}\mathcal{OCT}([n],D)$ is an $\mathcal{R}$-trivial semigroup. □

Now, as a consequence of Corollary 2.6, we readily have the following result.

Theorem 2.9. *Every* H_α ($\alpha\in E(\text{Fix}\mathcal{OCT}([n],D))$) *is a maximal subgroup of* $\text{Fix}\mathcal{OCT}([n],D)$ *and is isomorphic to the trivial group* $\mathbb{Z}_1$.

The next thing is to investigate when the whole semigroup $\text{Fix}\mathcal{OCT}([n],D)$ is regular. First, we note the following.

Remark 2.10. 1) *Obviously if* $n=1$, *then* $\mathcal{OCT}([1],D)=\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$, *which is a regular semigroup.*

2) *Moreover, if* $n=2$, *then*

$\text{Fix}\mathcal{OCT}([2],\{1\})=\left\{\begin{pmatrix}\{1,2\}\\1\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix}\right\}$ *and*

$\text{Fix}\mathcal{OCT}([2],\{2\})=\left\{\begin{pmatrix}\{1,2\}\\2\end{pmatrix},\begin{pmatrix}1&2\\2&2\end{pmatrix}\right\}$. *Evidently, the semigroup* $S\in\{\text{Fix}\mathcal{OCT}([2],\{1\}),\text{Fix}\mathcal{OCT}([2],\{2\})\}$ *is regular, since every element in* S *is a regular idempotent element.*

3) *However, if* $n=3$, *then* D *is in one of the following forms*

$\{1\},\{2\},\{3\},\{1,2\}$ *or* $\{2,3\}$. *Thus,*

$$\text{Fix}\mathcal{OCT}([3],\{1\})=\left\{\begin{pmatrix}\{1,2,3\}\\1\end{pmatrix},\begin{pmatrix}\{1,2\}&3\\1&2\end{pmatrix},\begin{pmatrix}1&\{2,3\}\\1&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}$$

and

$$\text{Fix}\mathcal{OCT}([3],\{3\})=\left\{\begin{pmatrix}\{1,2,3\}\\3\end{pmatrix},\begin{pmatrix}\{1,2\}&3\\2&3\end{pmatrix},\begin{pmatrix}1&\{2,3\}\\2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}$$

are non-regular monoids due to the fact that the elements $\begin{pmatrix}\{1,2\}&3\\1&2\end{pmatrix}\in\text{Fix}\mathcal{OCT}([3],\{1\})$ *and* $\begin{pmatrix}1&\{2,3\}\\2&3\end{pmatrix}\in\text{Fix}\mathcal{OCT}([3],\{3\})$ *are not regular. Furthermore, the monoids*

$$\text{Fix}\mathcal{OCT}([3],\{2\})=\left\{\begin{pmatrix}\{1,2,3\}\\2\end{pmatrix},\begin{pmatrix}\{1,2\}&3\\2&3\end{pmatrix},\begin{pmatrix}1&\{2,3\}\\1&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\},$$

$$\text{Fix}\mathcal{OCT}([3],\{1,2\})=\left\{\begin{pmatrix}1&\{2,3\}\\1&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}\ \textit{and}$$

$$\text{Fix}\mathcal{OCT}([3],\{2,3\})=\left\{\begin{pmatrix}\{1,2\}&3\\2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\right\}\ \textit{are regular.}$$

Further, we investigate when the whole semigroup $\text{Fix}\mathcal{OCT}([n],D)$ is regular for $n\geq 4$, in the Theorem below.

Theorem 2.11. *For* $n\geq 4$, *the monoid* $\text{Fix}\mathcal{OCT}([n],D)$ *is regular if and only if* $\bar{D}\in\{\{2,\cdots,n-1\},\{1,\cdots,n-1\},\{2,\cdots,n\},\{n\}\}$.

Proof. $\text{Fix}\mathcal{OCT}([n],D)$ is regular if and only if every element in $\text{Fix}\mathcal{OCT}([n],D)$ is regular if and only if every element in $\text{Fix}\mathcal{OCT}([n],D)$ is an idempotent (by Corollary 2.8) if and only if the kernel class of every element in $\text{Fix}\mathcal{OCT}([n],D)$ has a transversal T which is equal to $\bar{D}$ if and only if $\bar{D}=\{2,\cdots,n-1\}$ or $\bar{D}=\{1,\cdots,n-1\}$ or $\bar{D}=\{2,\cdots,n\}$ or $\bar{D}=\{n\}$.

Thus, we have the following remark:

Remark 2.12. *For* $n\geq 4$, *the monoid* $\text{Fix}\mathcal{OCT}([n],D)$ *is not regular if and only if* $\min D\geq 3$ *or* $\max D\leq n-2$.

Starred Green's Relations

There are five starred Green's equivalences defined on a semigroup S, namely $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{D}^*$, $\mathcal{H}^*$ and $\mathcal{J}^*$. In a semigroup S and for $a,b\in S$, $(a,b)\in\mathcal{L}^*$ if and only if $(a,b)\in\mathcal{L}$ in some over semigroup of S say T. The relation $(a,b)\in\mathcal{R}^*$ is defined dually. We shall use the notation $(a,b)\in\mathcal{L}(\mathcal{S})$ to mean $(a,b)\in\mathcal{L}$ in S and similarly, $(a,b)\in\mathcal{L}^*(\mathcal{S})$ to mean $(a,b)\in\mathcal{L}^*$ in S. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ have the following characterizations as in [24]:

$$\mathcal{L}^*(\mathcal{S})=\left\{(a,b):\left(\text{for all } x,y\in S^1\right)ax=ay\Leftrightarrow bx=by\right\}\tag{12}$$

and

$$\mathcal{R}^*(\mathcal{S})=\left\{(a,b):\left(\text{for all } x,y\in S^1\right)xa=ya\Leftrightarrow xb=yb\right\}.\tag{13}$$

The join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is $\mathcal{D}^*$ while their intersection is $\mathcal{H}^*$. For basic definitions of starred Green's relations, we refer the reader to [24] [25]. If a semigroup S is not regular, then there is a need to characterize its starred Green's relations in order to investigate the class to which it belongs. Now, in this section, we shall consider $D\subseteq[n]$ which does not belong to the set $\{\{1,n-1\},\{2,n-1\},\{2,n\},\{1,n\}\}$.

We now record the following result from [19].

Lemma 2.13. ([19], Ex. 2.6 (16)) *Let* $\alpha,\beta\in\mathcal{T}_n$. *Then*

1) $(\alpha,\beta)\in\mathcal{L}$ if and only if $\operatorname{Im}\alpha=\operatorname{Im}\beta$;
2) $(\alpha,\beta)\in\mathcal{R}$ if and only if $\ker\alpha=\ker\beta$;
3) $(\alpha,\beta)\in\mathcal{D}$ if and only if $|\operatorname{Im}\alpha|=|\operatorname{Im}\beta|$;
4) $\mathcal{D}=\mathcal{J}$.

Now, we give the characterization of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the monoid $\operatorname{Fix}\mathcal{OCT}([n],D)$ in the theorems below.

Theorem 2.14. *Let* $\alpha,\beta\in\operatorname{Fix}\mathcal{OCT}([n],D)$ *be expressed as in* (10) *and* (11), *respectively. Then,* $(\alpha,\beta)\in\mathcal{L}^*$ *if and only if* $\operatorname{Im}\alpha=\operatorname{Im}\beta$.

Proof. Suppose $(\alpha,\beta)\in\mathcal{L}^*$. Now define $\gamma:[n]\to[n]$ by

$$x\gamma=\begin{cases}a+1, & \text{if } 1\le x\le a+1;\\ x, & \text{if } a+1<x<a+p;\\ a+p, & \text{if } a+p<x\le n.\end{cases}$$

Then, obviously $\gamma\in\operatorname{Fix}\mathcal{OCT}([n],D)$ and one can easily verify that $\alpha\gamma=\alpha\mathrm{id}_{[n]}$ if and only if $\beta\gamma=\beta\mathrm{id}_{[n]}$ (by (12)). Obviously,

$$\operatorname{Im}\beta\subseteq\{a+1,\cdots,a+p\}=\operatorname{Im}\alpha.$$

Thus, $\operatorname{Im}\alpha\subseteq\operatorname{Im}\beta$. Similarly, one can show that $\operatorname{Im}\beta\subseteq\operatorname{Im}\alpha$. Therefore, $\operatorname{Im}\alpha=\operatorname{Im}\beta$.

Conversely, if $\operatorname{Im}\alpha=\operatorname{Im}\beta$, then by Lemma 2.13, $(\alpha,\beta)\in\mathcal{L}(\mathcal{T}_n)$. Thus, by definition it follows that $(\alpha,\beta)\in\mathcal{L}^A(\operatorname{Fix}\mathcal{OCT}([n],D))$. □

Theorem 2.15. *On the monoid* $\operatorname{Fix}\mathcal{OCT}([n],D)$, $\mathcal{R}^*=\mathcal{R}$.

Proof. The result follows from definition and Theorem 2.5-2). □

Theorem 2.16. *Let* $\alpha,\beta\in \mathit{Fix}\mathcal{OCT}([n],\mathcal{D})$. *Then,* $(\alpha,\beta)\in\mathcal{D}^*$ *if and only if* $\operatorname{Im}\alpha=\operatorname{Im}\beta$.

Proof. Suppose $(\alpha,\beta)\in\mathcal{D}^*$. This means that there exists $\gamma\in\operatorname{Fix}\mathcal{OCT}([n],D)$ such that $(\alpha,\gamma)\in\mathcal{L}^*$ and $(\gamma,\beta)\in\mathcal{R}^*$. Then, by Theorem 2.14, $\operatorname{Im}\alpha=\operatorname{Im}\gamma$ and by Theorem 2.15, $\gamma=\beta$. Thus, $\operatorname{Im}\alpha=\operatorname{Im}\beta$, as required.

Conversely, if $\operatorname{Im}\alpha=\operatorname{Im}\beta$. Then, since $\mathcal{D}^*$ is reflexive, it follows that $(\alpha,\beta)\in\mathcal{D}^*$. □

Now, we have the following remark:

Remark 2.17. *On the semigroup* $S\in\{D_n(p),L(n-r,p)\}$,

1) $\mathcal{R}^*=\mathcal{R}$;
2) $\mathcal{D}^*=\mathcal{L}^*$.

A semigroup S is said to be a *left abundant* (*resp., right abundant*) if both the

$\mathcal{L}^*$-class (resp., $\mathcal{R}^*$-class) contains an idempotent and S is said to be *abundant* if both $\mathcal{L}^*$-class and $\mathcal{R}^*$-class contains an idempotent [24]. A semigroup S is said to be *left quasi-adequate* (resp., *right quasi-adequate*) if it is left abundant (resp., right abundant) and its set of idempotent elements forms a subsemigroup, and it is *quasi-adequate* if it is both left and right quasi-adequate [24].

We now present the following results.

Theorem 2.18. *The monoid* $\mathrm{Fix}\mathcal{OCT}([n],D)$ *is left abundant.*

Proof. Let $\bar{D}=\{a+i,\cdots,a+i+r-1\}$ and $\alpha\in\mathrm{Fix}\mathcal{OCT}([n],D)$. Denote $\mathcal{L}^*_\alpha$ to be the $\mathcal{L}^*$-class of $\alpha\in\mathrm{Fix}\mathcal{OCT}([n],D)$. Now either $\bar{D}\le[n]\setminus\bar{D}$ or $[n]\setminus\bar{D}\le\bar{D}$ or there exist $i\in[n]$ such that $a+i-1<\min\bar{D}<\max\bar{D}<a+i+r$.

Case 1. If $\bar{D}\le[n]\setminus\bar{D}$, then $\bar{D}=\{1,\cdots,r\}$, so that

$$\alpha=\begin{pmatrix}1&\cdots&r-1&A_r&A_{r+1}&\cdots&A_p\\1&\cdots&r-1&r&r+1&\cdots&p\end{pmatrix},$$

where $r=\min A_r$. Thus, define γ as

$$\begin{pmatrix}1&\cdots&r-1&r&\cdots&p-1&\{p,\cdots,n\}\\1&\cdots&r-1&r&\cdots&p-1&p\end{pmatrix}.$$

Obviously, $\gamma\in\mathrm{Fix}\mathcal{OCT}([n],D)$ and clearly $\gamma^2=\gamma$.

Case 2. If $[n]\setminus\bar{D}\le\bar{D}$, then $\bar{D}=\{n-r+1,n-r+2,\cdots,n\}$ so that α is of the form

$$\begin{pmatrix}A_1&\cdots&A_{n-r+1}&n-r+2&\cdots&n\\n-p+1&\cdots&n-r+1&n-r+2&\cdots&n\end{pmatrix},$$

where $n-r+1=\max A_{n-r+1}$. So, define

$$\gamma=\begin{pmatrix}\{1,\cdots,n-p+1\}&n-p+2&\cdots&n\\n-p+1&n-p+2&\cdots&n\end{pmatrix}.$$

Clearly, $\gamma\in E(\mathrm{Fix}\mathcal{OCT}([n],D))$.

Case 3. If $1<\min D$ and $\max D<n$. Then, define

$$\gamma=\begin{pmatrix}\{1,\cdots,a+1\}&a+2&\cdots&a+p-1&\{a+p,\cdots,n\}\\a+1&a+2&\cdots&a+p-1&a+p\end{pmatrix}.$$

The element γ is clearly in $E(\mathrm{Fix}\mathcal{OCT}([n],D))$. Hence, in all the three cases $\mathrm{Im}\,\alpha=\mathrm{Im}\,\gamma$ and by Theorem 2.14, $(\alpha,\gamma)\in\mathcal{L}^*$, as such $\gamma\in\mathcal{L}^*_\alpha$, as required. □

Theorem 2.19. *For* $n\ge4$, *the monoid* $\mathrm{Fix}\mathcal{OCT}([n],D)$ *is not right abundant.*

Proof. If $\alpha\in\mathrm{Fix}\mathcal{OCT}([n],D)$, then obviously by Lemma 2.4, $\mathcal{R}^*_\alpha=\{\alpha\}$. □

The next lemma shows that the collection of idempotents in $\mathrm{Fix}\mathcal{OCT}([n],D)$ is a subsemigroup of $\mathrm{Fix}\mathcal{OCT}([n],D)$.

Lemma 2.20. $E(\mathrm{Fix}\mathcal{OCT}([n],D))$ *is a semilattice.*

Proof. The proof is the same as the proof of (11], Theorem 7). □

Consequently, we have proved the following theorem.

Theorem 2.21. *The monoid* $\text{Fix}\mathcal{OCT}([n],D)$ *is left adequate.*

3. The Combinatorial and Rank Properties of $\text{Fix}\mathcal{OCT}([n],D)$

In this section, we determine the cardinality and rank of the monoid $\text{Fix}\mathcal{OCT}([n],D)$. These properties heavily depend on the size of the subset D, as we shall see below.

We now present the following proposition, which counts the number of elements each of height p in $\text{Fix}\mathcal{OCT}([n],D)$.

Proposition 3.1. *Let* $D \subseteq [n]$ *such that* $|\bar{D}| = r$, *and let* $F(n,p)$ *be as defined in* (6). *Then, for* $1 \le r \le p \le n$

$$F(n,p) = \binom{n-r}{p-r}.$$

Proof. To count the number of elements of height p in $\text{Fix}\mathcal{OCT}([n],D)$, first is to partition the set $[n] \setminus \bar{D}$ (which obviously has $n-r$ elements since $|\bar{D}| = r$) into $p-r$ parts (since the rank of each of the elements is p). This is equivalent to selecting $p-r$ elements out of $n-r$ elements. The result now follows. □

Theorem 3.2. *Let* $D = \{i, \cdots, i+r-1\} \subseteq [n]$ *such that* $|\bar{D}| = r$ $(1 \le r < n)$. *Then*

$$|\text{Fix}\mathcal{OCT}([n],D)| = 2^{n-r}$$

Proof. Using Proposition 3.1, we see that

$$\begin{aligned} |Fix\mathcal{OCT}([n],\mathcal{D})| &= \sum_{p=r}^{n-r} F(n,p) = \sum_{p=r}^{n-r} \binom{n-r}{p-r} \\ &= \sum_{p=r}^{n-r} \binom{(i-1)+n-(i+r-1)}{p-r} \\ &= 2^{n-r}. \end{aligned}$$

□

Rank of $\text{Fix}\mathcal{OCT}([n],D)$

The semigroup $S \in \{\text{Fix}\mathcal{OCT}([n],D), D_n(p), L(n-r,p)\}$ is $\mathcal{J}$-trivial (from Corollary 2.6 and Theorem 2.7) and therefore, in line with [26], it admits a minimum generating set. Now, let J_p^* be defined as in (5). The next result shows that the collection of all the elements in J_p^* is the minimum generating set of $D_n(p)$.

Lemma 3.3. *Let* $\alpha, \beta \in D_n(p)$. *Then,* $\alpha\beta \in J_p^*$ *if and only if* $\alpha, \beta \in J_p^*$ *and* $\alpha\beta = \alpha$.

Proof. Let $\alpha, \beta \in D_n(p)$ and suppose $\alpha\beta \in J_p^*$. Thus, $h(\alpha) = h(\beta) = p$, as such $\alpha, \beta \in J_p^*$. Moreover, $h(\alpha\beta) = p$ implies $\text{Im}\,\alpha$ is one of the transversals of

$\ker\beta$ and $\ker\alpha\beta=\ker\alpha$. Thus, by Lemma 2.4 we have $\alpha\beta=\alpha$.

The converse is obvious. □

From Proposition 3.1, we record the following remark:

Remark 3.4. *For each* $r\le p\le n-1$, $\left|J_p^*\right|=\binom{n-r}{p-r}$.

We now present the following theorem:

Theorem 3.5. *Let* $D_n(p)$ *be as defined in* (7). *Then, the rank of* $D_n(p)$ *is given by*:

$$\operatorname{rank}\left(D_n(p)\right)=\binom{n-r}{p-r}.$$

Proof. Notice that J_p^* is the minimum generating set, as stated by Lemma 3.3, and it's order is given by Remark 3.4. □

The following lemma plays a crucial role in determining the rank of $L(n-r,p)$.

Lemma 3.6. *If* $\alpha\in J_p^*$ *then* $\alpha\in\left\langle J_{p+1}^*\right\rangle$ *for* $(1\le r\le p\le n-2)$.

Proof. Let $\alpha\in J_p^*$ be as expressed in (10). Now since $p\le n-2$, it means that there exists $A_k\in\mathbf{Ker}\,\alpha$ for some $k\in\{1,\cdots,i,i+r-1,\cdots,p\}$ such that $a+k\in A_k$ is a fixed point of α, *i.e.*, $(a+k)\alpha=(a+k)$. Now either $(a+k)=\min A_k$ or $a+k=\max A_k$ or $\min A_k<a+k<\max A_k$. We consider the following cases:

Case 1: If $a+k=\min A_k$. Then, $\max\bar{D}\le a+k$. We may without loss of generality suppose $\bar{D}=\{a+i,\cdots,a+i+r-1\}$, where $i+r-1\le k$. If $i+r-1=k$, then α is of the form

$$\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+k-1 & \{a+k,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p \end{pmatrix}.$$

In this case, define

$$\beta=\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+k-1 & a+k & \{a+k+1,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+k-1 & a+k & a+k+1 & a+k+2 & \cdots & a+p+1 \end{pmatrix}$$

and

$$\gamma=\begin{pmatrix} a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+k-1 & \{a+k,a+k+1\} & a+k+2 & \cdots & a+p+1 & \{a+p+2,\cdots,n\} \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p & a+p+1 \end{pmatrix}.$$

Now if $i+r-1<k$. Then, α has the form

$$\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r-1 & a+i+r & \cdots & a+k-1 & \{a+k,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r-1 & a+i+r & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p \end{pmatrix},$$

and so β and γ are defined as

$$\begin{pmatrix} A_1 & \cdots & A_{i-1} & \{\min A_i,\cdots,a+i\} & a+i+1 & \cdots & a+i+r & \cdots & a+k & \{a+k+1,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+i+r & \cdots & a+k & a+k+1 & a+k+2 & \cdots & a+p+1 \end{pmatrix}$$

and

$$\begin{pmatrix} a+1 & \cdots & a+i & \cdots & a+i+r & \cdots & a+k-1 & \{a+k,a+k+1\} & a+k+2 & \cdots & a+p+1 & \{a+p+2,\cdots,n\} \\ a+1 & \cdots & a+i & \cdots & a+i+r & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p & a+p+1 \end{pmatrix},$$

respectively.

Case 2. If $a+k=\max A_k$. Then, $a+k\le\min\bar{D}$. We may without loss of generality suppose $\bar{D}=\{a+i,\cdots,a+i+r-1\}$, where $a+k\le a+i$. If $a+i=a+k$ then α is of the form

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k\} & a+k+1 & \cdots & a+k+r-1 & \cdots & A_p \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+k+r-1 & \cdots & a+p \end{pmatrix}.$$

Now, define

$$\beta=\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k-1\} & a+k & \cdots & a+k+r-1 & \cdots & A_p \\ a & \cdots & a+k-2 & a+k-1 & a+k & \cdots & a+k+r-1 & \cdots & a+p \end{pmatrix},$$

and

$$\gamma=\begin{pmatrix} \{1,\cdots,a\} & \cdots & a+k-2 & \{a+k-1,a+k\} & a+k+1 & \cdots & a+k+r-1 & \cdots & a+p & \{a+p+1,\cdots,n\} \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+k+r-1 & \cdots & a+p & a+p+1 \end{pmatrix}.$$

If $a+k<a+i$ and $|A_i|\ge 2$ (in this case $\bar{D}=\{a+i\}$), then α has the form

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k\} & a+k+1 & \cdots & a+i-1 & \{a+i,\cdots,\max A_i\} & A_{i+1} & \cdots & A_p \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+p \end{pmatrix}.$$

and so β and γ are defined as

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k-1\} & a+k & \cdots & a+i-1 & \{a+i,\cdots,\max A_i\} & A_{i+1} & \cdots & A_p \\ a & \cdots & a+k-2 & a+k-1 & a+k & \cdots & a+i-1 & a+i & a+i+1 & \cdots & a+p \end{pmatrix}$$

and

$$\begin{pmatrix} \{1,\cdots,a\} & \cdots & a+k-2 & \{a+k-1,a+k\} & a+k+1 & \cdots & a+i & \cdots & a+p & \{a+p+1,\cdots,n\} \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+i & \cdots & a+p & a+p+1 \end{pmatrix},$$

respectively.

Also, if $a+k<a+i$ and $A_i=\{a+i\}$, then α is of the form:

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k\} & a+k+1 & \cdots & a+i & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+i & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix}.$$

Define β and γ as

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k-1\} & a+k & \cdots & a+i-1 & a+i & \cdots & a+i+r-2 & \{a+i+r-1,\cdots,\max A_{i+r-1}\} & A_{i+r} & \cdots & A_p \\ a & \cdots & a+k-2 & a+k-1 & a+k & \cdots & a+i-1 & a+i & \cdots & a+i+r-2 & a+i+r-1 & a+i+r & \cdots & a+p \end{pmatrix}$$

and

$$\begin{pmatrix} \{1,\cdots,a\} & \cdots & a+k-2 & \{a+k-1,a+k\} & a+k+1 & \cdots & a+i & \cdots & a+p & \{a+p+1,\cdots,n\} \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+i & \cdots & a+p & a+p+1 \end{pmatrix},$$

respectively.

Case 3: If $\min A_k<a+k<\max A_k$, then D must be singleton $\{a+k\}$. Thus, α is of the form

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p \end{pmatrix}.$$

Now, define β and γ as

$$\begin{pmatrix} A_1 & \cdots & A_{k-1} & \{\min A_k,\cdots,a+k\} & \{a+k+1,\cdots,\max A_k\} & A_{k+1} & \cdots & A_p \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & a+k+2 & \cdots & a+p+1 \end{pmatrix}$$

and

$$\begin{pmatrix} \{1,\cdots,a+1\} & \cdots & a+k-1 & \{a+k,a+k+1\} & a+k+2 & \cdots & a+p & a+p+1 & \{a+p+2,\cdots,n\} \\ a+1 & \cdots & a+k-1 & a+k & a+k+1 & \cdots & a+p-1 & a+p & a+p+1 \end{pmatrix}.$$

Now, clearly in each case, $\beta,\gamma \in J_{p+1}$ and also, $\beta\gamma = \alpha$. The proof is now complete. □

Consequently, we obtain the rank of the two-sided ideal $L(n-r,p)$ as stated in the following result:

Theorem 3.7. *Let* $L(n-r,p)$ *be as defined in* (4), *and let* $D_n(p)$ *be as defined in* (7). *Then, the* $\operatorname{rank}(L(n-r,p)) = \binom{n-r}{p-r}$.

Proof. The result follows from Lemma 3.6, Theorem 3.5 and Lemma 3.3. □

Next, we deduce the following result.

Corollary 3.8. *Let* $D \subseteq [n]$ *such that* $|\overline{D}| = r$, *and* $\mathrm{Fix}\mathcal{OCT}([n],D)$ *be as expressed in equation* (3). *Then,* $\operatorname{rank}(\mathrm{Fix}\mathcal{OCT}([n],D)) = n-r+1$.

Proof. The result follows from Theorem 3.7 and the fact that

$$\operatorname{rank}(\mathrm{Fix}\mathcal{OCT}([n],D)) = \operatorname{rank}(L(n-r,n-1)) + |\mathrm{id}_{[n]}|.$$ □

We conclude the paper with the following isomorphism result:

Theorem 3.9. *Let* D_1 *and* D_2 *be nonempty subsets of* $[n]$. *Then,*

$$\mathrm{Fix}\mathcal{OCT}([n],D_1) \cong \mathrm{Fix}\mathcal{OCT}([n],D_2)$$

if and only if $\overline{D}_1 = \overline{D}_2$.

Proof. The result follows easily from Lemma 1.7.

Acknowledgements

The authors would like to thank Prof. Abdullahi Umar for technical and financial support.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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