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k-Product Cordial Labeling of Path Graphs

Robinson Santrin Sabibha[1], Kruz Jeya Daisy[2], Pon Jeyanthi[3], Maged Zakaria Youssef[4,5]

[1]Department of Science and Humanities, Vins Christian College of Engineering, Nagercoil, India
[2]Department of Mathematics, Holy Cross College, Nagercoil, India
[3]Research Centre, Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur, India
[4]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh, Saudi Arabia
[5]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
Email: sanithazhi@gmail.com, jeyadaisy@holycrossngl.edu.in, jeyajeyanthi@rediffmail.com, mzyoussef11566@yahoo.com

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Abstract

In 2012, Ponraj *et al.* defined a concept of k-product cordial labeling as follows: Let f be a map from $V(G)$ to $\{0,1,\cdots,k-1\}$ where k is an integer, $1 \le k \le |V(G)|$. For each edge uv assign the label $f(u)f(v)(\bmod k)$. f is called a k-product cordial labeling if $|v_f(i)-v_f(j)| \le 1$, and $|e_f(i)-e_f(j)| \le 1$, $i,j \in \{0,1,\cdots,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x=0,1,\cdots,k-1$). Motivated by this concept, we further studied and established that several families of graphs admit k-product cordial labeling. In this paper, we show that the path graphs P_n admit k-product cordial labeling.

Keywords

Cordial Labeling, Product Cordial Labeling, *k*-Product Cordial Labeling, Path Graph

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [1]. While studying graph theory, one that has gained a lot of popularity during the last 60 years is the concept of labelings of graphs due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [2] published a pioneering paper on graph labeling problems. Thereafter, several authors have studied many types of graph labeling techniques. Gallian in his survey [3] beautifully classified them. Cordial labeling is one such labelings defined by Cahit [4] as follows: Let *f* be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. *f* is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. This concept was extended and defined product cordial labeling by Sundaram *et al*. [5] as follows: Let *f* be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then *f* is called product cordial labeling if $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i=0,1)$. Some of the researchers have shown interest in this topic and published their results. An interested reader can refer to [6]-[11].

Ponraj *et al*. [12] further extended the concept of product cordial labeling and defined a new labeling called k-product cordial labeling. The same authors [13] established that the 4-product cordial behaviour of graphs such as subdivision star, wheel, K_2+mK_1, $K_{2,n}$ and $K_n^c+2K_2$. For further results on 3-product and 4-product cordial labeling one can refer to [3]. Inspired by the concept of k-product cordial labeling and the results in [12], we further studied and showed that several families of graphs admit k-product cordial labeling in our published papers [14]-[21]. In [22], it is proved that P_n is 3-product cordial for every positive integer *n*. In this paper, we made an attempt to characterize the values of positive integer $k\ge 2$ for which the path graph P_n is k-product cordial for every positive integer *n*.

2. Terminology and Definitions

We use the following terminology and definitions to prove our main results.

The pigeonhole principle is, if *m* pigeons occupy *n* pigeonholes and $m>n$, then at least one pigeonhole has two or more pigeons roosting in it [23].

Let *x* be any real number. Then $\lfloor x\rfloor$ stands for the largest integer less than or equal to *x* and $\lceil x\rceil$ stands for the smallest integer greater than or equal to *x*.

Definition 1. An integer $p>1$ is called a prime number if 1 and *p* are the only divisor of *p*.

Definition 2. Let *f* be a map from $V(G)$ to $\{0,1,\cdots,k-1\}$ where *k* is an integer, $1\le k\le|V(G)|$. For each edge uv assign the label $f(u)f(v)(\bmod k)$. *f* is called a k-product cordial labeling if $|v_f(i)-v_f(j)|\le 1$, and $|e_f(i)-e_f(j)|\le 1$, $i,j\in\{0,1,\cdots,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with *x* ($x=0,1,\cdots,k-1$).

3. Main Results

In this section, we study the k-product cordial labeling of path graph P_n where $k=5,6,7,10,11$ and 15. Also, we show that not all paths are p^2-product cordial when *p* is odd prime. In all the results, we consider the vertex and edge set of P_n be $V(P_n)=\{v_i;1\le i\le n\}$ and $E(P_n)=\{(v_i,v_{i+1});1\le i\le n-1\}$ respectively.

Theorem 3. For $n\ge 3$, the path P_n is 5-product cordial.

Proof. Define $f: V(P_n) \rightarrow \{0,1,2,3,4\}$ as follows:

$$f(v_i) = 0;\ 1 \le i \le \left\lfloor \frac{n}{5} \right\rfloor.$$

For $i = \left\lfloor \frac{n}{5} \right\rfloor + j; 1 \le j \le n - \left\lfloor \frac{n}{5} \right\rfloor$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,6 \pmod 8 \\ 1 & \text{if } j \equiv 2,5 \pmod 8 \\ 2 & \text{if } j \equiv 3,7 \pmod 8 \\ 3 & \text{if } j \equiv 4,0 \pmod 8 \end{cases}$$

From the above labeling pattern, we have the following cases.

Case (i): If $n \equiv 1 \pmod 5$,

$$e_f(i) = \left\lfloor \frac{n}{5} \right\rfloor \text{ for } 0 \le i \le 4.$$

For n is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,2,3,4 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 1 \end{cases}$

For n is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,1,2,3 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 4 \end{cases}$

Hence, P_n is a 5-product cordial graph if $n \equiv 1 \pmod 5$.

Case (ii): If $n \equiv 2 \pmod 5$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,2,3 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 1,4 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,1,2,3 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 4 \end{cases}$$

Hence, P_n is a 5-product cordial graph if $n \equiv 2 \pmod 5$.

Case (iii): If $n \equiv 3 \pmod 5$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,3 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 1,2,4 \end{cases}$$

For n is even, $e_f(i) = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor & \text{if } i = 0,1,2 \\ \left\lfloor \frac{n}{5} \right\rfloor + 1 & \text{if } i = 3,4 \end{cases}$

For n is odd, $e_f(i)=\begin{cases}\left\lfloor \frac{n}{5}\right\rfloor & \text{if } i=0,1,3\\ \left\lfloor \frac{n}{5}\right\rfloor+1 & \text{if } i=2,4\end{cases}$

Hence, P_n is a 5-product cordial graph if $n\equiv 3(\bmod 5)$.

Case (iv): If $n\equiv 4(\bmod 5)$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{5}\right\rfloor & \text{if } i=0\\ \left\lfloor \frac{n}{5}\right\rfloor+1 & \text{if } i=1,2,3,4\end{cases}$$

For n is even, $e_f(i)=\begin{cases}\left\lfloor \frac{n}{5}\right\rfloor & \text{if } i=0,3\\ \left\lfloor \frac{n}{5}\right\rfloor+1 & \text{if } i=1,2,4\end{cases}$

For n is odd, $e_f(i)=\begin{cases}\left\lfloor \frac{n}{5}\right\rfloor & \text{if } i=0,2\\ \left\lfloor \frac{n}{5}\right\rfloor+1 & \text{if } i=1,3,4\end{cases}$

Hence, P_n is a 5-product cordial graph if $n\equiv 4(\bmod 5)$.

Case (v): If $n\equiv 0(\bmod 5)$,

$$v_f(i)=\frac{n}{5} \quad \text{for} \quad 0\le i\le 4.$$

For n is even, $e_f(i)=\begin{cases}\frac{n}{5}-1 & \text{if } i=2\\ \frac{n}{5} & \text{if } i=0,1,3,4\end{cases}$

For n is odd, $e_f(i)=\begin{cases}\frac{n}{5}-1 & \text{if } i=3\\ \frac{n}{5} & \text{if } i=0,1,2,4\end{cases}$

Hence, P_n is a 5-product cordial graph if $n\equiv 0(\bmod 5)$. □

An example of 5-product cordial labeling of P_{12} is shown in **Figure 1**.

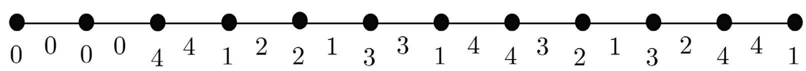

Figure 1. 5-product cordial labeling of P_{12}.

Theorem 4. For $n\ge 3$, the path P_n is 7-product cordial.

Proof. Define $f:V(P_n)\to\{0,1,2,3,4,5,6\}$ as follows:

$$f(v_i)=0;\ 1\le i\le \left\lfloor \frac{n}{7}\right\rfloor.$$

For $i=\left\lfloor \frac{n}{7}\right\rfloor+j;1\le j\le n-\left\lfloor \frac{n}{7}\right\rfloor$,

$$f(v_i)=\begin{cases}6 & \text{if } j\equiv 1,8(\text{mod}12)\\ 1 & \text{if } j\equiv 2,7(\text{mod}12)\\ 2 & \text{if } j\equiv 3,10(\text{mod}12)\\ 5 & \text{if } j\equiv 4,9(\text{mod}12)\\ 3 & \text{if } j\equiv 5,0(\text{mod}12)\\ 4 & \text{if } j\equiv 6,11(\text{mod}12)\end{cases}$$

From the above labeling pattern, we have the following cases.

Case (i): If $n\equiv 1(\text{mod}7)$,

$$e_f(i)=\left\lfloor\frac{n}{7}\right\rfloor \text{ for } 0\le i\le 6.$$

For n is even, $v_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,2,3,4,5,6\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=1\end{cases}$

For n is odd, $v_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,1,2,3,4,5\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=6\end{cases}$

Hence, P_n is a 7-product cordial graph if $n\equiv 1(\text{mod}7)$.

Case (ii): If $n\equiv 2(\text{mod}7)$.

$$v_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,2,3,4,5\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=1,6\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,1,2,3,4,5\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=6\end{cases}$$

Hence, P_n is a 7-product cordial graph if $n\equiv 2(\text{mod}7)$.

Case (iii): If $n\equiv 3(\text{mod}7)$,

$$e_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,1,3,4,5\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=2,6\end{cases}$$

For n is odd, $v_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,3,4,5\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=1,2,6\end{cases}$

For n is even, $v_f(i)=\begin{cases}\left\lfloor\frac{n}{7}\right\rfloor & \text{if } i=0,2,3,4\\ \left\lfloor\frac{n}{7}\right\rfloor+1 & \text{if } i=1,5,6\end{cases}$

Hence, P_n is a 7-product cordial graph if $n \equiv 3 \pmod 7$.

Case (iv): If $n \equiv 4 \pmod 7$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,3,4 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,5,6 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,1,4,5 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 2,3,6 \end{cases}$$

Hence, P_n is a 7-product cordial graph if $n \equiv 4 \pmod 7$.

Case (v): If $n \equiv 5 \pmod 7$,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,4,5 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,3,6 \end{cases}$$

For n is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,4 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,3,5,6 \end{cases}$

For n is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,3 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,4,5,6 \end{cases}$

Hence, P_n is a 7-product cordial graph if $n \equiv 5 \pmod 7$.

Case (vi): If $n \equiv 6 \pmod 7$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,3,4,5,6 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{7} \right\rfloor & \text{if } i = 0,4 \\ \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } i = 1,2,3,5,6 \end{cases}$$

Hence, P_n is a 7-product cordial graph if $n \equiv 6 \pmod 7$.

Case (vii): If $n \equiv 0 \pmod 7$,

$$v_f(i) = \frac{n}{7} \quad \text{for} \quad 0 \le i \le 6, \quad e_f(i) = \begin{cases} \frac{n}{7} - 1 & \text{if } i = 4 \\ \frac{n}{7} & \text{if } i = 0,1,2,3,5,6 \end{cases}$$

Hence, P_n is a 7-product cordial graph if $n \equiv 0 \pmod 7$. □

An example of 7-product cordial labeling of P_{11} is shown in **Figure 2**.

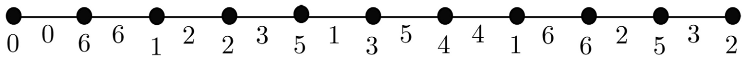

Figure 2. 7-product cordial labeling of P_{11}.

Theorem 5. For $n \geq 3$, the path P_n is 11-product cordial.

Proof. Define $f : V(P_n) \to \{0,1,2,\cdots,10\}$ as follows:

$$f(v_i) = 0;\ \ 1 \leq i \leq \left\lfloor \frac{n}{11} \right\rfloor.$$

For $i = \left\lfloor \frac{n}{11} \right\rfloor + j; 1 \leq j \leq n - \left\lfloor \frac{n}{11} \right\rfloor$,

$$f(v_i) = \begin{cases} 10 & \text{if } j \equiv 1,12 \pmod{20} \\ 1 & \text{if } j \equiv 2,11 \pmod{20} \\ 9 & \text{if } j \equiv 3,14 \pmod{20} \\ 2 & \text{if } j \equiv 4,13 \pmod{20} \\ 7 & \text{if } j \equiv 5,16 \pmod{20} \\ 4 & \text{if } j \equiv 6,15 \pmod{20} \\ 3 & \text{if } j \equiv 7,18 \pmod{20} \\ 8 & \text{if } j \equiv 8,17 \pmod{20} \\ 6 & \text{if } j \equiv 9,0 \pmod{20} \\ 5 & \text{if } j \equiv 10,19 \pmod{20} \end{cases}$$

From the above labeling pattern, we have the following cases.

Case (i): If $n \equiv 1 \pmod{11}$,

$$e_f(i) = \left\lfloor \frac{n}{11} \right\rfloor \text{ for } 0 \leq i \leq 10.$$

For n is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,2,3,4,5,6,7,8,9,10 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1 \end{cases}$

For n is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,1,2,3,4,5,6,7,8,9 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 10 \end{cases}$

Hence, P_n is a 11-product cordial graph if $n \equiv 1 \pmod{11}$.

Case (ii): If $n \equiv 2 \pmod{11}$.

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,2,3,4,5,6,7,8,9 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,10 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,1,2,3,4,5,6,7,8,9 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 10 \end{cases}$$

Hence, P_n is a 11-product cordial graph if $n \equiv 2 \pmod{11}$.

Case (iii): If $n \equiv 3 \pmod{11}$,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,1,2,3,4,5,6,7,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 9,10 \end{cases}$$

For n is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,3,4,5,6,7,8,9 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,2,10 \end{cases}$

For n is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,2,3,4,5,6,7,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,9,10 \end{cases}$

Hence, P_n is a 11-product cordial graph if $n \equiv 3 \pmod{11}$.

Case (iv): If $n \equiv 4 \pmod{11}$.

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,3,4,5,6,7,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,2,9,10 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,1,2,3,4,5,6,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 7,9,10 \end{cases}$$

Hence, P_n is a 11-product cordial graph if $n \equiv 4 \pmod{11}$.

Case (v): If $n \equiv 5 \pmod{11}$,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,1,2,4,5,6,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 3,7,9,10 \end{cases}$$

For n is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,3,5,6,7,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,2,4,9,10 \end{cases}$

For n is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{11} \right\rfloor & \text{if } i = 0,3,4,5,6,8 \\ \left\lfloor \frac{n}{11} \right\rfloor + 1 & \text{if } i = 1,2,7,9,10 \end{cases}$

Hence, P_n is a 11-product cordial graph if $n \equiv 5 \pmod{11}$.

Case (vi): If $n \equiv 6 \pmod{11}$.

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,3,5,6,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,4,7,9,10\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,1,2,4,5,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=3,6,7,9,10\end{cases}$$

Hence, P_n is a 11-product cordial graph if $n\equiv 6(\text{mod}11)$.

Case (vii): If $n\equiv 7(\text{mod}11)$,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,2,4,5,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,3,6,7,9,10\end{cases}$$

For n is even, $v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,3,5,6\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,4,7,8,9,10\end{cases}$

For n is odd, $v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,5,6,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,7,9,10\end{cases}$

Hence, P_n is a 11-product cordial graph if $n\equiv 7(\text{mod}11)$.

Case (viii): If $n\equiv 8(\text{mod}11)$.

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,5,6\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,7,8,9,10\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,4,5,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,6,7,9,10\end{cases}$$

Hence, P_n is a 11-product cordial graph if $n\equiv 8(\text{mod}11)$.

Case (ix): If $n\equiv 9(\text{mod}11)$,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,5,8\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,6,7,9,10\end{cases}$$

For n is even, $v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,6\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,5,7,8,9,10\end{cases}$

For n is odd, $v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,5\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,6,7,8,9,10\end{cases}$

Hence, P_n is a 11-product cordial graph if $n\equiv 9(\bmod 11)$.

Case (x): If $n\equiv 10(\bmod 11)$.

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } 1\le i\le 10\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{11}\right\rfloor & \text{if } i=0,5\\ \left\lfloor \frac{n}{11}\right\rfloor+1 & \text{if } i=1,2,3,4,6,7,8,9,10\end{cases}$$

Hence, P_n is a 11-product cordial graph if $n\equiv 10(\bmod 11)$.

Case (xi): If $n\equiv 0(\bmod 11)$.

$$v_f(i)=\frac{n}{11} \quad \text{for} \quad 0\le i\le 10.$$

$$e_f(i)=\begin{cases}\frac{n}{11}-1 & \text{if } i=5\\ \frac{n}{11} & \text{if } i=0,1,2,3,4,6,7,8,9,10\end{cases}$$

Hence, P_n is a 11-product cordial graph if $n\equiv 0(\bmod 11)$. □

An example of 11-product cordial labeling of P_8 is shown in **Figure 3**.

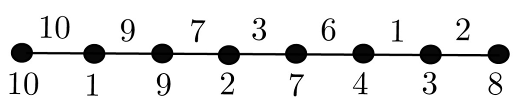

Figure 3. 11-product cordial labeling of P_8.

As a consequence of Theorems 3, 4 and 5 we propose Conjecture 6.

Conjecture 6. For all $n\ge 3$, the path P_n is *k*-product cordial graph if *k* is prime.

Theorem 7. For $n\ge 3$, the path P_n is 6-product cordial.

Proof. Define $f:V(P_n)\to\{0,1,2,3,4,5\}$.

We have the following six cases.

Case (i): If $n\equiv 1(\bmod 6)$,

$$f(v_i)=\begin{cases}0 & \text{if } 1\le i\le \left\lfloor \frac{n}{6}\right\rfloor\\ 3 & \text{if } \left\lfloor \frac{n}{6}\right\rfloor+1\le i\le 2\left\lfloor \frac{n}{6}\right\rfloor\end{cases}$$

For $i=4\left\lfloor \frac{n}{6} \right\rfloor+2-j; 1\le j\le 2\left\lfloor \frac{n}{6} \right\rfloor+1$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor+2-j}\right)=\begin{cases}1 & \text{if } j\equiv 1,0 \pmod 4\\ 5 & \text{if } j\equiv 2,3 \pmod 4\end{cases}$$

For $i=4\left\lfloor \frac{n}{6} \right\rfloor+1+j; 1\le j\le 2\left\lfloor \frac{n}{6} \right\rfloor$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor+1+j}\right)=\begin{cases}4 & \text{if } j\equiv 1,0 \pmod 4\\ 2 & \text{if } j\equiv 2,3 \pmod 4\end{cases}$$

Therefore,

$$e_f(i)=\left\lfloor \frac{n}{6} \right\rfloor \quad \text{for} \quad 0\le i\le 5.$$

For $\left\lfloor \frac{n}{6} \right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{6} \right\rfloor & \text{for } i=0,1,2,3,4\\ \left\lfloor \frac{n}{6} \right\rfloor+1 & \text{for } i=5\end{cases}$$

For $\left\lfloor \frac{n}{6} \right\rfloor$ is even,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{6} \right\rfloor & \text{for } i=0,2,3,4,5\\ \left\lfloor \frac{n}{6} \right\rfloor+1 & \text{for } i=1\end{cases}$$

Hence, P_n is a 6-product cordial graph if $n\equiv 1 \pmod 6$.

Case (ii): If $n\equiv 2 \pmod 6$,

$$f(v_i)=\begin{cases}0 & \text{for } 1\le i\le \left\lfloor \frac{n}{6} \right\rfloor\\ 3 & \text{for } \left\lfloor \frac{n}{6} \right\rfloor+1\le i\le 2\left\lfloor \frac{n}{6} \right\rfloor\end{cases}$$

For $i=4\left\lfloor \frac{n}{6} \right\rfloor+3-j; 1\le j\le 2\left\lfloor \frac{n}{6} \right\rfloor+2$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor+3-j}\right)=\begin{cases}1 & \text{for } j\equiv 1,0 \pmod 4\\ 5 & \text{for } j\equiv 2,3 \pmod 4\end{cases}$$

For $i=4\left\lfloor \frac{n}{6} \right\rfloor+2+j; 1\le j\le 2\left\lfloor \frac{n}{6} \right\rfloor$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor+2+j}\right)=\begin{cases}4 & \text{for } j\equiv 1,0 \pmod 4\\ 2 & \text{for } j\equiv 2,3 \pmod 4\end{cases}$$

Therefore,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,2,3,4\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=1,5\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,1,2,3,4\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=5\end{cases}$$

Hence, P_n is a 6-product cordial graph if $n\equiv 2(\bmod 6)$.

Case (iii): If $n\equiv 3(\bmod 6)$.

Subcase (i): If $\left\lfloor \frac{n}{6}\right\rfloor$ is odd.

We label the vertices $v_i(1\le i\le n-1)$ as in Case (ii), then assign 2 to v_n.

Subcase (ii): If $\left\lfloor \frac{n}{6}\right\rfloor$ is even.

We label the vertices $v_i(1\le i\le n-1)$ as in Case (ii), then assign 4 to v_n.

Therefore,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,1,2,3\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=4,5\end{cases}$$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is even,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,2,3\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=1,4,5\end{cases}$$

For $\left\lfloor \frac{n}{6}\right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,3,4\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=1,2,5\end{cases}$$

Hence, P_n is a 6-product cordial graph if $n\equiv 3(\bmod 6)$.

Case (iv): If $n\equiv 4(\bmod 6)$.

Subcase (i): If $\left\lfloor \frac{n}{6}\right\rfloor$ is odd.

We label the vertices $v_i(1\le i\le n-1)$ as in Case (iii) Subcase (i), then assign 4 to v_n.

Subcase (ii): If $\left\lfloor \frac{n}{6}\right\rfloor$ is even.

We label the vertices $v_i\left(1\le i\le n-1\right)$ as in Case (iii) Subcase (ii), then assign 2 to v_n.

Therefore,

$$v_f\left(i\right)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,3\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=1,2,4,5\end{cases}$$

$$e_f\left(i\right)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,1,3\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=2,4,5\end{cases}$$

Hence, P_n is a 6-product cordial graph if $n\equiv 4\left(\bmod 6\right)$.

Case (v): If $n\equiv 5\left(\bmod 6\right)$,

$$f\left(v_i\right)=\begin{cases}0 & \text{for } 1\le i\le \left\lfloor \frac{n}{6}\right\rfloor\\ 3 & \text{for } \left\lfloor \frac{n}{6}\right\rfloor+1\le i\le 2\left\lfloor \frac{n}{6}\right\rfloor+1\end{cases}$$

For $i=4\left\lfloor \frac{n}{6}\right\rfloor+4-j;1\le j\le 2\left\lfloor \frac{n}{6}\right\rfloor+2$,

$$f\left(v_{4\left\lfloor \frac{n}{6}\right\rfloor+4-j}\right)=\begin{cases}1 & \text{for } j\equiv 1,0\left(\bmod 4\right)\\ 5 & \text{for } j\equiv 2,3\left(\bmod 4\right)\end{cases}$$

For $i=4\left\lfloor \frac{n}{6}\right\rfloor+3+j;1\le j\le 2\left\lfloor \frac{n}{6}\right\rfloor+2$,

$$f\left(v_{4\left\lfloor \frac{n}{6}\right\rfloor+3+j}\right)=\begin{cases}4 & \text{for } j\equiv 1,0\left(\bmod 4\right)\\ 2 & \text{for } j\equiv 2,3\left(\bmod 4\right)\end{cases}$$

Therefore,

$$v_f\left(i\right)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=1,2,3,4,5\end{cases}$$

$$e_f\left(i\right)=\begin{cases}\left\lfloor \frac{n}{6}\right\rfloor & \text{for } i=0,1\\ \left\lfloor \frac{n}{6}\right\rfloor+1 & \text{for } i=2,3,4,5\end{cases}$$

Hence, P_n is a 6-product cordial graph if $n\equiv 5\left(\bmod 6\right)$.

Case (vi): If $n\equiv 0\left(\bmod 6\right)$,

$$f\left(v_i\right)=\begin{cases}0 & \text{if } 1\le i\le \left\lfloor \frac{n}{6}\right\rfloor\\ 3 & \text{if } \left\lfloor \frac{n}{6}\right\rfloor+1\le i\le 2\left\lfloor \frac{n}{6}\right\rfloor\end{cases}$$

For $i = 4\left\lfloor \frac{n}{6} \right\rfloor + 1 - j; 1 \le j \le 2\left\lfloor \frac{n}{6} \right\rfloor$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor + 1 - j}\right) = \begin{cases} 1 & \text{for } j \equiv 1, 0 \pmod 4 \\ 5 & \text{for } j \equiv 2, 3 \pmod 4 \end{cases}$$

For $i = 4\left\lfloor \frac{n}{6} \right\rfloor + j; 1 \le j \le 2\left\lfloor \frac{n}{6} \right\rfloor$,

$$f\left(v_{4\left\lfloor \frac{n}{6} \right\rfloor + j}\right) = \begin{cases} 4 & \text{for } j \equiv 1, 0 \pmod 4 \\ 2 & \text{for } j \equiv 2, 3 \pmod 4 \end{cases}$$

Therefore,

$$v_f(i) = \left\lfloor \frac{n}{6} \right\rfloor \quad \text{for} \quad 0 \le i \le 5,$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor & \text{for } i = 0, 2, 3, 4, 5 \\ \left\lfloor \frac{n}{6} \right\rfloor - 1 & \text{for } i = 1 \end{cases}$$

Hence, P_n is a 6-product cordial graph if $n \equiv 0 \pmod 6$. □

Figure 4 shows the 6-product cordial labeling of P_9.

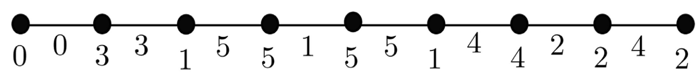

Figure 4. 6-product cordial labeling of P_9.

Theorem 8. For $n \ge 3$, the path P_n is 10-product cordial.

Proof. Define $f : V(P_n) \to \{0,1,2,3,4,5,6,7,8,9\}$.

We have the following four cases.

Case (i): If $n = 10\left\lfloor \frac{n}{10} \right\rfloor + t$ where $t = 1,2,3,4$ then

$$f(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le \left\lfloor \frac{n}{10} \right\rfloor \\ 5 & \text{if } \left\lfloor \frac{n}{10} \right\rfloor + 1 \le i \le 2\left\lfloor \frac{n}{10} \right\rfloor \end{cases}$$

For $i = 6\left\lfloor \frac{n}{10} \right\rfloor + 1 + t - j; 1 \le j \le 4\left\lfloor \frac{n}{10} \right\rfloor + t$ where $t = 1,2,3,4$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor + 1 + t - j}\right) = \begin{cases} 3 & \text{for } j \equiv 1, 6 \pmod 8 \\ 7 & \text{for } j \equiv 2, 5 \pmod 8 \\ 9 & \text{for } j \equiv 3, 7 \pmod 8 \\ 1 & \text{for } j \equiv 4, 0 \pmod 8 \end{cases}$$

For $i = 6\left\lfloor \frac{n}{10} \right\rfloor + t + j; 1 \le j \le 4\left\lfloor \frac{n}{10} \right\rfloor$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor+t+j}\right)=\begin{cases}4 & \text{for } j\equiv 1,6 \pmod 8\\ 6 & \text{for } j\equiv 2,5 \pmod 8\\ 8 & \text{for } j\equiv 3,0 \pmod 8\\ 2 & \text{for } j\equiv 4,7 \pmod 8\end{cases}$$

From the above labeling we have the following subcases:

Subcase (i): If $t=1$,

$$e_f(i)=\left\lfloor \frac{n}{10} \right\rfloor \quad \text{for} \quad 0\le i\le 9 .$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,1,2,4,5,6,7,8,9\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=3\end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,1,2,3,4,5,6,8,9\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=7\end{cases}$$

Subcase (ii): If $t=2$,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,2,3,4,5,6,7,8,9\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=1\end{cases}$$

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,1,2,4,5,6,8,9\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=3,7\end{cases}$$

Subcase (iii): If $t=3$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,1,2,4,5,6,8\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=3,7,9\end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10} \right\rfloor & \text{for } i=0,2,4,5,6,7,8,9\\ \left\lfloor \frac{n}{10} \right\rfloor+1 & \text{for } i=1,3\end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 0,2,3,4,5,6,8,9 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 1,7 \end{cases}$$

Subcase (iv): If $t = 4$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 0,1,2,4,5,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 1,3,7,9 \end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 0,2,4,5,6,7,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 1,3,9 \end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 0,2,3,4,5,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 1,7,9 \end{cases}$$

Hence, P_n is a 10-product cordial graph if $n = 10\left\lfloor \frac{n}{10} \right\rfloor + t$ where $t = 1,2,3,4$.

Case (ii): If $n = 10\left\lfloor \frac{n}{10} \right\rfloor + 5$,

$$f(v_i) = \begin{cases} 0 & \text{for } 1 \le i \le \left\lfloor \frac{n}{10} \right\rfloor \\ 5 & \text{for } \left\lfloor \frac{n}{10} \right\rfloor + 1 \le i \le 2\left\lfloor \frac{n}{10} \right\rfloor + 1 \end{cases}$$

For $i = 6\left\lfloor \frac{n}{10} \right\rfloor + 6 - j; 1 \le j \le 4\left\lfloor \frac{n}{10} \right\rfloor + 4$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor + 6 - j}\right) = \begin{cases} 3 & \text{for } j \equiv 1,6 \pmod 8 \\ 7 & \text{for } j \equiv 2,5 \pmod 8 \\ 9 & \text{for } j \equiv 3,7 \pmod 8 \\ 1 & \text{for } j \equiv 4,0 \pmod 8 \end{cases}$$

For $i = 6\left\lfloor \frac{n}{10} \right\rfloor + 5 + j; 1 \le j \le 4\left\lfloor \frac{n}{10} \right\rfloor$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor+5+j}\right)=\begin{cases}4 & \text{for } j\equiv 1,6 \pmod 8\\ 6 & \text{for } j\equiv 2,5 \pmod 8\\ 8 & \text{for } j\equiv 3,0 \pmod 8\\ 2 & \text{for } j\equiv 4,7 \pmod 8\end{cases}$$

Therefore, $v_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=0,2,4,6,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=1,3,5,7,9\end{cases}$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is even,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=0,2,4,6,7,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=1,3,5,9\end{cases}$$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is odd,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=0,2,3,4,6,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=1,5,7,9\end{cases}$$

Hence, P_n is a 10-product cordial graph if $n=10\left\lfloor \frac{n}{10}\right\rfloor+5$.

Case (iii): If $n=10\left\lfloor \frac{n}{10}\right\rfloor+t$ where $t=6,7,8,9$ then

$$f(v_i)=\begin{cases}0 & \text{if } 1\le i\le \left\lfloor \frac{n}{10}\right\rfloor+1\\ 5 & \text{if } \left\lfloor \frac{n}{10}\right\rfloor+2\le i\le 2\left\lfloor \frac{n}{10}\right\rfloor+2\end{cases}$$

For $i=6\left\lfloor \frac{n}{10}\right\rfloor+7-j; 1\le j\le 4\left\lfloor \frac{n}{10}\right\rfloor+4$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor+7-j}\right)=\begin{cases}3 & \text{for } j\equiv 1,6 \pmod 8\\ 7 & \text{for } j\equiv 2,5 \pmod 8\\ 9 & \text{for } j\equiv 3,7 \pmod 8\\ 1 & \text{for } j\equiv 4,0 \pmod 8\end{cases}$$

For $i=6\left\lfloor \frac{n}{10}\right\rfloor+6+j; 1\le j\le 4\left\lfloor \frac{n}{10}\right\rfloor+t-6$ where $t=6,7,8,9$,

$$f\left(v_{6\left\lfloor \frac{n}{10} \right\rfloor+6+j}\right)=\begin{cases}4 & \text{for } j\equiv 1,6 \pmod 8\\ 6 & \text{for } j\equiv 2,5 \pmod 8\\ 8 & \text{for } j\equiv 3,0 \pmod 8\\ 2 & \text{for } j\equiv 4,7 \pmod 8\end{cases}$$

From the above labeling we have the following subcases:

Subcase (i): If $t = 6$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 2,4,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,3,5,7,9 \end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 2,4,6,7,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,3,5,9 \end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 2,3,4,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,5,7,9 \end{cases}$$

Subcase (ii): If $t = 7$.

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 2,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,3,4,5,7,9 \end{cases},$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 4,6,7,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,2,3,5,9 \end{cases}$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 2,4,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,3,5,6,7,9 \end{cases},$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 3,4,6,8 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,2,5,7,9 \end{cases}$$

Subcase (iii): If $t = 8$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=2,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,3,4,5,6,7,9\end{cases}$$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is even,

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=6,7,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,2,3,4,5,9\end{cases}$$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=3,6,8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,2,4,5,7,9\end{cases}$$

Subcase (iv): If $t=9$.

For $\left\lfloor \frac{n}{10}\right\rfloor$ is even,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=2\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,3,4,5,6,7,8,9\end{cases},$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=6,7\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,2,3,4,5,8,9\end{cases}.$$

For $\left\lfloor \frac{n}{10}\right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=8\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,2,3,4,5,6,7,9\end{cases},$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{10}\right\rfloor & \text{for } i=3,6\\ \left\lfloor \frac{n}{10}\right\rfloor+1 & \text{for } i=0,1,2,4,5,7,8,9\end{cases}.$$

Hence, P_n is a 10-product cordial graph if $n=10\left\lfloor \frac{n}{10}\right\rfloor+t$ where $t=6,7,8,9$.

Case (iv): If $n=10\left\lfloor \frac{n}{10}\right\rfloor$.

Subcase (i): If $\left\lfloor \frac{n}{10} \right\rfloor$ is even.

We label the vertices $v_i \left(1 \le i \le n-1\right)$ as in Case (iii), then assign 8 to v_n.

Subcase (ii): If $\left\lfloor \frac{n}{10} \right\rfloor$ is odd.

We label the vertices $v_i \left(1 \le i \le n-1\right)$ as in Case (iii), then assign 2 to v_n.

From this label we get,

$$v_f\left(i\right) = \left\lfloor \frac{n}{10} \right\rfloor \text{ for } 0 \le i \le 9.$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is odd,

$$e_f\left(i\right) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor - 1 & \text{for } i = 7 \\ \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 0,1,2,3,4,5,6,8,9 \end{cases}.$$

For $\left\lfloor \frac{n}{10} \right\rfloor$ is even,

$$e_f\left(i\right) = \begin{cases} \left\lfloor \frac{n}{10} \right\rfloor & \text{for } i = 3 \\ \left\lfloor \frac{n}{10} \right\rfloor + 1 & \text{for } i = 0,1,2,4,5,6,7,8,9 \end{cases}.$$

Hence, P_n is a 10-product cordial graph if $n = 10\left\lfloor \frac{n}{10} \right\rfloor$. □

Figure 5 shows the 10-product cordial labeling of P_{10}.

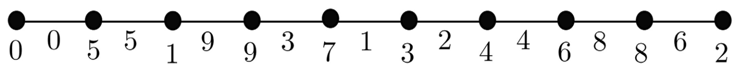

Figure 5. 10-product cordial labeling of P_{10}.

Theorem 9 For $n \ge 3$, the path P_n is 15-product cordial.

Proof. Define $f : V\left(P_n\right) \to \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$.

We have the following five cases.

Case (i): If $n = 15\left\lfloor \frac{n}{15} \right\rfloor + t$ where $1 \le t \le 8$ then

$$f\left(v_i\right) = 0 ; 1 \le i \le \left\lfloor \frac{n}{15} \right\rfloor.$$

For $t = 1,2,3,5$ and $i = 3\left\lfloor \frac{n}{15} \right\rfloor + 1 - j ; 1 \le j \le 2\left\lfloor \frac{n}{15} \right\rfloor$,

$$f\left(v_{3\left\lfloor \frac{n}{15} \right\rfloor + 1 - j}\right) = \begin{cases} 5 & \text{for } j \equiv 1,0 \pmod 4 \\ 10 & \text{for } j \equiv 2,3 \pmod 4 \end{cases}$$

For $t = 4,6,7,8$ and $i = 3\left\lfloor \frac{n}{15} \right\rfloor + 1 - j; 1 \le j \le 2\left\lfloor \frac{n}{15} \right\rfloor$,

$$f\left(v_{3\left\lfloor \frac{n}{15} \right\rfloor + 1 - j}\right) = \begin{cases} 10 & \text{for } j \equiv 1,0 \pmod 4 \\ 5 & \text{for } j \equiv 2,3 \pmod 4 \end{cases}$$

For $i = 11\left\lfloor \frac{n}{15} \right\rfloor + 1 + t - j; 1 \le j \le 8\left\lfloor \frac{n}{15} \right\rfloor + t$ where $1 \le t \le 8$,

$$f\left(v_{11\left\lfloor \frac{n}{15} \right\rfloor + 1 + t - j}\right) = \begin{cases} 2 & \text{for } j \equiv 1 \pmod 8 \\ 8 & \text{for } j \equiv 2 \pmod 8 \\ 11 & \text{for } j \equiv 3 \pmod 8 \\ 13 & \text{for } j \equiv 4 \pmod 8 \\ 14 & \text{for } j \equiv 5 \pmod 8 \\ 4 & \text{for } j \equiv 6 \pmod 8 \\ 1 & \text{for } j \equiv 7 \pmod 8 \\ 7 & \text{for } j \equiv 0 \pmod 8 \end{cases}$$

For $i = 11\left\lfloor \frac{n}{15} \right\rfloor + t + j; 1 \le j \le 4\left\lfloor \frac{n}{15} \right\rfloor$,

$$f\left(v_{11\left\lfloor \frac{n}{15} \right\rfloor + t + j}\right) = \begin{cases} 6 & \text{for } j \equiv 1,6 \pmod 8 \\ 9 & \text{for } j \equiv 2,5 \pmod 8 \\ 12 & \text{for } j \equiv 3,0 \pmod 8 \\ 3 & \text{for } j \equiv 4,7 \pmod 8 \end{cases}$$

From the above labeling we have the following subcases:

Subcase (i): If $t = 1$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,1,3,4,5,6,7,8,9,10,11,12,13,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 2 \end{cases}$$

$$e_f(i) = \left\lfloor \frac{n}{15} \right\rfloor; \quad 0 \le i \le 14.$$

Subcase (ii): If $t = 2$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,1,3,4,5,6,7,9,10,11,12,13,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 2,8 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,2,3,4,5,6,7,8,9,10,11,12,13,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1 \end{cases}$$

Subcase (iii): If $t = 3$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,1,3,4,5,6,7,9,10,12,13,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=2,8,11\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,2,3,4,5,6,7,8,9,10,11,12,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=1,13\end{cases}$$

Subcase (iv): If $t=4$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,1,3,4,5,6,7,9,10,12,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=2,8,11,13\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,2,3,4,5,6,7,9,10,11,12,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=1,13,8\end{cases}$$

Subcase (v): If $t=5$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,1,3,4,5,6,7,9,10,11,12 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=2,8,11,13,14\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,3,4,5,6,7,9,10,11,12,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=1,2,8,13\end{cases}$$

Subcase (vi): If $t=6$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,1,3,5,6,7,9,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=2,4,8,11,13,14\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,3,4,5,6,7,9,10,12,14 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=1,2,8,11,13\end{cases}$$

Subcase (vii): If $t=7$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15} \right\rfloor & \text{if } i=0,3,5,6,7,9,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor+1 & \text{if } i=1,2,4,8,11,13,14\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=0,3,5,6,7,9,10,12,14\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=1,2,4,8,11,13\end{cases}$$

Subcase (viii): If $t=8$,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=0,3,5,6,9,10,12\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=1,2,4,7,8,11,13,14\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=0,3,5,6,9,10,12,14\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=1,2,4,7,8,11,13\end{cases}$$

Therefore, P_n is a 15-product cordial graph if $n=15\left\lfloor \frac{n}{15}\right\rfloor+t$ where $1\le t\le 8$.

Case (ii): If $n=15\left\lfloor \frac{n}{15}\right\rfloor+t$, where $9\le t\le 12$ then

$$f(v_i)=0;1\le i\le\left\lfloor \frac{n}{15}\right\rfloor.$$

For $i=3\left\lfloor \frac{n}{15}\right\rfloor+1-j;1\le j\le 2\left\lfloor \frac{n}{15}\right\rfloor$,

$$f\left(v_{3\left\lfloor \frac{n}{15}\right\rfloor+1-j}\right)=\begin{cases}10 & \text{for } j\equiv 1,0 \pmod 4\\ 5 & \text{for } j\equiv 2,3 \pmod 4\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+9-j;1\le j\le 8\left\lfloor \frac{n}{15}\right\rfloor+8$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+9-j}\right)=\begin{cases}2 & \text{for } j\equiv 1 \pmod 8\\ 8 & \text{for } j\equiv 2 \pmod 8\\ 11 & \text{for } j\equiv 3 \pmod 8\\ 13 & \text{for } j\equiv 4 \pmod 8\\ 14 & \text{for } j\equiv 5 \pmod 8\\ 4 & \text{for } j\equiv 6 \pmod 8\\ 1 & \text{for } j\equiv 7 \pmod 8\\ 7 & \text{for } j\equiv 0 \pmod 8\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+8+j;1\le j\le 4\left\lfloor \frac{n}{15}\right\rfloor+t-8$ where $9\le t\le 12$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+8+j}\right)=\begin{cases}6 & \text{for } j\equiv 1,6 \pmod 8\\ 9 & \text{for } j\equiv 2,5 \pmod 8\\ 12 & \text{for } j\equiv 3,0 \pmod 8\\ 3 & \text{for } j\equiv 4,7 \pmod 8\end{cases}$$

From the above labeling we have the following subcases:

Subcase (i): If $t = 9$,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,6,9,10,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,7,8,11,12,13 \end{cases}$$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,9,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,6,7,8,11,13,14 \end{cases}$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,6,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,7,8,9,11,13,14 \end{cases}$

Subcase (ii): If $t = 10$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,6,7,8,9,11,13,14 \end{cases}$$

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,6,10,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,7,8,9,11,12,13 \end{cases}$$

Subcase (iii): If $t = 11$,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,5,6,10,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,3,4,7,8,9,11,12,13 \end{cases}$$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is even, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,3,5,10 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,4,6,7,8,9,11,12,13,14 \end{cases}$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is odd, $v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,5,10,12 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,3,4,6,7,8,9,11,13,14 \end{cases}$

Subcase (iv): If $t = 12$,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 0,5,10 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 1,2,3,4,6,7,8,9,11,12,13,14 \end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=0,5,10,14\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=1,2,3,4,6,7,8,9,11,12,13\end{cases}$$

Therefore, P_n is a 15-product cordial graph if $n=15\left\lfloor \frac{n}{15}\right\rfloor+t$ where $9\le t\le 12$.

Case (iii): If $n=15\left\lfloor \frac{n}{15}\right\rfloor+13$, then

$$f(v_i)=0;1\le i\le \left\lfloor \frac{n}{15}\right\rfloor+1.$$

For $i=3\left\lfloor \frac{n}{15}\right\rfloor+2-j;1\le j\le 2\left\lfloor \frac{n}{15}\right\rfloor$,

$$f\left(v_{3\left\lfloor \frac{n}{15}\right\rfloor+2-j}\right)=\begin{cases}10 & \text{for } j\equiv 1,0(\bmod 4)\\ 5 & \text{for } j\equiv 2,3(\bmod 4)\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+10-j;1\le j\le 8\left\lfloor \frac{n}{15}\right\rfloor+8$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+10-j}\right)=\begin{cases}2 & \text{for } j\equiv 1(\bmod 8)\\ 8 & \text{for } j\equiv 2(\bmod 8)\\ 11 & \text{for } j\equiv 3(\bmod 8)\\ 13 & \text{for } j\equiv 4(\bmod 8)\\ 14 & \text{for } j\equiv 5(\bmod 8)\\ 4 & \text{for } j\equiv 6(\bmod 8)\\ 1 & \text{for } j\equiv 7(\bmod 8)\\ 7 & \text{for } j\equiv 0(\bmod 8)\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+9+j;1\le j\le 4\left\lfloor \frac{n}{15}\right\rfloor+4$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+9+j}\right)=\begin{cases}6 & \text{for } j\equiv 1,6(\bmod 8)\\ 9 & \text{for } j\equiv 2,5(\bmod 8)\\ 12 & \text{for } j\equiv 3,0(\bmod 8)\\ 3 & \text{for } j\equiv 4,7(\bmod 8)\end{cases}$$

Therefore,

$$v_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=5,10\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=0,1,2,3,4,6,7,8,9,11,12,13,14\end{cases}$$

$$e_f(i)=\begin{cases}\left\lfloor \frac{n}{15}\right\rfloor & \text{if } i=5,10,14\\ \left\lfloor \frac{n}{15}\right\rfloor+1 & \text{if } i=0,1,2,3,4,6,7,8,9,11,12,13\end{cases}$$

Hence, P_n is a 15-product cordial graph if $n = 15\left\lfloor \frac{n}{15} \right\rfloor + 13$.

Case (iv): If $n = 15\left\lfloor \frac{n}{15} \right\rfloor + 14$, then

$$f(v_i) = 0; 1 \le i \le \left\lfloor \frac{n}{15} \right\rfloor + 1.$$

For $i = 3\left\lfloor \frac{n}{15} \right\rfloor + 2 - j; 1 \le j \le 2\left\lfloor \frac{n}{15} \right\rfloor + 1$,

$$f\left(v_{3\left\lfloor \frac{n}{15} \right\rfloor + 2 - j}\right) = \begin{cases} 10 & \text{for } j \equiv 1,0 \pmod 4 \\ 5 & \text{for } j \equiv 2,3 \pmod 4 \end{cases}$$

For $i = 11\left\lfloor \frac{n}{15} \right\rfloor + 11 - j; 1 \le j \le 8\left\lfloor \frac{n}{15} \right\rfloor + 8$,

$$f\left(v_{11\left\lfloor \frac{n}{15} \right\rfloor + 11 - j}\right) = \begin{cases} 2 & \text{for } j \equiv 1 \pmod 8 \\ 8 & \text{for } j \equiv 2 \pmod 8 \\ 11 & \text{for } j \equiv 3 \pmod 8 \\ 13 & \text{for } j \equiv 4 \pmod 8 \\ 14 & \text{for } j \equiv 5 \pmod 8 \\ 4 & \text{for } j \equiv 6 \pmod 8 \\ 1 & \text{for } j \equiv 7 \pmod 8 \\ 7 & \text{for } j \equiv 0 \pmod 8 \end{cases}$$

For $i = 11\left\lfloor \frac{n}{15} \right\rfloor + 10 + j; 1 \le j \le 4\left\lfloor \frac{n}{15} \right\rfloor + 4$,

$$f\left(v_{11\left\lfloor \frac{n}{15} \right\rfloor + 10 + j}\right) = \begin{cases} 6 & \text{for } j \equiv 1,6 \pmod 8 \\ 9 & \text{for } j \equiv 2,5 \pmod 8 \\ 12 & \text{for } j \equiv 3,0 \pmod 8 \\ 3 & \text{for } j \equiv 4,7 \pmod 8 \end{cases}$$

Therefore,

$$e_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 5,14 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 0,1,2,3,4,6,7,8,9,10,11,12,13 \end{cases}$$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is even,

$$v_f(i) = \begin{cases} \left\lfloor \frac{n}{15} \right\rfloor & \text{if } i = 5 \\ \left\lfloor \frac{n}{15} \right\rfloor + 1 & \text{if } i = 0,1,2,3,4,6,7,8,9,10,11,12,13,14 \end{cases}$$

For $\left\lfloor \frac{n}{15} \right\rfloor$ is odd,

$$v_f(i)=\begin{cases}\left\lfloor \dfrac{n}{15}\right\rfloor & \text{if } i=10\\ \left\lfloor \dfrac{n}{15}\right\rfloor+1 & \text{if } i=0,1,2,3,4,5,6,7,8,9,11,12,13,14\end{cases}$$

Hence, P_n is a 15-product cordial graph if $n=15\left\lfloor \frac{n}{15}\right\rfloor+14$.

Case (v): If $n=15\left\lfloor \frac{n}{15}\right\rfloor$, then

$$f(v_i)=0;1\le i\le\left\lfloor \frac{n}{15}\right\rfloor.$$

For $i=3\left\lfloor \frac{n}{15}\right\rfloor+1-j;1\le j\le 2\left\lfloor \frac{n}{15}\right\rfloor$,

$$f\left(v_{3\left\lfloor \frac{n}{15}\right\rfloor+1-j}\right)=\begin{cases}10 & \text{if } j\equiv 1,0(\text{mod}\,4)\\ 5 & \text{if } j\equiv 2,3(\text{mod}\,4)\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+1-j;1\le j\le 8\left\lfloor \frac{n}{15}\right\rfloor$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+1-j}\right)=\begin{cases}2 & \text{for } j\equiv 1(\text{mod}\,8)\\ 8 & \text{for } j\equiv 2(\text{mod}\,8)\\ 11 & \text{for } j\equiv 3(\text{mod}\,8)\\ 13 & \text{for } j\equiv 4(\text{mod}\,8)\\ 14 & \text{for } j\equiv 5(\text{mod}\,8)\\ 4 & \text{for } j\equiv 6(\text{mod}\,8)\\ 1 & \text{for } j\equiv 7(\text{mod}\,8)\\ 7 & \text{for } j\equiv 0(\text{mod}\,8)\end{cases}$$

For $i=11\left\lfloor \frac{n}{15}\right\rfloor+j;1\le j\le 4\left\lfloor \frac{n}{15}\right\rfloor$,

$$f\left(v_{11\left\lfloor \frac{n}{15}\right\rfloor+j}\right)=\begin{cases}6 & \text{for } j\equiv 1,6(\text{mod}\,8)\\ 9 & \text{for } j\equiv 2,5(\text{mod}\,8)\\ 12 & \text{for } j\equiv 3,0(\text{mod}\,8)\\ 3 & \text{for } j\equiv 4,7(\text{mod}\,8)\end{cases}$$

Therefore,

$$v_f(i)=\left\lfloor \frac{n}{15}\right\rfloor;\quad 0\le i\le 14.$$

$$e_f(i)=\begin{cases}\left\lfloor \dfrac{n}{15}\right\rfloor & \text{if } i=14\\ \left\lfloor \dfrac{n}{15}\right\rfloor+1 & \text{if } i=0,1,2,3,4,5,6,7,8,9,10,11,12,13\end{cases}$$

Hence, P_n is a 15-product cordial graph if $n=15\left\lfloor \frac{n}{15}\right\rfloor$. □

Figure 6 shows the 15-product cordial labeling of P_{16}.

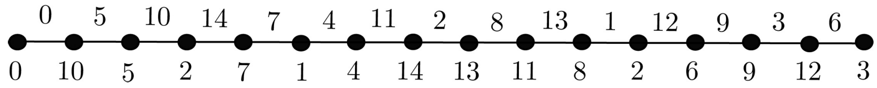

Figure 6. 15-product cordial labeling of P_{16}.

Remark 10. [12] The path P_n is 4-product cordial if and only if $n \le 11$.

Lemma 11. The path P_{2p^2} does not admit p^2-product cordial labeling for every odd prime p.

Proof. Suppose that f is a p^2-product cordial labeling of P_{2p^2}. Then $v_f(i) = 2$ for ($i = 0,1,2,3,\cdots,p^2-1$) and $e_f(i) = 2$ or 1 for ($i = 0,1,2,3,\cdots,p^2-1$). Obviously, $v_f(0) = 2$ and 0 must be assigned consecutively at the beginning or end of the path or beginning and end of the path together. Otherwise $e_f(0) > 2$, which is not possible. Thus, $e_f(0) = 2$. Now $v_f(ip) = 2$ for ($i = 1,2,3,\cdots,p-1$) and each vertex labels ip must be labeled inconsecutively, otherwise $e_f(0) > 2$, which is not possible. These vertex labels with the other vertex labels produce the edge labels ip and ($i = 1,2,3,\cdots,p-1$), so $\sum_{i=1}^{p-1} e_f(ip) \ge 4p-6$ and using the pigeonhole principle, we get at least i from the set $\{p, 2p, \cdots, (p-1)p\}$ such that $e_f(i) > 2$ which contradicts that f is a p^2-product cordial labeling of P_{2p^2}.

As a consequence of Theorems 7, 8, 9, Remark 10 and Lemma 11 we propose the following conjecture:

Conjecture 12. For all $n \ge 3$, the path P_n is k-product cordial graph if k is the product of two distinct prime numbers.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Harary, F. (1972) Graph Theory. Addison-Wesley.

[2] Rosa, A. (1967) On Certain Valuations of the Vertices of a Graph. *Theory of Graphs* (*International Symposium, Rome, July* 1966), Dunod Gordon & Breach Science Publishers, Inc., New York and Dunod Paris, 349-355.

[3] Gallian, J.A. (2021) A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, 1-644.

[4] Cahit, I. (1987) Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs. *Ars Combinatoria*, **23**, 201-207.

[5] Sundaram, M., Ponraj, R. and Somasundaram, S. (2004) Product Cordial Labeling of Graphs. *Bulletin of Pure and Applied Sciences*, **23**, 155-163.

[6] Rokad, A.H. (2019) Product Cordial Labeling of Double Wheel and Double Fan Related Graphs. *Kragujevac Journal of Mathematics*, **43**, 7-13.

[7] Seoud, M.A. and Helmi, E.F. (2011) On Product Cordial Graphs. *Ars Combinatoria*, **101**, 519-529.

[8] Vaidya, S.K. and Barasara, C.M. (2011) Product Cordial Labeling for Some New Graphs. *Journal of Mathematics Research*, **3**, 206-211. https://doi.org/10.5539/jmr.v3n2p206

[9] Vaidya, S.K. and Barasara, C.M. (2016) Product Cordial Labeling of Line Graph of Some Graphs. *Kragujevac Journal of Mathematics*, **40**, 290-297.

[10] Vaidya, S.K. and Kanani, K.K. (2010) Some Cycle Related Product Cordial Graphs. *International Journal of Algorithms*, *Computing and Mathematics*, **3**, 109-116.

[11] Vaidya, S.K. and Kanani, K.K. (2011) Some New Product Cordial Graphs. *Mathematics Today*, **27**, 64-70.

[12] Ponraj, R., Sivakumar, M. and Sundaram, M. (2012) k-Product Cordial Labeling of Graphs. *International Journal of Contemporary Mathematical Sciences*, **7**, 733-742.

[13] Ponraj, R., Sivakumar, M. and Sundaram, M. (2012) On 4-Product Cordial Graphs. *International Journal of Mathematical Archive*, **7**, 2809-2814.

[14] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2022) k-Product Cordial Behaviour of Union of Graphs. *Journal of the Indonesian Mathematical Society*, **28**, 1-7. https://doi.org/10.22342/jims.28.1.1025.1-7

[15] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2022) k-Product Cordial Labeling of Cone Graphs. *International Journal of Mathematical Combinatorics*, **2**, 72-80.

[16] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2022) k-Product Cordial Labeling of Napier Bridge Graphs. *Nepal Journal of Mathematical Sciences*, **3**, 59-70. https://doi.org/10.3126/njmathsci.v3i2.49201

[17] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2024) k-Product Cordial Labeling of Product of Graphs. *Discrete Mathematics*, *Algorithms and Applications*, **16**, Article ID: 2250187. https://doi.org/10.1142/S1793830922501877

[18] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2022) k-Product Cordial Labeling of Powers of Paths. *Jordan Journal of Mathematics and Statistics*, **15**, 911-924.

[19] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2023) k-Product Cordial Labeling of Fan Graphs. *Turkic World Mathematical Society Journal of Applied and Engineering Mathematics*, **13**, 11-20.

[20] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2024) k-Product Cordial Labeling of Splitting Graph of Star Graphs. *Communications in Combinatorics*, *Cryptography & Computer Science*, **1**, 68-80.

[21] Jeya Daisy, K., Santrin Sabibha, R., Jeyanthi, P. and Youssef, M.Z. (2024) Further Results on k-Product Cordial Labeling. *Turkic World Mathematical Society Journal of Applied and Engineering Mathematics*, **14**, 981-990.

[22] Jeyanthi, P. and Maheswari, A. (2012) 3-Product Cordial Labeling. *SUT Journal of Mathematics*, **48**, 231-240.

[23] Grimaldi, R.P. (1994) Discrete and Combinatorial Mathematics: An Applied Introduction. 3rd Edition, Addison-Wesley.