Open Journal of Discrete Mathematics, 2024, 14, 16-27
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

Subplanes of $PG(2,q^3)$ and the Ruled Varieties V_2^5 of $PG(6,q)$

Rita Vincenti

Department of Mathematics and Computer Science, University of Perugia, Perugia, Italy
Email: aliceiw213@gmail.com

How to cite this paper: Vincenti, R. (2024) Subplanes of $PG(2,q^3)$ and the Ruled Varieties V_2^5 of $PG(6,q)$. *Open Journal of Discrete Mathematics*, **14**, 16-27.
https://doi.org/10.4236/ojdm.2024.142003

Received: March 15, 2024
Accepted: April 23, 2024
Published: April 26, 2024

Abstract

In this note we study subplanes of order q of the projective plane $\Pi = PG(2,q^3)$ and the ruled varieties V_2^5 of $\Sigma = PG(6,q)$ using the spatial representation of Π in Σ, by fixing a hyperplane Σ' with a regular spread of planes. First are shown some configurations of the affine q-subplanes. Then to prove that a variety V_2^5 of Σ represents a non-affine subplane of order q of Π, after having shown basic incidence properties of it, such a variety V_2^5 is constructed by choosing appropriately the two directrix curves in two complementary subspaces of Σ. The result can be translated into further incidence properties of the affine points of V_2^5. Then a maximal bundle of varieties V_2^5 having in common one directrix cubic curve is constructed.

Keywords

Finite Geometry, Translation Planes, Spreads, Varieties

1. Introduction

It is known that a projective translation plane Π of order q^r and kernel $F = GF(q)$ can be represented by a $2r$-dimensional projective space $\Sigma = PG(2r,q)$ over F, fixing a hyperplane $\Sigma' = PG(2r-1,q)$ and a spread (partition) $\mathcal{S}$ of Σ' with $(r-1)$-dimensional subspaces (cf. [1] and [2]).

The points of Π are represented by 1) the points of $\Sigma \setminus \Sigma'$ (the affine points) and by 2) the elements of $\mathcal{S}$ (the points at infinity). The lines of Π are represented by 1) the r-dimensional subspaces S of $\Sigma \setminus \Sigma'$ such that $S \cap \Sigma'$ belongs to $\mathcal{S}$ and by 2) the spread $\mathcal{S}$. The translation line l_∞ of Π (the line at infinity) is represented by $\mathcal{S}$ (cf. Lemma 2.6).

If a subplane of Π meets l_∞ in a *subline*, then such a subplane is *affine*, if it meets l_∞ in one point is *non-affine*.

An *affine* subplane $\mathcal{A}$ of order q is represented by every *transversal* plane α to $\mathcal{S}$, that is, by a plane $\alpha \subset \Sigma \setminus \Sigma'$ such that the line $t = \alpha \cap \Sigma'$ meets $q+1$ elements of $\mathcal{S}$, t is a *transversal* line to $\mathcal{S}$. In such a way l_∞ is a line of the projective completion of $\mathcal{A}$. Of course all that holds also in case Π is the Desarguesian plane $PG(2,q^r)$ when $\mathcal{S}$ is a *regular* spread (cf. [1] [2] [3] [4] [5] for $r=2$).

Fix $r=3$ so that $\Pi = PG(2,q^3)$, $\Sigma = PG(6,q)$, $\Sigma' = PG(5,q)$ and $\mathcal{S}$ is a regular spread of planes.

About the affine subplanes of $\Pi = PG(2,q^3)$ of order q (having the same subline at infinity) we prove there exist q^2+q+1 through one fixed affine point, while q^4 partition the affine points of Π (cf. Proposition 2.11, Theorem 2.12).

A variety V_2^5 of Σ is a ruled variety of $PG(6,q)$ with the minimum order directrix a conic and a maximum order directrix a skew cubic in a 3-dimensional subspace, the two curves lying in two complementary spaces (cf. [6], Capter 13, 8., 9.). The variety can be obtained by joining points of the two directrix curves corresponding via a projectivity.

We choose a conic in a plane $\pi_\infty \in \mathcal{S}$ and a cubic in a 3-dimensional subspace $S_0 \in \Sigma \setminus \Sigma'$ with $\pi_0 = S_0 \cap \Sigma' \in \mathcal{S}$ and $\pi_0 \neq \pi_\infty$. Some fundamental incidence properties of V_2^5 are shown (cf. Paragraph 3.1). Then, if $q \equiv 1, (\mathrm{mod}\, 3)$, we can prove that the variety V_2^5 represents a non-affine subplane Π_q of order q of $PG(2,q^3)$ (cf. Theorem 3.6).

The properties of Π_q of being a plane, translate into further incidence properties of the affine points of V_2^5 (cf. Theorem 3.8).

By fixing the 3-subspace S_0 with the chosen directrix cubic curve $\mathcal{C}_0$, a maximal bundle of varieties V_2^5 having in common $\mathcal{C}_0$ is constructed (cf. Theorem 3.9).

At the end is formulated the conjecture that a ruled variety V_2^{2r-1} of $PG(2r,q)$ represents a non affine subplane of order q of $PG(2,q^r)$, via the spatial representation.

2. Preliminary Notes and Results

Let $F = GF(q)$ be a finite field, $q = p^s$, p an odd prime. Denote F^{r+1} the $(r+1)$-dimensional vector space over *F*, $PG(r,q) = PrF^{r+1}$ the *r*-dimensional projective space contraction of F^{r+1} over *F*. Let $\bar{F}$ be the algebraic closure of the field $F = GF(q)$.

The geometry $PG(r,q)$ is considered a sub-geometry of $\overline{PG(r,q)}$, the projective geometry over $\bar{F}$. We refer to the points of $PG(r,q)$ as the *rational points* of $\overline{PG(r,q)}$.

Denote S_h or *h*-space with $1 \le h \le r-1$ a subspace of $PG(r,q)$ of dimension *h*. A hyperplane S_{r-1} will be denoted also by *H*, a plane by π. If $A, B, C, \cdots$

are subspaces denote $A+B+C+\cdots=\langle A,B,C,\cdots\rangle$, the subspace generated by them. More simply, when A,B are points, AB denote the line defined by them.

Definition 2.1 *A variety* V_u^v *of dimension u and of order v of* $PG(r,q)$ *is the set of the rational points of a projective variety* $\bar{V}_u^v$ *of* $\overline{PG(r,q)}$ *defined by a finite set of polynomials with coefficients in the field F.*

From [6], pp. 290, 7., for $r\geq 4$ follows.

Lemma 2.2 *The ruled variety* V_2^{r-1} *of* $PG(r,q)$ *is generated by the lines joining the corresponding points of two birationally (projectively) equivalent curves of order m and* $r-1-m$, *respectively, lying in two complementary subspaces of the same dimensions. As the directrix curves have no point in common, then the number of points of* V_2^{r-1} *is* $(q+1)^2$ *and the order is the sum of the orders of the curves.*

Let Σ' be the projective space $PG(2r-1,q)$ over the field $F=GF(q)$, $r>1$ an integer.

Definition 2.3 *A spread of* Σ' *is a partition* $\mathcal{S}$ *with* $(r-1)$*-dimensional subspaces (that is, every point of* Σ' *lies in one element of* $\mathcal{S}$*). A regulus* $\mathcal{R}$ *of* $\mathcal{S}$ *is a collection of subspaces of* $\mathcal{S}$ *such that:*

1) $\mathcal{R}$ contains at least 3 elements,

2) Every line meeting 3 elements of $\mathcal{R}$, a *transversal line*, meets every element of $\mathcal{R}$,

3) Every point of a transversal line to $\mathcal{R}$ lies in one element of $\mathcal{R}$,

4) Every plane through a transversal line is a *transversal plane* to $\mathcal{S}$.

Any three pairwise disjoint $(r-1)$-dimensional subspaces of $\mathcal{S}$ lie in a unique regulus. Any two distinct transversal lines are skew.

A spread is *regular* if for any three distinct elements of $\mathcal{S}$, all the members of the unique regulus determined by them are in $\mathcal{S}$.

Regular spreads represent Desarguesian planes $PG(2,q^r)$ (cf. [2], pp. 162-163).

The construction of a regular spread can be described as follows.

Choose a coordinate system in Σ' so that for a point P of Σ',
$P\approx(\mathbf{x},\mathbf{y})=(x_1,x_2,\cdots,x_r;y_1,y_2,\cdots,y_r)=F^*(\mathbf{x},\mathbf{y})$, $F^*=F\setminus\{0\}$.

A regular spread of Σ' is given by the set
$\{J_\infty=(\mathbf{0},\mathbf{y})\mid\mathbf{y}\in F^r\}\cup\{J_\mathbf{m}=(\mathbf{x},\mathbf{xm})\mid\mathbf{x},\mathbf{m}\in F^r\}$ where $\mathbf{y}=\mathbf{xm}$ is the multiplication in the field F^r. Such a multiplication can be represented also by $\mathbf{y}=\mathbf{x}M$ with M a $r\times r$ matrix over F so that $\mathbf{x}M=\mathbf{xm}$. The set $\mathcal{M}=\{M\mid\mathbf{x}M=\mathbf{xm}\}$ is a field isomorphic to $(F^r)^2$, acting strictly transitively over F^r.

In case of a projective plane over a skew-field, a spread can be constructed in the same way. The set of matrices is not a field, anyway it operates strictly transitively over F^r (cf. [1] and [2]).

Let Σ be the projective geometry $PG(6,q)$.

Denote π_∞ and S_0 a plane and a 3-space, respectively, of Σ. Assume π_∞ and S_0 are complementary. Let $\mathcal{C}^2$ be an irreducible conic of π_∞, $\mathcal{C}^3$ a skew cubic curve of S_0. The curves $\mathcal{C}^2$ and $\mathcal{C}^3$ are projectively equivalent so that their points can be connected by a projectivity.

Lemma 2.4 *A variety* V_2^5 *of* Σ *is obtained by joining the corresponding points of* $\mathcal{C}^2$ *and of* $\mathcal{C}^3$.

Proof. See [6], p. 291.

Corollary 2.5 *The variety* V_2^5 *consists of* $(q+1)^2$ *points of the* $q+1$ *generatrix lines including the points of the minimum order directrix* $\mathcal{C}^2$ *and of the maximum order directrix* $\mathcal{C}^3$.

Note that the set of the $q+1$ generatrix lines partition the variety.

Choose a hyperplane $\Sigma' = PG(5,q)$ of Σ as the hyperplane at infinity.

Fix a coordinate system in Σ so that it is a coordinate system also for $\bar{\Sigma}$. Denote a point $P \approx (\mathbf{x},\mathbf{y},t) = (x_1,x_2,x_3,y_1,y_2,y_3,t) := \bar{F}^*(\mathbf{x},\mathbf{y},t)$, $\bar{F}^* = \bar{F}\setminus\{0\}$. Let $t=0$ be the equation of Σ'.

P is a *rational point* if there exists $(\mathbf{x},\mathbf{y},t) \in F^7$ such that $P \approx (\mathbf{x},\mathbf{y},t)$.

A variety V of Σ is the set of the rational points of $\bar{\Sigma}$ solutions of a finite set of polynomials of $F[\mathbf{x},\mathbf{y},t]$.

Let $\Pi = PG(2,q^3)$ be the Desarguesian plane over the field $GF(q^3)$. Denote l_∞ the line at infinity of Π. Represent Π in $\Sigma = PG(6,q)$ by a regular spread $\mathcal{S}$ of planes of Σ', with $|\mathcal{S}| = q^3+1$.

More precisely define the following incidence structure $\Pi' = (\mathcal{P},\mathcal{L},\mathcal{I})$ (points, lines, incidence, respectively) where

$$\mathcal{P} = \{P \in \Sigma\setminus\Sigma'\} \cup \{\pi \in \mathcal{S}\},$$

$$\mathcal{L} = \left\{\mathcal{L}_0 = \{S_3 \subset \Sigma\setminus\Sigma' \mid S_3 \cap \Sigma' \in \mathcal{S}\}\right\} \cup \{l_\infty = \mathcal{S}\},$$

$\mathcal{I}$ is defined as follows

if $P \in \Sigma\setminus\Sigma'$, $l \in \mathcal{L}_0$ then $P\mathcal{I}l \Leftrightarrow P \in l$, no point of $\Sigma\setminus\Sigma'$ incides l_∞, $\pi\mathcal{I}l_\infty$ for all $\pi \in \mathcal{S}$, $\pi\mathcal{I}l$ where $l \in \mathcal{L}_0 \Leftrightarrow l \cap \Sigma' = \pi$.

From [1] [2] [3] and [4] pp. 38-39 follows.

Lemma 2.6 $\Pi' \cong \Pi$.

In short, the affine points of Π are represented by the q^6 affine points of $\Sigma\setminus\Sigma'$, the points at infinity by the q^3+1 planes of $\mathcal{S}$. The affine lines of Π are represented by the 3-spaces S of $\Sigma\setminus\Sigma'$ such that the plane $S\cap\Sigma'$ belongs to $\mathcal{S}$, the line at infinity l_∞ by the spread $\mathcal{S}$.

Definition 2.7 *A subplane* $\pi' = (P',L',I')$ *of a plane* $\pi = (P,L,I)$ *is a subgeometry of* π*, that is, an incidence substructure for which* $P' \subset P$, *for each line* $l' \subset L'$, *there exists a line* $l \in L$ *such that* $l' \subset L$ *and* $I' = I$.

Definition 2.8 *A subplane of* $\Pi = PG(2,q^3)$ *of order q is affine if it meets the line* l_∞ *of* Π *in a subline consisting of* $q+1$ *points, it is non-affine if it meets the line* l_∞ *in one point.*

Let r be any transversal line to $\mathcal{S}$. As $\mathcal{S}$ is regular, the $q+1$ planes that r meets form a regulus $\mathcal{R} \subset \mathcal{S}$ (cf. [2], Lemma 12.2). Choose and fix a transversal plane α through the line r.

It is easy to prove the following.

Proposition 2.9 *The plane* α *is isomorphic to a subplane* $\pi \cong PG(2,q)$ *of* Π

whose points at infinity are represented by the $q+1$ *planes of* $\mathcal{R}$ *, the lines of* π *being represented by the sublines intersections of* α *with the 3-spaces of* Σ *through the planes of* $\mathcal{R}$ *. As the line at infinity of* π *is a subline of the infinity line of* Π*, then* π *is an affine subplane.*

To construct transversal lines to $\mathcal{R}$ in Σ' in a synthetic way the procedure is similar to the one used for dimension 3.

Proposition 2.10 *The transversal lines to* $\mathcal{R}$ *are* q^2+q+1.

Proof. Denote π_1,π_2,π_3 three planes of the regulus $\mathcal{R}$. Fix a point $P\in\pi_1$ and denote $S=\langle P,\pi_2\rangle$ the 3-space of Σ' direct sum of P and π_2 and $S'=\langle P,\pi_3\rangle$ the 3-space of Σ' direct sum of P and π_3. Lying in a 5-dimensional subspace, then $S\cap S'=r$ is a line. As a line of *S*, *r* meets π_2 in a point, as a line of S', *r* meets π_3 in a point. Therefore *r* is a transversal line to the planes π_1,π_2,π_3. As π_1,π_2,π_3 belong to the regulus $\mathcal{R}$, the line *r* meets each of the $q+1$ elements of $\mathcal{R}$. In such a way one can construct a transversal line for every point *P* chosen in π_1, that is, q^2+q+1.

Proposition 2.11 *The cardinality of the set of all affine subplanes of* Π *isomorphic to* $PG(2,q)$ *having the same subline of* $q+1$ *points at infinity and containing one affine point is* q^2+q+1.

Proof. Let r_0 be a transversal line to $\mathcal{R}$, $\alpha_0\supset r_0$ a transversal plane, *O* an affine point of α_0. Denote $\{r_i\,|\,i=0,\cdots,q^2+q\}$ the transversal lines to the regulus $\mathcal{R}$. Each of the q^2+q+1 planes $\alpha_i=\langle O,r_i\rangle$ represents an affine subplane π_i of Π, $\pi_i\cong PG(2,q)$ (cf. Proposition 2.9).

Choose and fix a transversal line *r*. Consider the bundle (*r*) of the planes of $\Sigma\setminus\Sigma'$ having the line *r* as axis. Each plane $\alpha\in(r)$ is isomorphic to $PG(2,q)$ (cf. Proposition 2.9) and it is an affine subplane of Π having the same subline of $q+1$ points at infinity.

Theorem 2.12 *The planes of* (*r*) *partition the* q^6 *affine points of* Π.

Proof. The planes of (*r*) through the transversal line *r* are *parallel*, therefore they have no affine point in common otherwise they would coincide. Each such a plane contains q^2 affine points.

To prove the statement it is appropriate to calculate the number of the planes of $\Sigma\setminus\Sigma'$ of the bundle through the line *r*.

It is known that the number of *k*-subspaces in a *n*-space (vector notation) is

$$[n\,|\,k]=\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

The number *h* of the planes through a line in $PG(6,q)\setminus PG(5,q)$ equals the number of 3-spaces in a 4-space, minus the number of the planes through a line in $PG(5,q)$, which equals the number of the planes in a 3-space.

Therefore, after simplification of the two ratios, follows

$$h=\frac{q^5-1}{q-1}-\frac{q^4-1}{q-1}=q^4.$$

More simply, as a line and an independent point define a plane, fixed the line

r, there are q^6 choices for a point in $\Sigma \setminus \Sigma'$ to get the plane $\langle r, P \rangle \subset \Sigma \setminus \Sigma'$, this number to be divided by q^2, which equals the choices of an affine point on a same plane, hence again $h = q^4$.

As each plane of (r) contains q^2 points of $\Sigma \setminus \Sigma'$ and $|(r)| = q^4$, then the total number of points of $\Sigma \setminus \Sigma'$ covered by them is q^6, which are all the affine points of Π.

3. Main Results

3.1. The Variety V_2^5 and Its Sections

Denote $\Sigma = PG(6,q)$, $\Sigma' = PG(5,q)$ a hyperplane of Σ, $\mathcal{S}$ a regular spread of Σ'. Choose and fix a plane $\pi_\infty \in \mathcal{S}$ and a 3-space S_0 in $\Sigma \setminus \Sigma'$ such that $S_0 \cap \Sigma' = \pi_0 \in \mathcal{S} \setminus \pi_\infty$. Then choose and fix:

1) A non-degenerate conic $\mathcal{C}^2 \subset \pi_\infty$,

2) A skew cubic curve $\mathcal{C}^3$ in S_0 such that $\mathcal{C}^3 \cap \pi_0 = \varnothing$ (cf. [7], p. 234, Corollary 4, $\mathcal{N}_5$).

Let $\Lambda : \mathcal{C}^2 \to \mathcal{C}^3$ be a projectivity. Represent $\mathcal{C}^2 = \{G_{i_\infty} \mid i = 1, \cdots, q+1\}$, $\mathcal{C}^3 = \{G_i = \Lambda G_{i_\infty} \mid i = 1, \cdots, q+1\}$. Denote V_2^5 the variety arising by connecting corresponding points of $\mathcal{C}^2$ and $\mathcal{C}^3$ via Λ (cf. [6], p. 291). The curves $\mathcal{C}^2$ and $\mathcal{C}^3$ are directrix curves of V_2^5, the set $\mathcal{G} = \{g_i = G_i G_{i_\infty} \mid i = 1, \cdots, q+1\}$ is the set of the generatrix lines of V_2^5. The set $\mathcal{G}$ partitions the variety.

Let H be any hyperplane. In a suitable complexification of Σ, $H \cap V_2^5$ is a curve of order 5 (cf. [6], p. 288, 5.).

Proposition 3.1 *The variety V_2^5 consists of $q+1$ skew affine generatrix lines and of $q^2 + q$ affine points.*

1) A directrix curve $\mathcal{C} \neq \mathcal{C}^2$ cut by a hyperplane on V_2^5 cannot lie in a plane. The conic $\mathcal{C}^2$ is the unique minimum order 2 directrix.

2) The 3-space joining two generatrix lines cannot contain the plane π_∞.

3) The 3-space joining one generatrix line and the plane π_∞ meets no other generatrix.

4) Three generatrices g_1, g_2, g_3 are joint by a hyperplane H that contains the plane π_∞, so that $H \cap V_2^5 = \{g_1, g_2, g_3\} \cup \mathcal{C}^2$.

5) A hyperplane contains neither a fixed directrix, nor a fixed generatrix.

Proof. Let g, g' be two generatrix lines. Denote $G_\infty = g \cap \pi_\infty$, $G'_\infty = g' \cap \pi_\infty$, with $G_\infty, G'_\infty \in \mathcal{C}^2$, $G = g \cap \mathcal{C}^3$, $G' = g' \cap \mathcal{C}^3$, $G, G' \in S_0$. Let r be the line $G_\infty G'_\infty$, $r \subset \pi_\infty$, r' the line GG', $r' \subset S_0$. Assume g, g' meet in a point. Hence they define a plane $\tau = \langle r, r' \rangle$ so that $r \cap r' = P$ with $P \in \pi_\infty \cap S_0$, a contradiction.

As $\mathcal{C}^3$ has no points in π_0, then all the $q+1$ generatrices are affine and the points of them are affine except the points of $\mathcal{C}^2$, therefore the affine points of V_2^5 are $q(q+1) = q^2 + q$.

1) Assume a hyperplane H meets V_2^5 in a directrix curve $\mathcal{C} \neq \mathcal{C}^2$ lying in a plane τ. Then V_2^5 is contained at most in the 5-space generated by τ and π_∞, a contradiction. The conic $\mathcal{C}^2$ is the unique minimum order 2 directrix,

otherwise the variety generated by the two conics would have order at most 4.

2) Assume there exists a 3-dimensional subspace S containing π_∞ and two generatrix lines, g_1, g_2. Denote $G_i = g_i \cap \mathcal{C}^3$, $i = 1,2$ and $G_1G_2 \cap \Sigma' = G$. The line G_1G_2 belongs to S_0 and to S, so that the point G is a common point of π_∞ and π_0, a contradiction.

3) Let $S = \langle g, \pi_\infty \rangle$, with $g \in \mathcal{G}$, be a 3-space. If $S \cap g' \neq \emptyset$ with $g' \neq g$, $g' \in \mathcal{G}$, then $g' \subset S$, so that S contains two generatrix lines and the plane π_∞, a contradiction to (2).

4) Assume three generatrices g_1, g_2, g_3 are joint by a 4-space S'. As S' contains the three points $G_{1_\infty}, G_{2_\infty}, G_{3_\infty} \in \mathcal{C}^2$, $G_{i\infty} = g_i \cap \mathcal{C}^2$, $i = 1,2,3$, then $\pi_\infty \subset S'$ and $\mathcal{C}^2 \subset S'$. As S' cannot contain V_2^5, a hyperplane $H \supset S'$ and through a further point $P \in V_2^5 \setminus S'$ should contain also the generatrix g_P through P. Hence H would meet V_2^5 in 4 generatrix lines and a conic, that is, in a variety of order 6, a contradiction (cf. [6], p. 288, 5.). Therefore a hyperplane H containing three generatrices g_1, g_2, g_3, contains the non collinear points $G_{1_\infty}, G_{2_\infty}, G_{3_\infty}$ hence the whole plane π_∞ and then the conic directrix $\mathcal{C}^2$. Therefore $H \cap V_2^5 = \{g_1, g_2, g_3\} \cup \mathcal{C}^2$ that is a curve of order 5 (and H contains no further point of V_2^5).

5) Let g, g' two generatrices of V_2^5. Denote S the 3-space containing them. Let H be a generic hyperplane with $H \supset S$ and assume H contains a fix directrix $\mathcal{C}$. Let P be a point of V_2^5, $P \in H \setminus \mathcal{C}$. Denote $S_4 = \langle S, P \rangle$. Then every hyperplane containing S_4 and S_4 itself, would contain the generatrix g_P through P, so that $g, g', g_P \subset S_4$, a contradiction to (4). An analogous contradiction is reached if we assume a generic hyperplane containing g, g' contained a fix generatrix (cf. [6], 6., pp. 289-290).

The following propositions are a *rereading* in the current case of [6], pp. 287-290.

Proposition 3.2 *A hyperplane H containing two generatrices, contains a residual cubic curve* $\mathcal{C}$ *lying in a 3-space* $S \subset H$, *S skew to* π_∞. $\mathcal{C}$ *is irreducible and is a directrix.*

Proof. In [6], 3., p. 287, the 2nd paragraph, is affirmed that a hyperplane H meets V_2^5 in a rational normal curve of order 5 (as it lives in a 5-space) or in a curve of order $m < 5$ met by all the generatrix lines and in $5 - m$ generatrices. In our case it is $m = 2$ or $m = 3$. Set $m = 2$. If a hyperplane H contains the unique conic directrix $\mathcal{C}^2$ (cf. 1), Proposition 3.1), then it must contain $5 - m = 3$ generatrix lines and viceversa (cf. 4), Proposition 3.1).

Set $m = 3$. If a hyperplane contains $5 - m = 2$ generatrix lines, then it meets V_2^5 in a residual cubic curve $\mathcal{C}$ and viceversa.

Assume the cubic $\mathcal{C}$ exists in a plane π. Let H be a hyperplane containing π and three further points $P, Q, R \in V_2^5 \setminus \pi$. Then H contains also the 3 generatrix lines g_P, g_Q, g_R as all the generatrix meet $\mathcal{C}$. In such a way $H \cap V_2^5 \supset \{\mathcal{C}, g_P, g_Q, g_R\}$, that is a curve of order 6, a contradiction. Hence $\mathcal{C}$ lies in a 3-space S.

If such a space $\mathcal{S}$ met π_∞, then a hyperplane $H \supseteq \langle S, \pi_\infty \rangle$ would contain V_2^5, a contradiction, therefore $S \cap \pi_\infty = \varnothing$.

Assume a cubic curve $\mathcal{C}$ is reducible. Of course, the unique possibility is that $\mathcal{C}$ consists of at most 3 generatrix lines. In such a case $\mathcal{C}$ would meet the conic $\mathcal{C}^2$ and then the plane π_∞, a contradiction with $S \cap \pi_\infty = \varnothing$.

Each irreducible curve of order 3 lying in V_2^5, meets each generatrix lines (as they partition V_2^5) that is, it is a directrix curve.

Corollary 3.3 *All the directrix cubic curves are obtaining by cutting* V_2^5 *with the hyperplanes through any two generatrix lines. The maximal hyperplane section of* V_2^5 *consists* $4q+1$ *points.*

Proof. An irreducible cubic curve $\mathcal{C} \subset V_2^5$ is a *rational normal curve* that is, it lies in a 3-space $\mathcal{S}$ (cf. Proposition 3.2). If $\mathcal{C} \subset V_2^5$ is a cubic curve, for any two generatrix lines $g, g' \in \mathcal{G}$ is uniquely defined the hyperplane $H = \langle g, g', S \rangle$.

Let H be a hyperplane. If $H \cap V_2^5$ were an irreducible curve of order 5, then $\left| H \cap V_2^5 \right| = q+1$. If $H \cap V_2^5 = \{ g_1, g_2, \mathcal{C}^3 \}$, $g_1, g_2 \in \mathcal{G}$, then $\left| H \cap V_2^5 \right| = 3q+1$. If $H \cap V_2^5 = \{ g_1, g_2, g_3, \mathcal{C}^2 \}$, $g_1, g_2, g_3 \in \mathcal{G}$, then $\left| H \cap V_2^5 \right| = 4q+1$.

Proposition 3.4 1) *No two directrix cubic curves belong to a same 3-space.*

2) *Two directrix cubic curves meet in at most one point.*

Proof. 1) Assume two directrix cubic curves $\mathcal{C}, \mathcal{C}' \subset V_2^5$ belong to a same 3-space $\mathcal{S}$. Then any hyperplane $H \supset S$ meets V_2^5 in a curve of order at least 6, a contradiction.

2) Let $\mathcal{C}$ and $\mathcal{C}'$ be two cubic curves with $\mathcal{C} \subset S$, $\mathcal{C}' \subset S'$, where S, S' are 3-spaces, $S \neq S'$ from (1). Assume the curves have at least 2 points in common, $P, Q \in \mathcal{C} \cap \mathcal{C}'$, $P \neq Q$. Then $S \cap S' \supset PQ$ so that the hyperplane $H = \langle S, S' \rangle$ meets V_2^5 in a curve of order 6, a contradiction.

3.2. Bundles of Cubics on V_2^5 and a Non-Affine Subplane

Denote $F = GF(q)$, $\Sigma = PG(6,q)$. Let $\Sigma' = PG(5,q)$ be a hyperplane of Σ, $\mathcal{S}$ a regular spread of Σ'.

Choose a coordinate system $(\mathbf{x}, \mathbf{y}, t) = (x_1, x_2, x_3, y_1, y_2, y_3, t)$ in Σ so that $t = 0$ represents Σ', $(\mathbf{x}, \mathbf{y})$ are *internal coordinates* for Σ' and for a point $P \in \Sigma \setminus \Sigma'$, $P \approx (\mathbf{x}, \mathbf{y}, t) = (x_1, x_2, x_3, y_1, y_2, y_3, t) = F^* (\mathbf{x}, \mathbf{y}, t)$, $F^* = F \setminus \{0\}$.

The spread $\mathcal{S}$ can be represented as follows

$$\mathcal{S} = \left\{ \pi_\infty = (0, \mathbf{y}) \mid \mathbf{y} \in F^3 \right\} \cup \left\{ \pi_{\mathbf{m}} = (\mathbf{x}, \mathbf{xm}) \mid \mathbf{x}, \mathbf{m} \in F^3 \right\}$$

where $\mathbf{y} = \mathbf{xm}$ is the multiplication in the field F^3. Such a multiplication can be represented also by $\mathbf{y} = \mathbf{x}M$ with M a 3 × 3 matrix over F so that $\mathbf{x}M = \mathbf{xm}$. The set $\mathcal{M} = \{ M \mid \mathbf{x}M = \mathbf{xm} \}$ is a field isomorphic to $\left(F^3\right)^2$, strictly transitive over F^3.

Denote $\mathcal{R} = \{ \pi_\infty, \pi_k \mid k \in F \}$ the regulus of $\mathcal{S}$ represented by the scalar matrices kI.

From now on we choose $q \equiv 1 (\bmod 3)$, so that in $F = GF(q)$ $\dfrac{q-1}{3}$ elements are *cubes*, while the remaining ones are *non cubes*.

Let $\mathcal{C}_\infty$ be an irreducible conic of π_∞ and $\mathcal{C}_0$ a skew cubic curve in the 3-space S_0 of $\Sigma \setminus \Sigma'$ through π_0 so that $\mathcal{C}_0 \cap \pi_0 = \varnothing$ (cf. [7], p. 234, Corollary 4, $\mathcal{N}_5$):

$$\mathcal{C}_\infty = \left\{ \left(0,0,0,1,\lambda,\lambda^2,0\right) \mid \lambda \in GF(q) \right\} \cup \left\{ O' = \left(0,0,0,0,0,1,0\right) \right\}$$

$$\mathcal{C}_0 = \left\{ \left(1,\lambda,\lambda^2,0,0,0,s-\lambda^3\right) \mid \lambda \in GF(q) \right\} \cup \left\{ O = \left(0,0,0,0,0,0,1\right) \right\},$$

where $s \in GF(q)$ is a non cube.

The two curves are referred through a projectivity $\Lambda : \mathcal{C}_0 \to \mathcal{C}_\infty$ represented by having inserted the same parameter λ for which it is agreed that the points are considered corresponding to each other, plus $\Lambda(O) = O'$.

Denote $\mathcal{V}$ the ruled variety V_2^5 defined by $\mathcal{C}_\infty$ and $\mathcal{C}_0$. The curves $\mathcal{C}_\infty$ and $\mathcal{C}_0$ are directrix curves of $\mathcal{V}$, the set $\mathcal{G}$ of the lines connecting corresponding points are the generatrix lines of $\mathcal{V}$.

Let us consider the affinity φ of Σ represented by the following 6 × 6 matrix in 3 × 3 blocks

$$M_\varphi = \left(\begin{array}{c|c} I & kI \\ \hline 0 & I \end{array} \right)$$

so that the extended projectivity $\bar{\varphi}$ is represented by the 7 × 7 matrix $M_{\bar{\varphi}}$ obtained from M_φ by adding the vector $(0,0,0,0,0,0,1)$ as the 7th column and the 7th row.

Theorem 3.5 1) *For every point $P \in OO' \setminus \{O'\}$ there exists a bundle C_P of q cubic curves on the variety $\mathcal{V} = V_2^5$ having the point P in common, each curve lying in one 3-space intersecting a plane of $\mathcal{R} \setminus \pi_\infty$. Each bundle cover the q^2 points of $\mathcal{V} \setminus OO'$.*

2) Such cubic curves are q^2.

Proof. 1) For each point $A = (\mathbf{0},\mathbf{a},0) = (0,0,0,a_1,a_2,a_3,0) \in \pi_\infty$ it is $\bar{\varphi}(A) := (A)M_{\bar{\varphi}} = A$, that is, π_∞ is pointwise fixed. For a point $B = (\mathbf{b},\mathbf{0},0) = (b_1,b_2,b_3,0,0,0,0) \in \pi_0$ it is $\bar{\varphi}(B) := (B)M_{\bar{\varphi}} = (\mathbf{b},k\mathbf{b},0)$, that is, $\bar{\varphi}(\pi_0) = \pi_k$, and $\bar{\varphi}(O) = O$. Hence $\bar{\varphi}(S_0)$ is a 3-space S_k through O with $S_k \cap \Sigma' = \pi_k$. The cubic $\mathcal{C}_0 \subset S_0$ is mapped onto a cubic $\mathcal{C}_k \subset S_k$ with $O \in \mathcal{C}_k$ and $\mathcal{C}_k \cap \pi_k = \varnothing$. Therefore there exists a bundle C_0 of q cubic curves through O collecting the q^2 points of $\mathcal{V} \setminus OO'$.

Let $P = (0,0,0,0,0,h,1)$ be a point of $OO' \setminus \{O,O'\}$ and denote τ_h the associated translation. Therefore $\tau_h(O) = P$ and $\tau_h(\mathrm{C}_0) = \mathrm{C}_P$.

2) The cardinality of $\{\mathrm{C}_P \mid P \in OO' \setminus \{O'\}\}$ is q^2 as for each point $P \in OO' \setminus \{O'\}$ it is $|\mathrm{C}_P| = q$ and the points of $OO' \setminus \{O'\}$ are q.

Note that, chosen π_∞ and S_0, the variety V_2^5 selections in the spread $\mathcal{S}$ the regulus $\mathcal{R}$ to which π_∞ and π_0 belong.

Denote Π the projective plane $PG(2,q^3)$. Represent Π in Σ, $\Pi = (\mathcal{P},\mathcal{L},\mathcal{I})$ as in Lemma 2.6.

Denote $\mathcal{V}'$ the set of the q^2 affine points of $\mathcal{V} = V_2^5$.

Let $\Pi_q = (\mathcal{P}',\mathcal{L}',\mathcal{I}')$ be the incidence substructure of Π defined as follows:

$$\mathcal{P}' = \{P \in \mathcal{V}'\} \cup \{\pi_\infty\},$$

$$\mathcal{L}' = \left\{\mathcal{C} \in C_P \mid P \in OO' \setminus \{O'\}\right\} \cup \mathcal{G},$$

$\mathcal{I}'$ is defined as follows

$\mathcal{I}' = \mathcal{I}$ restricted to the affine points and lines, $\pi_\infty \mathcal{I}' g$ for all $g \in \mathcal{G}$.

Theorem 3.6 Π_q *is a non-affine subplane of* Π *of order q*.

Proof. It is known from [8] [9] pp. 160-161 and [4] pp. 40-41, that if in an incidence structure the following four properties hold

$$\begin{pmatrix} 1 & 3 & - \\ 2 & - & 6^2 \end{pmatrix}$$

where

1: the number of the points is $q^2 + q + 1$,

2: the number of the lines is $q^2 + q + 1$,

3: each line contains $q + 1$ points,

6^2: two lines meet in at most one point,

then the structure is a projective plane of order *q*.

The affine points of $\mathcal{V}$ are the affine points of the $q + 1$ generatrix lines of $\mathcal{G}$, that is, they are $q(q+1) = q^2 + q$ to which the *point at infinity* π_∞ has to be added. Hence $|\mathcal{P}'| = q^2 + q + 1$, that is, 1 - holds.

From Theorem 3.5 follows $\left|\left\{\mathcal{C} \in C_P \mid P \in OO' \setminus \{O'\}\right\}\right| = q^2$. As $|\mathcal{G}| = q + 1$ then $|\mathcal{L}'| = q^2 + q + 1$, that is, 2 - holds.

Each cubic curve of C_P has as many points as $\mathcal{C}_0$ has, that is $q + 1$. Each generatrix line $g \in \mathcal{G}$ has *q* affine points and the point ad infinity π_∞, hence 3 - holds.

From Proposition 3.4, (2) follows that two cubic curves meet in at most one point. Moreover each cubic curve, being a directrix, meets a generatrix line in one point. Two generatrix lines meet only in the point π_∞. Hence 6^2 holds.

To end proving that Π_q is a subplane of Π it needs to verify that Π_q is a subgeometry of Π (cf. Definition 2.7). Its set of points is clearly a subset of the points of Π. Moreover, every line $g \in \mathcal{G}$ is contained in a unique 3-space $S = \langle g, \pi_\infty \rangle$ which meets no other generatrix (cf. Proposition 3.1, 3)) and every cubic of C_P lies in a unique 3-space (cf. Proposition 3.4, 1)) meeting Σ' in a plane of $\mathcal{R}$ (cf. Theorem 3.5, 1)).

From Theorem 3.6 follows.

Corollary 3.7 *Let P, Q be two points of* $\mathcal{V}'$. *If PQ is not a generatrix line then P, Q belong to one directrix cubic, if* $P \in \mathcal{V}'$ *and* $Q = \pi_\infty$ *then the line PQ of* Π_q *is the generatrix* g_P.

The properties of Π_q of being a plane can be translated into further incidence properties of the affine points of V_2^5.

Theorem 3.8 *Let P, Q be two points of* $V_2^5 \setminus \mathcal{C}_\infty$. *Then P, Q are joined by one generatrix line or by one directrix cubic* $\mathcal{C} \subset S$, *where S is a 3-space with* $S \cap \Sigma' = \pi_k \in \mathcal{R} \setminus \pi_\infty$. *Every two directrix cubic curves of* $V_2^5 \setminus \mathcal{C}_\infty$ *meet in one point.*

In Theorem 3.5, 1), is shown that the variety $\mathcal{V}$ selects a regulus $\mathcal{R}$ to which both π_∞, π_0 belong. Denote $\pi_\infty := \pi_{1_\infty}$, $\mathcal{C}_\infty := \mathcal{C}_{1_\infty}$. $\mathcal{R} := \mathcal{R}_1$. Fix the directrix cubic curve $\mathcal{C}_0 \subset S_0$.

Theorem 3.9 *There exists a bundle* $\mathcal{B}$ *of varieties* V_2^5 *with the cubic* $\mathcal{C}_0$ *as directrix, any to varieties having* $\mathcal{C}_0$ *in common,* $|\mathcal{B}| = q^2$.

Proof. The construction is done step by step, by choosing at each step a plane of the spread $\mathcal{S}$ out the regulus identified by the variety of the previous step, and a directrix conic in it.

Step 1 - Construct the variety $\mathcal{V}_1 = V_2^5$ starting from the conic $\mathcal{C}_{1_\infty}$ and the cubic $\mathcal{C}_0 \subset S_0$. In $\mathcal{S} \setminus \mathcal{R}_1$ are $q^3 - q$ possible choices for the next step.

Step 2 - Choose a plane $\pi_{2_\infty} \in \mathcal{S} \setminus \mathcal{R}_1$. Fix a conic $\mathcal{C}_{2_\infty}$ in it and construct the variety $\mathcal{V}_2 = V_2^5$ starting from the conic $\mathcal{C}_{2_\infty}$ and the cubic $\mathcal{C}_0 \subset S_0$. Let $\mathcal{R}_2$ be the regulus of $\mathcal{S}$ to which π_{2_∞} and π_0 belong. In $\mathcal{S} \setminus \{\mathcal{R}_1, \mathcal{R}_2\}$ are $q^3 - 2q$ possible choices for the next step.

Step 3 - Choose a plane $\pi_{3_\infty} \in \mathcal{S} \setminus \{\mathcal{R}_1, \mathcal{R}_2\}$. Fix a conic $\mathcal{C}_{3_\infty}$ in it and construct the variety $\mathcal{V}_3 = V_2^5$ starting from the conic $\mathcal{C}_{3_\infty}$ and $\mathcal{C}_0 \subset S_0$. Let $\mathcal{R}_3$ be the regulus of $\mathcal{S}$ to which π_{3_∞} and π_0 belong. In $\mathcal{S} \setminus \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$ are $q^3 - 3q$ possible choices for the next step. And so on.

The procedure ends evidently at the q^2-th step. Therefore $\mathcal{B} = \{\mathcal{V}_i \mid i = 1, 2, \cdots, q^2\}$ and $|\mathcal{B}| = q^2$.

Conjecture - A variety V_2^{2r-1} of $PG(2r, q)$ represents a non-affine subplane of order q of $PG(2, q^r)$ via the spatial representation. This is partially addressed in Theorem 11 of the following paper: M. Lavrauw, C. Zanella, *Subspaces intersecting each element of a regulus in one point, Andr-Bruck-Bose Representation and Clubs*, Electron. J. Combin. 23 (2016), Paper 1.37, pp. 1-11.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] André, J. (1954) Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Mathematische Zeitschrift*, **60**, 156-186. https://doi.org/10.1007/BF01187370

[2] Bruck, R.H. and Bose, R.C. (1966) Linear Representation of Projective Planes in Projective Spaces. *Journal of Algebra*, **4**, 117-172. https://doi.org/10.1016/0021-8693(66)90054-8

[3] Casse, R. and Quinn, C.T. (2002) Ruled Cubic Surfaces in $PG(4, q)$, Baer Subplanes of $PG(2, q^2)$ and Hermitian Curves. *Discrete Mathematics*, **248**, 17-25. https://doi.org/10.1016/S0012-365X(01)00182-0

[4] Vincenti, R. (1980) Alcuni tipi di varietá V_2^3 di $S_{4,q}$ e sottopiani di Baer, *Algebra e Geometria*, **2**, 31-44.

[5] Vincenti, R. (1983) A Survey on Varieties of $PG(4, q)$ and Baer Subplanes of Translation Planes. *Annals of Discrete Math*, **18**, 775-780.

https://doi.org/10.1016/S0304-0208(08)73355-3

[6] Bertini, E. (1907) Introduzione alla geometria proiettiva degli iperspazi. Enrico Spoerri Editore, Pisa.

[7] Hirschfeld, J.W.P. (1985) Finite Projective Spaces of Three Dimensions. Clarendon Press, Oxford.

[8] Barlotti, A. (1962) Un'osservazione sulle proprietá che caratterizzano un piano grafico finito, *Boll*, No.4, 394-398.

[9] Corsi, G. (1963) Sui sistemi minimi di assiomi atti a definire un piano grafico finito. Nota dell'Istituto Matematico di Firenze, Marzo.