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Pareto-Optimal Reinsurance Based on TVaR Premium Principle and Vajda Condition

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Abstract

Reinsurance is an effective risk management tool for insurers to stabilize their profitability. In a typical reinsurance treaty, an insurer cedes part of the loss to a reinsurer. As the insurer faces an increasing number of total losses in the insurance market, the insurer might expect the reinsurer to bear an increasing proportion of the total loss, that is the insurer might expect the reinsurer to pay an increasing proportion of the total claim amount when he faces an increasing number of total claims in the insurance market. Motivated by this, we study the optimal reinsurance problem under the Vajda condition. To prevent moral hazard and reflect the spirit of reinsurance, we assume that the retained loss function is increasing and the ceded loss function satisfies the Vajda condition. We derive the explicit expression of the optimal reinsurance under the TVaR risk measure and TVaR premium principle from the perspective of both an insurer and a reinsurer. Our results show that the explicit expression of the optimal reinsurance is in the form of two or three interconnected line segments. Under an additional mild constraint, we get the optimal parameters and find the optimal reinsurance strategy is full reinsurance, no reinsurance, stop loss reinsurance, or quota-share reinsurance. Finally, we gave an example to analyze the impact of the weighting factor on optimal reinsurance.

Keywords

Pareto-Optimal Reinsurance, TVaR Risk Measure, Vajda Condition, TVaR Premium Principle

1. Introduction

Reinsurance is one of the effective means for insurers to control their risks. To prevent losses or even bankruptcy due to huge risks, the insurer transfers a portion of the risk to the reinsurer, while the insurer is required to pay a premium accordingly. Over the past decades, the optimal reinsurance strategy has always been an issue of great concern. With the optimization objective of minimizing the VaR of the insurer's risk, Borch [1] obtains that the optimal reinsurance strategy under the expected premium principle is stop-loss reinsurance. With the optimization objective of maximizing the expected utility of the insurer's ultimate wealth, Arrow [2] reaches the same conclusion as Borch, *i.e.*, the optimal reinsurance strategy is stop-loss reinsurance. Since Borch and Arrow's seminal study, many papers have expanded from the perspective of changing premium principles or choosing different optimization criteria. In general, most of the papers studied the optimal reinsurance strategy from the perspective of the insurer or the reinsurer. For instance, Cai *et al.* [3] minimize the insurer's VaR and CTE and derive that the optimal reinsurance strategy can be stop-loss reinsurance, quota-share reinsurance, or change-loss reinsurance. Zheng *et al.* [4] study the optimal reinsurance strategy under the distortion risk measure and expect value premium principle for the reinsurer. Kaluszka [5] derives optimal reinsurance under premium principles based on the mean and variance of the reinsurer's share of the total claim amount. However, in insurance practice, both parties should negotiate the reinsurance contract and have conflicting interests. The optimal reinsurance strategy that the insurer believes may not be optimal for the reinsurer. So it is necessary to study the optimal reinsurance strategy based on the perspective of both the insurer and the reinsurer, see Liu *et al.* [6] and Fang *et al.* [7]. In this paper, we consider the interests of both parties and study the optimal reinsurance strategy under certain constraints.

It is typically necessary for both the insurer and the reinsurer to pay more for a bigger loss in order to avoid moral hazard. As a result, numerous studies examine the optimal reinsurance strategy when incentives are considered. For instance, Cai *et al.* [8] and Jiang *et al.* [9] investigate the optimal reinsurance strategy under the distortion risk measure from the viewpoints of both insurers and reinsurers. In reality, though, the insurer tends to ask the reinsurer to cover an increasing percentage of the entire amount of claims when the overall number of claims faced rises. Vajda [10] suggest the Vajda condition, assuming that the proportion paid by reinsurers grows with the increase in risk, to represent the original design of reinsurance to protect the insurer. Chi *et al.* [11] study the optimal reinsurance strategy under the Vajda condition from the perspective of the insurer.

In this paper, we study the optimal reinsurance strategy under the TVaR risk measure and TVaR premium principle. The three main implications of this study are as follows. Firstly, most of the existing literature assumes that the set of ceded loss functions can only avoid moral hazard and cannot reflect the spirit of reinsurance. In this study, the set of ceded loss functions can successfully avoid moral hazard and accurately capture the essence of reinsurance. Secondly, whereas the majority of earlier research looked at the optimal reinsurance with VaR risk measure from the insurer's perspective, this study examines the optimal reinsurance with TVaR risk measure while taking into account the interests of both the insurer and the reinsurer. Finally, the expected value premium principle is typically chosen in prior works for calculation convenience, however, this study picks the TVaR premium principle since it is consistent and contains a variety of different premium principles.

The rest of the paper is organized as follows. In section 2, we will introduce some preliminaries and basic knowledge. The optimal reinsurance strategy under TVaR risk measure and TVaR premium principle are given in Section 3. Section 4 gives some numerical examples to further illustrate our results. Section 5 concludes the paper and puts forward the research direction in the future.

2. Preliminaries

Reinsurance contracts are a common strategy for managing risk in the financial markets. We assume that the initial claim faced by the insurer in a given period is a non-negative random variable X on the probability space and X has a finite expectation. After underwriting, the insurer's safety may be jeopardized due to the excessive risk of insurance contract claims, etc. For their safety, insurers will consider spreading the risk. Thus, reinsurance arises to meet the needs of the insurer. By entering into a reinsurance contract with a reinsurer, the insurer cedes a portion of the risk $f(X)$ to the reinsurer, while the reinsurer charges the insurer a fee, also known as a reinsurance premium $\Pi_f(X)$.

In the case of reinsurance, the cumulative distribution function of X is written as $F(x)=\mathbb{P}\{X \le x\}$, and the survival function of X is written as $S(x)=\mathbb{P}\{X > x\}$. $f(X)$ is the loss allocated to the reinsurer, and $R_f(X)=X-f(X)$ is the part reserved by the insurer. Functions $f(x)$ and $R_f(x)$ are usually referred to as the ceded loss function and the retained loss function, respectively. The insurer's risk is represented by $T_f(X)$ with the following expression

$$T_f(X)=R_f(X)+\Pi_f(X). \tag{2.1}$$

Most papers assume that the set of ceded loss functions is monotonically increasing and 1-Lipschitz continuous for both the ceded loss function and the retained loss function. This means that as the amount of claims increases, the amount of benefits borne by both the insurer and the reinsurer will increase. At this point, the reinsurance parties are incentive compatible, which can effectively prevent the occurrence of moral hazard. However, it does not reflect the spirit of reinsurance that protects insurers very well. For this reason, this paper follows Vajda [10] and assumes that both the insurer's retained loss and the proportion paid by the reinsurer are increasing in indemnity. Therefore, we need to find the optimal reinsurance strategy in the following set of ceded loss functions.

$$\mathfrak{C} := \left\{ 0 \le f(x) \le x : R_f(x) \text{ and } \frac{f(x)}{x} \text{are increasing in } x \right\} \tag{2.2}$$

Definition 2.1. (*Vajda condition*). *If* $f(x)$ *is the indemnity function and* *the loss is depicted by a non-negative random variable X, the Vajda condition refers to the increasingness of* $\frac{f(x)}{x}$ *on the support of X.*

This condition, joint with the no-sabotage condition, implies that the retained risk $R_f(x)$ is increasing and the relative retained risk $\frac{x-f(x)}{x}$ is decreasing on the support of *X*. The functions satisfying the Vajda condition are called the Vajda functions. For instance, convex functions are Vajda functions. It is worth noting that Vajda functions could effectively reflect the spirit of reinsurance of protecting the insurer. That is, as the insurer faces an increasing number of total risk *X*, the reinsurer will bear an increasing proportion of the total risk *X* of the insurer to protect the insurer.

The total risk faced by the insurer is a random variable that cannot be quantified, so we use risk measurement to measure the risk faced by the insurer. The optimal reinsurance problem under VaR risk measure can be found in Cai *et al.* [12] and the references therein. Compared to the VaR risk measure, TVaR risk measure has the following advantages: Firstly, TVaR risk measure satisfies subadditivity. However, VaR risk measure does not satisfy subadditivity. Secondly, TVaR risk measure takes into account tail losses and represents the average level of excess losses, which reflects the size of the average losses of VaR risk measure and better reflects the potential risk value. Finally, VaR risk measure is a special case of TVaR risk measure. The mathematical definition of TVaR risk measure and the expressions are given as follows.

Definition 2.2. *TVaR of random variable X with confidence level* $p \in (0,1)$ *is defined as*

$$TVaR_p(X) := \frac{1}{1-p}\int_p^1 VaR_q(X)\,\mathrm{d}q. \tag{2.3}$$

Note that $TVaR_p(X)$ can also be written as

$$TVaR_p(X) := VaR_p(X) + \frac{1}{1-p}E\left[\left(X - VaR_p(X)\right)_+\right]. \tag{2.4}$$

TVaR is translation invariant. For any $C \in R$,
$TVaR_p(X+C) = TVaR_p(X) + C$

TVaR is homologous additive. If Z_1, Z_2 are non decreasing functions of common random variables in the sense of homology, then
$TVaR_p(Z_1+Z_2) = TVaR_p(Z_1) + TVaR_p(Z_2)$.

Most of the previous studies on optimal reinsurance problem are restricted to a specific reinsurance premium principle. For example, it is very common to assume that the reinsurance premium is calculated according to the expected value premium principle. However, this paper uses a more general premium principle, the TVaR premium principle. Specifically, the expected premium principle is a special case of the TVaR premium principle. TVaR premium principle is defined as follows.

Definition 2.3. *For random variable X, TVaR premium principle is defined as* follows

$$P_X = (1+\delta) TVaR_\gamma(X), \tag{2.5}$$

where $\delta \geq 0$ is the relative safety loading.

In this paper, we assume that the premium principle is the TVaR premium principle and derive the optimal form of ceded loss function. Thus, we transform the infinite dimensional optimal reinsurance problem into a finite dimensional optimal reinsurance problem and derive the explicit expressions for the optimal reinsurance treaties. To take into account the interests of both the insurer and the reinsurer, it is assumed that the insurer and the reinsurer use $TVaR_\alpha$ and $TVaR_\beta$ to measure risk, where $\alpha, \beta \in (0,1)$ and generally $\gamma < \alpha$, $\gamma < \beta$. Generally, an approach to identify Pareto-optimal reinsurance treaties is to minimize the convex combination of both the insurer's loss and the reinsurer's loss. Thus, we obtain a Pareto-optimal reinsurance treaty by minimizing the convex combination of the TVaRs of the insurer and the reinsurer. Next, we state the optimal reinsurance problem that is the focus of this paper.

$$\min_{f \in \mathfrak{C}} \lambda TVaR_\alpha\left(X - f(X) + P_{f(X)}\right) + (1-\lambda) TVaR_\beta\left(f(X) - P_{f(X)}\right)$$

where $\lambda \in [0,1]$ is the weight factor. Based on the properties of the TVaR risk measure and the definition of the TVaR premium, the optimization problem becomes the following

$$\begin{aligned}\min_{f \in \mathfrak{C}} & \lambda TVaR_\alpha(X) - \lambda TVaR_\alpha(f(X)) + (1-\lambda) TVaR_\beta(f(X)) \\ & + (2\lambda - 1)(1+\delta) TVaR_\gamma(f(X)).\end{aligned}$$

Denote $v_p = VaR_p(X)$ for $p \in (0,1)$ and

$$\begin{aligned}Q(f) = & -\lambda TVaR_\alpha(f(X)) + (1-\lambda) TVaR_\beta(f(X)) \\ & + (2\lambda - 1)(1+\delta) TVaR_\gamma(f(X)),\end{aligned}$$

then the optimization problem becomes

$$\begin{aligned}\min_{f \in \mathfrak{C}} Q(f) = & -\lambda TVaR_\alpha(f(X)) + (1-\lambda) TVaR_\beta(f(X)) \\ & + (2\lambda - 1)(1+\delta) TVaR_\gamma(f(X))\end{aligned} \tag{2.6}$$

With the expression (2.4) for TVaR, we have

$$\begin{aligned}Q(f) = & -\lambda\left\{VaR_\alpha(f(X)) + \frac{1}{1-\alpha} E\left[f(X) - VaR_\alpha(f(X))\right]_+\right\} \\ & + (1-\lambda)\left\{VaR_\beta(f(X)) + \frac{1}{1-\beta} E\left[f(X) - VaR_\beta(f(X))\right]_+\right\} \\ & + (2\lambda-1)(1+\delta)\left\{VaR_\gamma(f(X)) + \frac{1}{1-\gamma} E\left[f(X) - VaR_\gamma(f(X))\right]_+\right\}\end{aligned} \tag{2.7}$$

3. Pareto-Optimal Reinsurance Strategy

In the case of $\lambda = \frac{1}{2}$, the optimal reinsurance treaty have been studied by Cai *et al.* [8]. Next, we give the optimal reinsurance strategy of problem (2.6), where the weight factor $\lambda \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$. This paper only provides the proof of Theorem 3.1, while the proof of the other theorems are provided in the supplementary materials. For mathematical convenience, we define the following notation. Let

$$m_0 = \frac{(2\lambda - 1)(1+\delta)}{1-\gamma}, m = m_0 + \frac{1-\lambda}{1-\beta} - \frac{\lambda}{1-\alpha}, n = m_0 - \frac{\lambda}{1-\alpha},$$

$$s = m_0 + \frac{1-\lambda}{1-\beta}, \Delta = 1 - \frac{\lambda}{s},$$

$$t_2 = (2\lambda - 1)\delta v_\beta,$$

$$t_3 = (2\lambda - 1)(1+\delta)v_\gamma + (1-\lambda)v_\beta - \lambda v_\alpha,$$

$$t_4 = (2\lambda - 1)(1+\delta)v_\gamma + (1-\lambda)v_\beta - \lambda v_\Delta.$$

Theorem 3.1. *Under the conditions of* $\frac{1}{2} < \lambda \le 1$ *and* $0 < \beta \le \alpha < 1$, *the optimal ceded loss functions of the optimal reinsurance model* (2.6) *are given as follows.*

(a) If $m \ge 0$ and $c_1 > 0$, the optimal ceded loss function is $f^*(x) = 0$. The parameter $\theta_1 = 0$ is optimal. That is, no reinsurance is the optimal reinsurance strategy. The insurer bears all risks themselves and does not sign a reinsurance contract.

(b) If $m \ge 0$ and $c_1 < 0$, the optimal ceded loss function is $f^*(x) = x$. The parameter $\theta_1 = 1$ is optimal. That is, full reinsurance is the optimal reinsurance strategy. The insurer does not bear the risk and transfers all risks to the reinsurer.

(c) If $m \ge 0$ and $c_1 = 0$, the optimal ceded loss function is $f^*(x) = \theta_1 x$, where θ_1 can be any constant in $[0,1]$. That is, quota-share reinsurance is the optimal reinsurance strategy. The insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1.

(d) If $m < 0$, the optimal ceded loss function is $f^*(x) = \begin{cases} (x - v_\alpha)_+, & s \ge 0 \\ (x - v_\beta)_+, & s < 0 \end{cases}$.

That is, stop-loss reinsurance is the optimal reinsurance strategy. Note that $c_1 = m\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t - \lambda v_\alpha$. Under the condition of $s \ge 0$, when risks $x < v_\alpha$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\alpha$, the insurer only bears risks v_α and divides out remaining risks. Under the condition of $s < 0$, when risks $x < v_\beta$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\beta$, the insurer only bears risks v_β and divides out remaining risks.

Proof.

Under the conditions of $\frac{1}{2} < \lambda \le 1$ and $0 < \beta \le \alpha < 1$, recall the definition of m_0, m, and s, equality (2.7) reduces to

$$Q(f) = m_0\int_\gamma^\beta f(v_t(X))\mathrm{d}t + s\int_\beta^\alpha f(v_t(X))\mathrm{d}t + m\int_\alpha^1 f(v_t(X))\mathrm{d}t. \quad (3.1)$$

Clearly $f(v_\alpha) \geq f(v_\beta)$ and $v_\alpha - f(v_\alpha) \geq v_\beta - f(v_\beta)$, as $f(x)$ and $R_f(x)$ are nondecreasing for all $x \geq 0$ and $\alpha \geq \beta$. Note that $0 \leq v_\alpha \leq f(v_\alpha)$ and $0 \leq v_\beta \leq f(v_\beta)$ since $0 \leq f(x) \leq x$ for all $x \geq 0$. Recall the definition of m in (3.2). Equality (2.7) reduces to

$$\begin{aligned} Q(f) &= m_0 \int_{f(v_\gamma)}^{f(v_\beta)} S_{f(X)}(t)\mathrm{d}t + s\int_{f(v_\beta)}^{f(v_\alpha)} S_{f(X)}(t)\mathrm{d}t + m\int_{f(v_\alpha)}^{\infty} S_{f(X)}(t)\mathrm{d}t \\ &\quad -\lambda f(v_\alpha) + (1-\lambda) f(v_\beta) + (2\lambda-1)(1+\delta) f(v_\gamma). \end{aligned} \tag{3.2}$$

(i) If $m > 0$, then $s > 0$. For the above $f \in \mathfrak{C}$, define

$$J(X) = \begin{cases} 0, & 0 \leq x < v_\alpha - f(v_\alpha), \\ x - v_\alpha + f(v_\alpha), & v_\alpha - f(v_\alpha) \leq x < v_\alpha \\ \dfrac{f(v_\alpha)}{v_\alpha} x, & x \geq v_\alpha. \end{cases} \tag{3.3}$$

Denote $\theta_1 = \dfrac{f(v_\alpha)}{v_\alpha}$, $d_1 = v_\alpha - f(v_\alpha)$, we have

$$J(X) = \begin{cases} 0, & 0 \leq x < d_1, \\ x - d_1, & d_1 \leq x < v_\alpha, \\ \theta_1 x, & x \geq v_\alpha. \end{cases} \tag{3.4}$$

where $0 \leq \theta_1 \leq 1$, $0 \leq d_1 \leq v_\alpha$.

The relationship between $f(x)$ and $J(x)$ is illustrated by **Figure 1**. One can show that $J(x) \in \mathfrak{C}$ and $Q(f) > Q(J)$ for any $f \in \mathfrak{C}$. Indeed, from **Figure 1**, we conclude that for $x \geq 0$, $f(x) > J(x)$. Moreover, since $m \geq 0$, we have

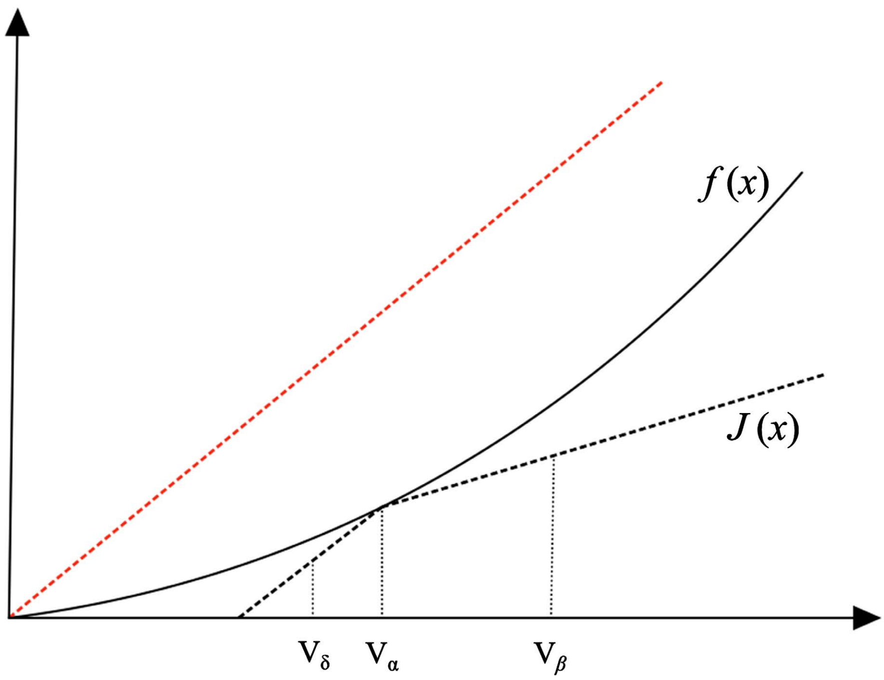

Figure 1. Relationship between $f(x)$ and $J(x)$ in case (i).

$$m_0\int_{\gamma}^{\beta} f\left(v_t\left(X\right)\right)\mathrm{d}t > m_0\int_{\gamma}^{\beta} J\left(v_t\left(X\right)\right)\mathrm{d}t,$$

$$s\int_{\beta}^{\alpha} f\left(v_t\left(X\right)\right)\mathrm{d}t > s\int_{\beta}^{\alpha} J\left(v_t\left(X\right)\right)\mathrm{d}t,$$

$$m\int_{\alpha}^{1} f\left(v_t\left(X\right)\right)\mathrm{d}t > m\int_{\alpha}^{1} J\left(v_t\left(X\right)\right)\mathrm{d}t.$$

Hence, it follows immediately from (3.1) that $Q(f) > Q(J)$, where the inequality is strictly hold. If f and J are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (3.3) in case (i). The equivalence of (3.1) implies that $\min_{f\in\mathfrak{C}} Q(f) = \min Q(J)$. The expression of $Q(J)$ is as follows.

$$\begin{aligned} Q(J) = &-\frac{\lambda}{1-\alpha}\theta_1\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t + m_0\left[\int_{v_\gamma}^{\alpha} S_X(t)\mathrm{d}t\right. \\ &\left.+\theta_1\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t\right] - \lambda\theta_1 v_\alpha + (1-\lambda)(v_\beta - d_1) + (2\lambda-1)(1+\delta)(v_\gamma - d_1). \end{aligned} \tag{3.5}$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(J)}{\partial d_1} = -(1-\lambda) - (2\lambda-1)(1+\delta) < 0.$$

We can imply that $Q(J)$ is decreasing in $d_1 \in [0, v_\alpha]$, thus,

$$Q(J) \geq Q(J_1) = \theta_1\left[m\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t - \lambda v_\alpha\right].$$

Taking the derivative of $Q(J_1)$ with respect to θ_1, we have

$$\frac{\partial Q(J_1)}{\partial \theta_1} = c_1.$$

If $c_1 > 0$, $Q(J_1)$ is increasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = 0$ in this situation.

If $c_1 < 0$, $Q(J_1)$ is decreasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = x$ in this situation.

If $c_1 = 0$, the optimal ceded function is $f^*(x) = \theta_1 x$, where θ_1 can be any constant in $[0,1]$.

(ii) If $m < 0$ and $s < 0$, for the above $f \in \mathfrak{C}$, define

$$J(X) = \begin{cases} 0, & 0 \leq x < v_\beta - f(v_\beta), \\ x - \left(v_\beta - f(v_\beta)\right), & x \geq v_\beta - f(v_\beta). \end{cases} \tag{3.6}$$

Denote $d_1 = v_\beta - f(v_\beta)$, we have

$$J(X) = \begin{cases} 0, & 0 \leq x < d_1, \\ x - d_1, & x \geq d_1. \end{cases} \tag{3.7}$$

where $0 \leq d_1 \leq v_\beta$.

The relationship between $f(x)$ and $J(x)$ is illustrated by **Figure 2**. One can show that $J(x) \in \mathfrak{C}$ and $Q(f) > Q(J)$ for any $f \in \mathfrak{C}$. Indeed, from **Figure 2**, we conclude that for $0 \leq x \leq v_\beta$, $f(x) > J(x)$, and for $x > v_\beta$, $f(x) < J(x)$. Moreover, since $m < 0$ and $s < 0$, we have

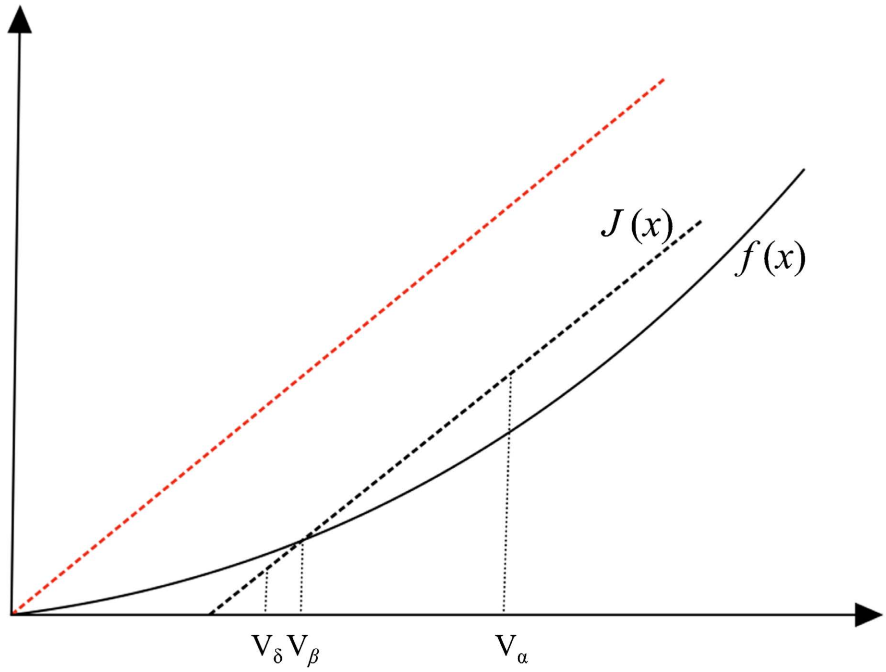

Figure 2. Relationship between $f(x)$ and $J(x)$ in case (ii).

$$m_0\int_\gamma^\beta f\left(v_t\left(X\right)\right)\mathrm{d}t > m_0\int_\gamma^\beta J\left(v_t\left(X\right)\right)\mathrm{d}t$$

$$s\int_\beta^\alpha f\left(v_t\left(X\right)\right)\mathrm{d}t > s\int_\beta^\alpha J\left(v_t\left(X\right)\right)\mathrm{d}t$$

$$m\int_\alpha^1 f\left(v_t\left(X\right)\right)\mathrm{d}t > m\int_\alpha^1 J\left(v_t\left(X\right)\right)\mathrm{d}t$$

Hence, it follows immediately from (3.1) that $Q(f) > Q(J)$, where the inequality is strict if *f* and *J* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (3.6) in case (ii). The equivalence of (3.1) implies that $\min_{f\in\mathfrak{C}} Q(f) = \min Q(J)$. The expression of Q(J) is as follows.

$$\begin{aligned} Q(J) = & -\frac{\lambda}{1-\alpha}\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\int_{v_\beta}^\infty S_X(t)\mathrm{d}t + m_0\int_{v_\gamma}^\infty S_X(t)\mathrm{d}t \\ & -\lambda\left(v_\alpha - d_1\right) + (1-\lambda)\left(v_\beta - d_1\right) + (2\lambda-1)(1+\delta)\left(v_\gamma - d_1\right). \end{aligned} \tag{3.8}$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(J)}{\partial d_1} = -\delta(2\lambda-1) < 0$$

We can imply that $Q(J)$ is decreasing in $d_1 \in \left[0, v_\beta\right]$, thus, the optimal ceded function is $f^*(x) = \left(x - v_\beta\right)_+$ in this situation.

(iii) If $m < 0$ and $s \geq 0$, for the above $f \in \mathfrak{C}$, define

$$J(X) = \begin{cases} 0, & 0 \leq x < v_\alpha - f\left(v_\alpha\right), \\ x - \left(v_\alpha - f\left(v_\alpha\right)\right), & x \geq v_\alpha - f\left(v_\alpha\right). \end{cases} \tag{3.9}$$

Denote $d_1 = v_\alpha - f\left(v_\alpha\right)$, we have

$$J(X)=\begin{cases}0, & 0\le x<d_1,\\ x-d_1, & x\ge d_1.\end{cases} \tag{3.10}$$

where $0\le d_1\le v_\alpha$.

The relationship between $f(x)$ and $J(x)$ is illustrated by **Figure 3**. One can show that $J(x)\in\mathfrak{C}$ and $Q(f)>Q(J)$ for any $f\in\mathfrak{C}$. Indeed, from **Figure 2**, we conclude that for $0\le x\le v_\alpha$, $f(x)>J(x)$, and for $x>v_\alpha$, $f(x)<J(x)$. Moreover, since $m<0$ and $s\ge 0$, we have

$$m_0\int_\gamma^\beta f\left(v_t(X)\right)\mathrm{d}t>m_0\int_\gamma^\beta J\left(v_t(X)\right)\mathrm{d}t$$

$$s\int_\beta^\alpha f\left(v_t(X)\right)\mathrm{d}t>s\int_\beta^\alpha J\left(v_t(X)\right)\mathrm{d}t$$

$$m\int_\alpha^1 f\left(v_t(X)\right)\mathrm{d}t>m\int_\alpha^1 J\left(v_t(X)\right)\mathrm{d}t$$

Hence, it follows immediately from (3.1) that $Q(f)>Q(J)$, where the inequality is strict if *f* and *J* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (3.10) in case (iii). The equivalence of (3.1) implies that $\min_{f\in\mathfrak{C}}Q(f)=\min Q(J)$. The expression of Q(J) is as follows.

$$\begin{aligned}Q(J)=&-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^\infty S_X(t)\mathrm{d}t+m_0\int_{v_\gamma}^\infty S_X(t)\mathrm{d}t\\&-\lambda(v_\alpha-d_1)+(1-\lambda)(v_\beta-d_1)+(2\lambda-1)(1+\delta)(v_\gamma-d_1).\end{aligned} \tag{3.11}$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(J)}{\partial d_1}=-\delta(2\lambda-1)<0$$

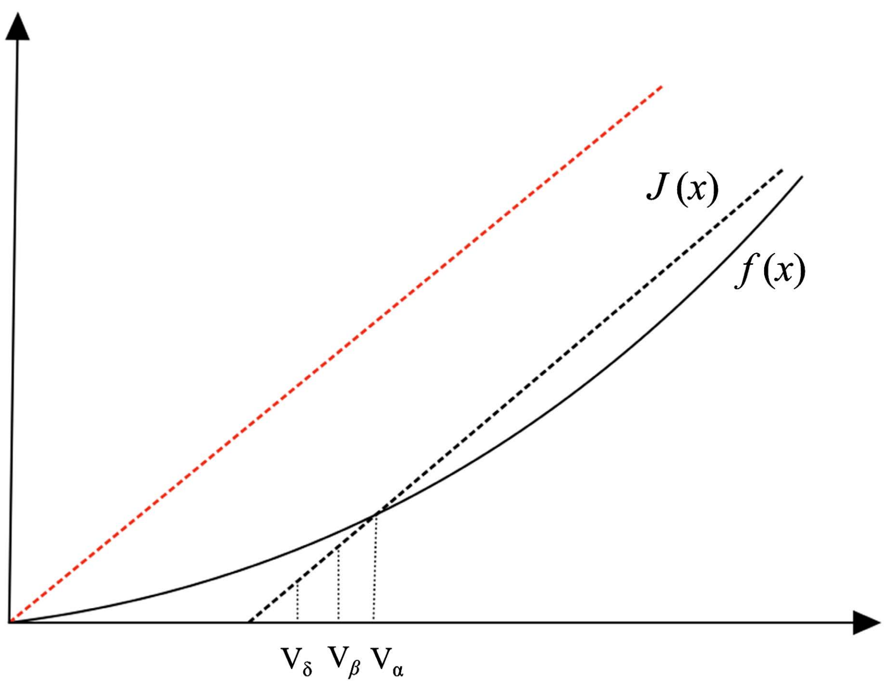

Figure 3. Relationship between $f(x)$ and $J(x)$ in case (iii).

We can imply that $Q(J)$ is decreasing in $d_1 \in [0, v_\alpha]$, thus, the optimal ceded function is $f^*(x) = (x - v_\alpha)_+$ in this situation. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Under the conditions of* $\frac{1}{2} < \lambda \le 1$ *and* $0 < \alpha \le \beta < 1$, *the optimal ceded loss functions of the optimal reinsurance model* 6 *are given as follows.*

(a) If $m \le 0$, the optimal ceded loss function is $f^*(x) = (x - v_\alpha)_+$. The parameter $\theta_1 = 1$ is optimal. That is, stop-loss reinsurance is the optimal reinsurance strategy. When risks $x < v_\alpha$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\alpha$, the insurer only bears risks v_α and divides out remaining risks.

(b) If $m > 0$, $n \ge 0$, or $m > 0$, $n < 0$, $t_1 > 0$ and $c_2 > 0$, the optimal ceded loss function is $f^*(x) = 0$. The parameter $\theta_1 = 0$ is optimal. That is, no reinsurance is the optimal reinsurance strategy. The insurer bears all risks themselves and does not sign a reinsurance contract.

(c) If $m > 0$, $n < 0$, $t_1 > 0$ and $c_2 < 0$, the optimal ceded loss function is $f^*(x) = x$. The parameter $\theta_1 = 1$ is optimal. That is, full reinsurance is the optimal reinsurance strategy. The insurer does not bear the risk and transfers all risks to the reinsurer.

(d) If $m > 0$, $n < 0$, $t_1 > 0$ and $c_2 = 0$, the optimal ceded loss function is

$$f^*(x) = \begin{cases} \left(x - (1-\theta_1) v_\beta\right)_+, & x < v_\beta, \\ \theta_1 x, & x \ge v_\beta, \end{cases}$$

where θ_1 can be any constant in $[0,1]$. That is, two-layer reinsurance is the optimal reinsurance strategy. When risks $x < (1-\theta_1) v_\beta$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $(1-\theta_1) v_\beta \le x \le v_\beta$, the insurer only bears risks $(1-\theta_1) v_\beta$ and divides out remaining risks. When risk $x > v_\beta$, the insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1.

(e) If $m > 0$, $n < 0$, and $t_1 \le 0$, then the optimal ceded loss function is

$$f^*(x) = \begin{cases} (x - v_\alpha)_+, & x < \dfrac{v_\alpha}{1-\theta_1^*}, \\ \theta_1^* x, & x \ge \dfrac{v_\alpha}{1-\theta_1^*}, \end{cases}$$

where $\theta_1^* = \arg\min_{\theta_1 \in [0,1]} \{Q(J_1)\}$. That is, two-layer reinsurance is the optimal reinsurance strategy. When risks $x < v_\alpha$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $v_\alpha \le x \le \frac{v_\alpha}{1-\theta_1^*}$, the insurer only bears risks v_α and divides out remaining risks. When risk $x > \frac{v_\alpha}{1-\theta_1^*}$, The insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1^*.

Note that $t_1 = \lambda - (2\lambda - 1)(1+\delta)$, $c_2 = m\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t - m_0\int_{v_\gamma}^{v_\alpha} S_X(t)\mathrm{d}t + t_3$.

Theorem 3.3. *Under the conditions of* $0 \le \lambda < \frac{1}{2}$ *and* $0 < \beta \le \alpha < 1$, *the optimal ceded loss functions of the optimal reinsurance model* (2.6) *are given as follows.*

(a) If $m > 0$, $s > 0$, and $c_3 > 0$, the optimal ceded loss function is $f^*(x) = 0$. The parameter $\theta_1 = 0$ is optimal. That is, no reinsurance is the optimal reinsurance strategy. The insurer bears all risks themselves and does not sign a reinsurance contract.

(b) If $m > 0$, $s > 0$, and $c_3 < 0$, or $m \le 0$, $0 \le s \le \lambda$, and $c_4 < 0$, or $m \le 0$, $s > \lambda$, and $c_5 < 0$, or $m \le 0$ and $s < 0$, the optimal ceded loss function is $f^*(x) = x$. The parameter $\theta_1 = 1$ is optimal. That is, full reinsurance is the optimal reinsurance strategy. The insurer does not bear the risk and transfers all risks to the reinsurer.

(c) If $m > 0$, $s > 0$, and $c_3 = 0$, then the optimal ceded function is $f^*(x) = \theta_1 x$, where θ_1 can be any constant in $[0,1]$. That is, quota-share reinsurance is the optimal reinsurance strategy. The insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1.

(d) If $m \le 0$, $0 \le s \le \lambda$, and $c_4 > 0$, then the optimal ceded loss function is $f^*(x) = (x - v_\beta)_+$. The parameter $\theta_1 = 1$ is optimal. That is, stop-loss reinsurance is the optimal reinsurance strategy. When risks $x < v_\beta$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\beta$, the insurer only bears risks v_β and divides out remaining risks.

(e) If $m \le 0$, $s > \lambda$, and $c_5 > 0$, then the optimal ceded loss function is $f^*(x) = (x - v_\Delta)_+$. The parameter $\theta_1 = 1$ is optimal. That is, stop-loss reinsurance is the optimal reinsurance strategy. When risks $x < v_\Delta$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\Delta$, the insurer only bears risks v_Δ and divides out remaining risks.

(f) If $m \le 0$, $0 \le s \le \lambda$, and $c_4 = 0$, then the optimal ceded loss function is

$$f^*(x) = \begin{cases} \theta_1 x, & 0 \le x < v_\beta, \\ x - (1-\theta_1)v_\beta, & x \ge v_\beta, \end{cases}$$

where θ_1 can be any constant in $[0,1]$. That is, two-layer reinsurance is the optimal reinsurance strategy. When risks $x < v_\beta$, the insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1. When risk $x \ge v_\beta$, the insurer only bears risks $(1-\theta_1)v_\beta$ and divides out remaining risks.

(g) If $m \le 0$, $s > \lambda$, and $c_5 = 0$, then the optimal ceded loss function is

$$f^*(x) = \begin{cases} \theta_1 x, & 0 \le x < v_\Delta, \\ x - (1-\theta_1)v_\Delta, & x \ge v_\Delta, \end{cases}$$

where θ_1 can be any constant in $[0,1]$. That is, two-layer reinsurance is the optimal reinsurance strategy. When risks $x < v_\Delta$, the insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1. When risk $x \ge v_\Delta$, the insurer only bears risks $(1-\theta_1)v_\Delta$ and divides out remaining risks.

Note that $c_3 = m_0\int_{v_\gamma}^{\infty} S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t - \frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t + t_3$, $c_4 = m_0\int_{v_\gamma}^{v_\beta} S_X(t)\mathrm{d}t + (1-2\lambda)v_\beta + (2\lambda-1)(1+\delta)v_\gamma$ and $c_5 = m_0\int_{v_\gamma}^{v_\Delta} S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\int_{v_\beta}^{v_\Delta} S_X(t)\mathrm{d}t + t_4$.

Theorem 3.4. *Under the conditions of* $0 \le \lambda < \frac{1}{2}$ *and* $0 < \alpha \le \beta < 1$, *the optimal ceded loss functions of the optimal reinsurance model* (2.6) *are given as follows.*

(a) If $m > 0$ and $c_6 > 0$, the optimal ceded loss function is $f^*(x) = 0$. The parameter $\theta_1 = 0$ is optimal. That is, no reinsurance is the optimal reinsurance strategy. The insurer bears all risks themselves and does not sign a reinsurance contract.

(b) If $m > 0$ and $c_6 < 0$, or $m \le 0$ and $c_7 < 0$, the optimal ceded loss function is $f^*(x) = x$. The parameter $\theta_1 = 1$ is optimal. That is, full reinsurance is the optimal reinsurance strategy. The insurer does not bear the risk and transfers all risks to the reinsurer.

(c) If $m > 0$ and $c_6 = 0$, the optimal ceded loss function is $f^*(x) = \theta_1 x$, where θ_1 can be any constant in $[0,1]$. That is, quota-share reinsurance is the optimal reinsurance strategy. The insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1.

(d) If $m \le 0$ and $c_7 > 0$, the optimal ceded loss function is $f^*(x) = (x - v_\beta)_+$. The parameter $\theta_1 = 1$ is optimal. That is, stop-loss reinsurance is the optimal reinsurance strategy. When risks $x < v_\beta$, the insurer bears all risks themselves and does not sign a reinsurance contract. When risk $x \ge v_\beta$, the insurer only bears risks v_β and divides out remaining risks.

(e) If $m \le 0$ and $c_7 = 0$, the optimal ceded loss function is

$f^*(x) = \begin{cases} \theta_1 x, 0 \le x < v_\beta \\ x - (1-\theta_1)v_\beta, x \ge v_\beta \end{cases}$, where θ_1 can be any constant in $[0,1]$. That is, two-layer reinsurance is the optimal reinsurance strategy. When risks $x < v_\beta$, the insurer cedes the risk to the reinsurer in proportion. The proportion depends on parameter θ_1. When risk $x \ge v_\beta$, the insurer only bears risks $(1-\theta_1)v_\beta$ and divides out remaining risks.

Note that $c_6 = m_0\int_{v_\gamma}^{v_\alpha} S_X(t)\mathrm{d}t + m\int_{v_\beta}^{v_\infty} S_X(t)\mathrm{d}t + t_3$, $c_7 = m_0\int_{v_\gamma}^{v_\beta} S_X(t)\mathrm{d}t - \frac{\lambda}{1-\alpha}\int_{v_\alpha}^{v_\beta} S_X(t)\mathrm{d}t + t_3$.

4. Numerical Examples

This section presents some examples to analyze the conclusions of section 3. We will analyze the impact of the weighting factor λ on optimal reinsurance. Assume that $\delta = 0.2$ and $S_X(x) = e^{-0.001x}$ for $x \ge 0$.

Example 4.1. We analyze the influence of the weighting factor λ on optimal reinsurance by choosing $\alpha = 0.99$, $\beta = 0.95$ and $\gamma = 0.9$. Then $v_\alpha = 4605.17$, $v_\beta = 2995.73$ and $v_\gamma = 2302.59$. By choosing $\lambda = 0.2, 0.3, 0.4, 0.6$, different optimal ceded loss functions are obtained and the results are reported in Table 1.

(i) If $\lambda = 0.2$, we have $m_0 = -0.8 < 0$, $m = -4.8 < 0$, $s = 15.2 > \lambda$ and $c_5 = -203.36 < 0$. Hence, from part (b) of Theorem 3.3, it follows that $f^*(x) = x$ is the optimal ceded loss function. That is, full reinsurance is the optimal reinsurance strategy.

(ii) If $\lambda = 0.3$, we have $m_0 = -4.8 < 0$, $m = -20.8 < 0$, $s = 9.2 > \lambda$ and $c_5 = 316.89 > 0$. Hence, from part (e) of Theorem 3.3, it follows that $f^*(x) = (x - 3423.18)_+$ is the optimal ceded loss function. That is, stop-loss reinsurance is the optimal reinsurance strategy.

(iii) If $\lambda = 0.4$, we have $m_0 = -2.4 < 0$, $m = -30.4 < 0$, $s = 0.6 > \lambda$ and $c_5 = 560.18 > 0$. Hence, from part (e) of Theorem 3.3, it follows that $f^*(x) = (x - 3170.09)_+$ is the optimal ceded loss function. That is, stop-loss reinsurance is the optimal reinsurance strategy.

(iv) If $\lambda = 0.6$, we have $m_0 = 2.4 > 0$, $m = -49.6 < 0$ and $s = 10.4 > 0$. Hence, from part (d) of Theorem 3.1, it follows that $f^*(x) = (x - 4605.17)_+$ is the optimal ceded loss function. That is, stop-loss reinsurance is the optimal reinsurance strategy.

Example 4.2. We analyze the influence of the weighting factor λ on optimal reinsurance by choosing $\alpha = 0.95$, $\beta = 0.99$ and $\gamma = 0.9$. Then $v_\alpha = 2995.73$, $v_\beta = 4605.17$ and $v_\gamma = 2302.59$. By choosing $\lambda = 0.4, 0.7, 0.8, 0.99$, different optimal ceded loss functions are obtained and the results are reported in Table 2.

Table 1. $f^*(x)$ under different λ.

λ	$f^*(x)$
$\lambda = 0.2$	$f^*(x) = x$
$\lambda = 0.3$	$f^*(x) = (x - 3423.18)_+$
$\lambda = 0.4$	$f^*(x) = (x - 3170.09)_+$
$\lambda = 0.6$	$f^*(x) = (x - 4605.17)_+$

Table 2. $f^*(x)$ under dual premium principle.

λ	$f^*(x)$
$\lambda = 0.4$	$f^*(x) = 0$
$\lambda = 0.7$	$f^*(x) = x$
$\lambda = 0.8$	$f^*(x) = 0$
$\lambda = 0.99$	$f^*(x) = (x - 2995.73)_+$

(i) If $\lambda = 0.4$, we have $m_0 = -2.4 < 0$, $m = 49.6 > 0$, $t_3 = 1012.19$ and $c_6 = 1388.19 > 0$. Hence, from part (a) of Theorem 3.4, it follows that $f^*(x) = 0$ is the optimal ceded loss function. That is, no reinsurance is the optimal reinsurance strategy.

(ii) If $\lambda = 0.7$, we have $m_0 = 4.8 > 0$, $m = 20.8 > 0$, $t_3 = -390.08$ and $c_2 = -566.08 < 0$. Hence, from part (c) of Theorem 3.2, it follows that $f^*(x) = x$ is the optimal ceded loss function. That is, full reinsurance is the optimal reinsurance strategy.

(iii) If $\lambda = 0.8$, we have $m_0 = 7.2 > 0$, $m = 11.2 > 0$, $n = -8.8 < 0$, $t_1 = 0.08$ and $c_2 = 304.62 > 0$. Hence, from part (b) of Theorem 3.2, it follows that $f^*(x) = 0$ is the optimal ceded loss function. That is, no reinsurance is the optimal reinsurance strategy.

(iv) If $\lambda = 0.99$, we have $m_0 = 11.76 > 0$ and $m = -7.04 < 0$. Hence, from part (a) of Theorem 3.2, it follows that $f^*(x) = (x - 2995.73)_+$ is the optimal ceded loss function. That is, stop-loss reinsurance is the optimal reinsurance strategy.

From Table 1 and Table 2, it follows that the optimal ceded loss functions are changing as the weighting factor λ changes. Thus, we know that the weighting factor λ influence the optimal reinsurance strategy. However, the optimal ceded loss function does not monotonically increase or decrease with the increasing monotonicity of λ.

5. Conclusion

From the perspective of both insurers and reinsurers, we study the optimal reinsurance strategy under the TVaR risk measure and the TVaR premium principle. To prevent moral hazard and reflect the spirit of reinsurance, the set of admissible ceded loss functions is assumed to satisfy Vajda condition, that is the insurer's retained loss and the proportion paid by the reinsurer are increasing in indemnity. We get the explicit expression of the optimal reinsurance strategy and give the optimal parameters. Our results show that the optimal loss function is piecewise linear. Finally, we analyze the impact of the weighting factor λ on optimal reinsurance. Further research can discuss the optimal reinsurance strategy under heterogenous beliefs and asymmetric information. We could assume the insurers who faced different types of risks are allowed to apply different preference measures and different probability measures, and the reinsurer is subject to adverse selection issue. By adding these two conditions, the model will become more complex but more in line with the real insurance market. In the future, we can further study the optimal reinsurance problem under this model with more constraints.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Proof of Theorem 3.2

Under the condition of $\frac{1}{2}<\lambda\le 1$ and $0<\alpha\le\beta<1$, recall the definition of m_0, m, and n, equality (2.7) reduces to

$$Q(f)=m_0\int_{\gamma}^{\beta} f\left(v_t(X)\right)\mathrm{d}t+n\int_{\alpha}^{\beta} f\left(v_t(X)\right)\mathrm{d}t+m\int_{\beta}^{1} f\left(v_t(X)\right)\mathrm{d}t \tag{1}$$

Clearly $f(v_\alpha)\le f(v_\beta)$ and $v_\alpha-f(v_\alpha)\le v_\beta-f(v_\beta)$, as $f(x)$ and $R_f(x)$ are nondecreasing for all $x\ge 0$ and $\alpha\le\beta$. Note that $0\le v_\alpha\le f(v_\alpha)$ and $0\le v_\beta\le f(v_\beta)$ since $0\le f(x)\le x$ for all $x\ge 0$. Recall the definition of m in (3.2). Equality (1) reduces to

$$\begin{aligned}Q(f)=&\,m_0\int_{f(v_\gamma)}^{f(v_\beta)} S_{f(X)}(t)\mathrm{d}t+n\int_{f(v_\alpha)}^{f(v_\beta)} S_{f(X)}(t)\mathrm{d}t+m\int_{f(v_\beta)}^{\infty} S_{f(X)}(t)\mathrm{d}t-(2\lambda-1)f(v_\alpha)\\&+(1-\lambda)\left[f(v_\beta)-f(v_\alpha)\right]+(2\lambda-1)(1+\gamma)f(v_\gamma)\end{aligned} \tag{2}$$

(i) If $m>0$ and $n<0$. For the above $f\in\mathfrak{C}$, define

$$J(X)=\begin{cases}0, & 0\le x<v_\alpha-f(v_\alpha),\\ x-v_\alpha-f(v_\alpha), & v_\alpha-f(v_\alpha)\le x<\dfrac{v_\alpha-f(v_\alpha)}{v_\beta-f(v_\beta)}v_\beta\\ \dfrac{f(v_\beta)}{v_\beta}x, & x\ge\dfrac{v_\alpha-f(v_\alpha)}{v_\beta-f(v_\beta)}v_\beta.\end{cases} \tag{3}$$

Denote $d_1=v_\alpha-f(v_\alpha)$ and $\theta_1=\dfrac{f(v_\beta)}{v_\beta}$, we have

$$J(X)=\begin{cases}0, & 0\le x<d_1,\\ x-d_1, & d_1\le x<\dfrac{d_1}{1-\theta_1}\\ \theta_1 x, & x\ge\dfrac{d_1}{1-\theta_1}.\end{cases} \tag{4}$$

where $0\le\theta_1\le 1$, $(1-\theta_1)v_\beta\le d_1\le v_\alpha$.

The relationship between $f(x)$ and $J(x)$ is illustrated by **Figure A1**. One can show that $J(x)\in\mathfrak{C}$ and $Q(f)>Q(J)$ for any $f\in\mathfrak{C}$. Indeed, from **Figure A1**, we conclude that for $0\le x\le v_\alpha$ and $x\ge v_\beta$, $f(x)>J(x)$. Moreover, since $m\ge 0$, we have

$$m_0\int_{\gamma}^{\alpha} f\left(v_t(X)\right)\mathrm{d}t>m_0\int_{\gamma}^{\alpha} J\left(v_t(X)\right)\mathrm{d}t$$

$$n\int_{\alpha}^{\beta} f\left(v_t(X)\right)\mathrm{d}t>n\int_{\alpha}^{\beta} J\left(v_t(X)\right)\mathrm{d}t$$

$$m\int_{\beta}^{1} f\left(v_t(X)\right)\mathrm{d}t>m\int_{\beta}^{1} J\left(v_t(X)\right)\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(J)$, where the inequality is strict if f and J are not identical almost everywhere, which means that

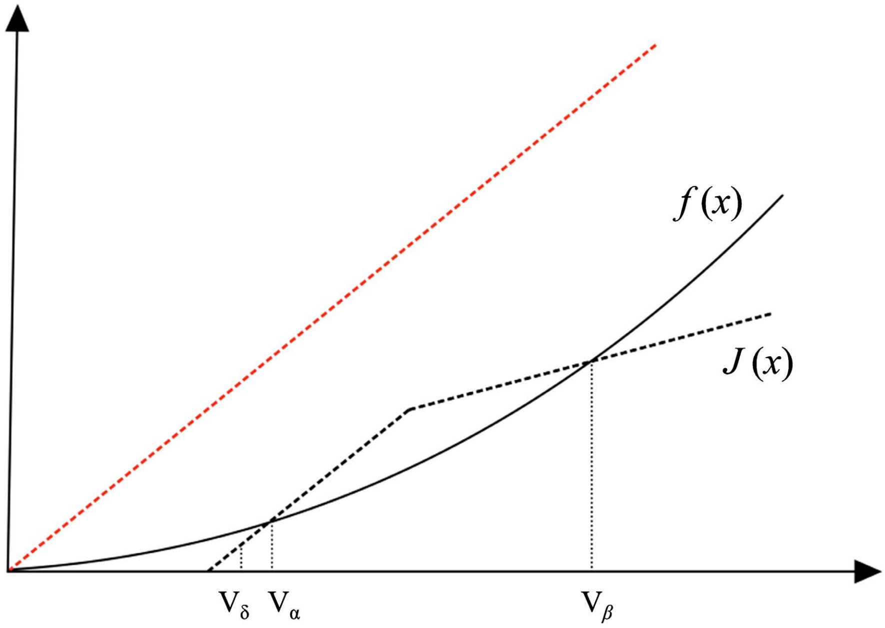

Figure A1. Relationship between $f(x)$ and $J(x)$ in case (i).

the optimal reinsurance contract can only take the form of (3) in case (i). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}} Q(f) = \min Q(J)$. The expression of $Q(J)$ is as follows.

$$\begin{aligned} Q(J) = & -\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{\frac{d_1}{1-\theta_1}} S_X(t)\mathrm{d}t + \theta_1\int_{\frac{d_1}{1-\theta_1}}^{\infty} S_X(t)\mathrm{d}t\right] + \frac{1-\lambda}{1-\beta}\theta_1\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t \\ & + m_0\left[\int_{v_\gamma}^{\frac{d_1}{1-\theta_1}} S_X(t)\mathrm{d}t + \theta_1\int_{\frac{d_1}{1-\theta_1}}^{\infty} S_X(t)\mathrm{d}t\right] \\ & -\lambda(v_\alpha - d_1) + (1-\lambda)\theta_1 v_\beta + (2\lambda-1)(1+\delta)(v_\gamma - d_1). \end{aligned} \quad (5)$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(J)}{\partial d_1} = nS_X\left(\frac{d_1}{1-\theta_1}\right) + t_1$$

We next consider the minimum value of $Q(J)$ in the case of $t_1 \le 0$ and $t_1 > 0$.

Case 1. Assume that $t_1 \le 0$

Note that $Q(J)$ is decreasing in $d_1 \in \left[(1-\theta_1)v_\beta, v_\alpha\right]$, thus,

$$\begin{aligned} Q(J) &\ge Q(J_1) \\ &= -\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{\frac{v_\alpha}{1-\theta_1}} S_X(t)\mathrm{d}t + \theta_1\int_{\frac{v_\alpha}{1-\theta_1}}^{\infty} S_X(t)\mathrm{d}t\right] + \frac{1-\lambda}{1-\beta}\theta_1\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t \\ &\quad + m_0\left[\int_{v_\gamma}^{\frac{v_\alpha}{1-\theta_1}} S_X(t)\mathrm{d}t + \theta_1\int_{\frac{v_\alpha}{1-\theta_1}}^{\infty} S_X(t)\mathrm{d}t\right] + (1-\lambda)\theta_1 v_\beta. \end{aligned}$$

Hence, $Q(J_1)$ obtains its minimum at $\theta_1 = \theta_1^*$. That is,

$$f^*(x)=\begin{cases} x-v_\alpha, & v_\alpha \le x < \dfrac{v_\alpha}{1-\theta_1^*}, \\ \theta_1^* x, & x\dfrac{v_\alpha}{1-\theta_1^*}. \end{cases}$$

where $\theta_1^* = \arg\min_{\theta_1\in[0,1]}\{Q(J_1)\}$.

Case 2. Assume that $t_1 > 0$

Note that $Q(J)$ is increasing in $d_1 \in [(1-\theta_1)v_\beta, v_\alpha]$, thus,

$$\begin{aligned} Q(J) &\ge Q(J_1) \\ &= -\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{v_\beta} S_X(t)\mathrm{d}t + \theta_1\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t\right] + \frac{1-\lambda}{1-\beta}\theta_1\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t \\ &\quad + m_0\left[\int_{v_\gamma}^{v_\beta} S_X(t)\mathrm{d}t + \theta_1\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t\right] - \lambda\left(v_\alpha - (1-\theta_1)v_\beta\right) \\ &\quad + (1-\lambda)\theta_1 v_\beta + (2\lambda-1)(1+\delta)\left(v_\gamma - (1-\theta_1)v_\beta\right). \end{aligned}$$

Taking the derivative of $Q(J_1)$ with respect to θ_1, we have

$$\frac{\partial Q(J_1)}{\partial \theta_1} = c_2.$$

If $c_2 > 0$, $Q(J_1)$ is increasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = 0$ in this situation.

If $c_2 < 0$, $Q(J_1)$ is decreasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = x$ in this situation.

If $c_2 = 0$, the optimal ceded function is

$$f^*(x)=\begin{cases} x-(1-\theta_1)v_\beta, & (1-\theta_1)v_\beta \le x < v_\beta, \\ \theta_1 x, & x \ge v_\beta. \end{cases}$$

where θ_1 can be any constant in $[0,1]$.

(ii) If $m > 0$, and $n \ge 0$, for the above $f \in \mathfrak{C}$, define

$$J(X)=\begin{cases} 0, & 0 \le x < v_\alpha - f(v_\alpha), \\ x-(v_\alpha - f(v_\alpha)), & v_\alpha - f(v_\alpha) \le x < v_\alpha, \\ \dfrac{f(v_\alpha)}{v_\alpha}x, & x \ge v_\alpha. \end{cases} \tag{6}$$

Denote $d_1 = v_\alpha - f(v_\alpha)$ and $\theta_1 = \dfrac{f(v_\alpha)}{v_\alpha}$, we have

$$J(X)=\begin{cases} 0, & 0 \le x < d_1, \\ x-d_1, & d_1 \le x < \dfrac{d_1}{1-\theta_1}, \\ \theta_1 x, & x \ge \dfrac{d_1}{1-\theta_1}. \end{cases} \tag{7}$$

where $0 \le \theta_1 \le 1$, $0 \le d_1 \le v_\alpha$.

The relationship between $f(x)$ and $J(x)$ is illustrated by **Figure A2**. One can show that $J(x) \in \mathfrak{C}$ and $Q(f) > Q(J)$ for any $f \in \mathfrak{C}$. Indeed, from **Figure A2**, we conclude that for $x \ge 0$, $f(x) > J(x)$. Moreover, since $m > 0$

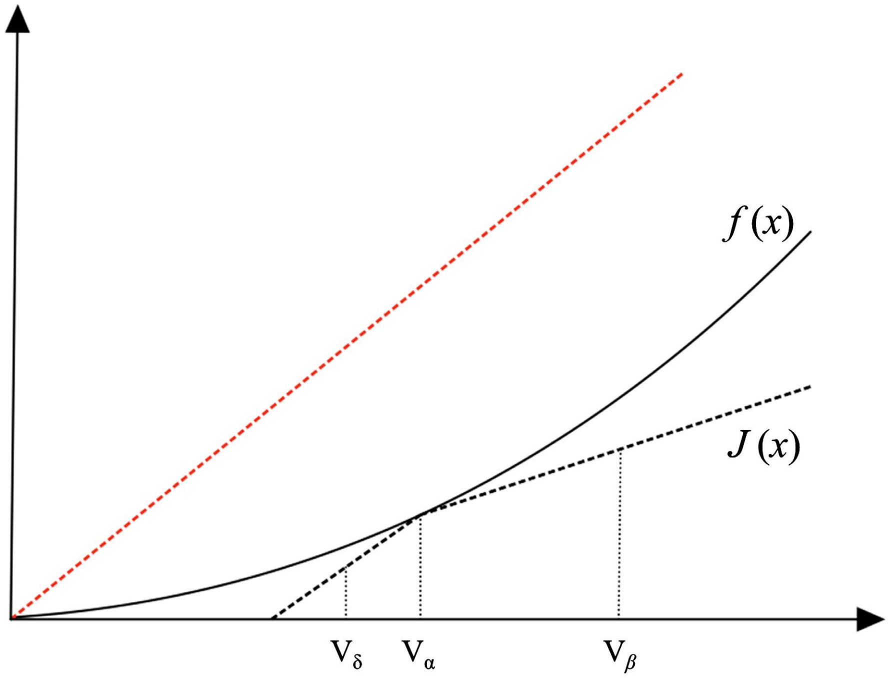

Figure A2. Relationship between $f(x)$ and $J(x)$ in case (ii).

and $n \geq 0$, we have

$$m_0 \int_\gamma^\alpha f(v_t(X))\,\mathrm{d}t > m_0 \int_\gamma^\alpha J(v_t(X))\,\mathrm{d}t$$

$$n \int_\alpha^\beta f(v_t(X))\,\mathrm{d}t > n \int_\alpha^\beta J(v_t(X))\,\mathrm{d}t$$

$$m \int_\beta^1 f(v_t(X))\,\mathrm{d}t > m \int_\beta^1 J(v_t(X))\,\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f) > Q(J)$, where the inequality is strict if *f* and *J* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (6) in case (ii). The equivalence of (24) implies that $\min_{f \in \mathfrak{C}} Q(f) = \min Q(J)$. The expression of $Q(J)$ is as follows.

$$Q(J) = -\frac{\lambda}{1-\alpha}\theta_1 \int_{v_\alpha}^\infty S_X(t)\,\mathrm{d}t + \frac{1-\lambda}{1-\beta}\theta_1 \int_{v_\beta}^\infty S_X(t)\,\mathrm{d}t + m_0\left[\int_{v_\gamma}^{v_\alpha} S_X(t)\,\mathrm{d}t + \theta_1 \int_{v_\alpha}^\infty S_X(t)\,\mathrm{d}t\right] - \lambda\theta_1 v_\alpha + (1-\lambda)\theta_1 v_\beta + (2\lambda-1)(1+\delta)(v_\gamma - d_1). \quad (8)$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(J)}{\partial d_1} = -(2\lambda-1)(1+\delta) < 0$$

We can imply that $Q(J)$ is decreasing in $d_1 \in [0, v_\alpha]$, thus,

$$Q(J) \geq Q(J_1) = -\frac{\lambda}{1-\alpha}\theta_1 \int_{v_\alpha}^\infty S_X(t)\,\mathrm{d}t + \frac{1-\lambda}{1-\beta}\theta_1 \int_{v_\beta}^\infty S_X(t)\,\mathrm{d}t + m_0\theta_1 \int_{v_\alpha}^\infty S_X(t)\,\mathrm{d}t + (1-\lambda)\theta_1 v_\beta.$$

Taking the derivative of $Q(J_1)$ with respect to θ_1, we have

$$\frac{\partial Q(J)}{\partial \theta_1} = n\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t + (1-\lambda)v_\beta > 0$$

We can imply that $Q(J_1)$ is increasing in $\theta_1 \in [0,1]$, thus, the optimal ceded function is $f^*(x) = 0$ in this situation.

(iii) If $m < 0$, then $n < 0$. For the above $f \in \mathfrak{C}$, define

$$J(X) = \begin{cases} 0, & 0 \le x < v_\alpha - f(v_\alpha), \\ x - (v_\alpha - f(v_\alpha)), & v_\alpha - f(v_\alpha) \le x < \dfrac{v_\alpha - f(v_\alpha)}{v_\beta - f(v_\beta)} v_\beta, \\ \dfrac{f(v_\beta)}{v_\beta} x, & \dfrac{v_\alpha - f(v_\alpha)}{v_\beta - f(v_\beta)} v_\beta \le x < v_\beta, \\ x - (v_\beta - f(v_\beta)), & x \ge v_\beta. \end{cases} \tag{9}$$

Denote $d_1 = v_\alpha - f(v_\alpha), d_2 = v_\beta - f(v_\beta)$ and $\theta_1 = \dfrac{f(v_\beta)}{v_\beta}$, we have

$$J(X) = \begin{cases} 0, & 0 \le x < d_1, \\ x - d_1, & d_1 \le x < \dfrac{d_1}{1-\theta_1}, \\ \theta_1 x, & \dfrac{d_1}{1-\theta_1} \le x < v_\beta, \\ x - d_2, & x \ge v_\beta. \end{cases} \tag{10}$$

where $0 \le d_1 \le v_\alpha, (1-\theta_1)v_\alpha \le d_2 \le (1-\theta_1)v_\beta$ and $0 \le \theta_1 \le 1$.

The relationship between $f(x)$ and $J(x)$ is illustrated by Figure A3. One can show that $J(x) \in \mathfrak{C}$ and $Q(f) > Q(J)$ for any $f \in \mathfrak{C}$. Moreover, since

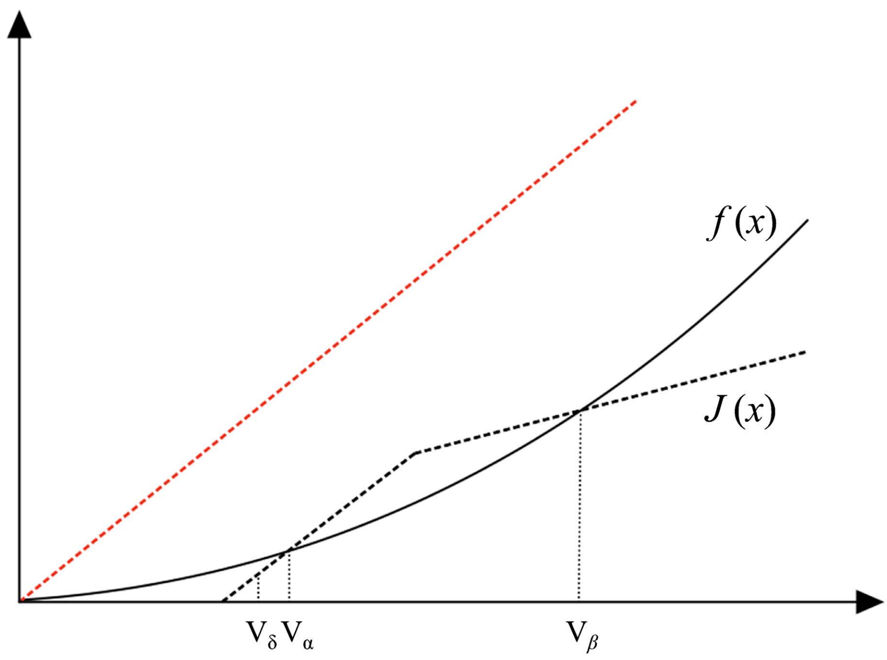

Figure A3. Relationship between $f(x)$ and $J(x)$ in case (iii).

$m<0$ and $s\ge 0$, we have

$$m_0\int_\gamma^\alpha f\left(v_t\left(X\right)\right)\mathrm{d}t > m_0\int_\gamma^\alpha J\left(v_t\left(X\right)\right)\mathrm{d}t$$

$$n\int_\alpha^\beta f\left(v_t\left(X\right)\right)\mathrm{d}t > n\int_\alpha^\beta J\left(v_t\left(X\right)\right)\mathrm{d}t$$

$$m\int_\beta^1 f\left(v_t\left(X\right)\right)\mathrm{d}t > m\int_\beta^1 J\left(v_t\left(X\right)\right)\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(J)$, where the inequality is strict if *f* and *J* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (9) in case (iii). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}} Q(f)=\min Q(J)$. The expression of $Q(J)$ is as follows.

$$\begin{aligned} Q(J) = & -\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{\frac{d_1}{1-\theta_1}} S_X(t)\mathrm{d}t+\theta_1\int_{\frac{d_1}{1-\theta_1}}^{v_\beta} S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t\right]+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t \\ & +m_0\left[\int_{v_\gamma}^{\frac{d_1}{1-\theta_1}} S_X(t)\mathrm{d}t+\theta_1\int_{\frac{d_1}{1-\theta_1}}^{v_\beta} S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t\right] \\ & -\lambda\left(v_\alpha-d_1\right)+\left(1-\lambda\right)\left(v_\beta-d_2\right)+\left(2\lambda-1\right)\left(1+\delta\right)\left(v_\gamma-d_1\right). \end{aligned} \tag{11}$$

Taking the derivative of $Q(J)$ with respect to θ_1, we have

$$\frac{\partial Q(J)}{\partial\theta_1}=n\left[\frac{d_1}{1-\theta_1}S_X\left(\frac{d_1}{1-\theta_1}\right)+\int_{\frac{d_1}{1-\theta_1}}^{\infty} S_X(t)\mathrm{d}t\right]<0$$

We can imply that $Q(J)$ is decreasing in $\theta_1\in[0,1]$, thus,

$$\begin{aligned} & Q(J)\ge Q(J_1) \\ & =-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty} S_X(t)\mathrm{d}t+m_0\int_{v_\gamma}^{\infty} S_X(t)\mathrm{d}t \\ & \quad -\lambda\left(v_\alpha-d_1\right)+\left(1-\lambda\right)\left(v_\beta-d_1\right)+\left(2\lambda-1\right)\left(1+\delta\right)\left(v_\gamma-d_1\right). \end{aligned}$$

Taking the derivative of $Q(J_1)$ with respect to d_1, we have

$$\frac{\partial Q(J_1)}{\partial d_1}=-\delta\left(2\lambda-1\right)<0$$

We can imply that $Q(J)$ is decreasing in $d_1\in[0,v_\alpha]$, thus, the optimal ceded function is $f^*(x)=(x-v_\alpha)_+$ in this situation. The proof of Theorem 3.2 is completed. □

Proof of Theorem 3.3.

Under the condition of $0\le\lambda<\frac{1}{2}$ and $0<\beta\le\alpha<1$, recall the definition of m_0, m, and s, equality (2.7) reduces to

$$Q(f)=m_0\int_\gamma^\beta f\left(v_t\left(X\right)\right)\mathrm{d}t+s\int_\beta^\alpha f\left(v_t\left(X\right)\right)\mathrm{d}t+m\int_\alpha^1 f\left(v_t\left(X\right)\right)\mathrm{d}t \tag{12}$$

Clearly $f(v_\alpha)\ge f(v_\beta)$ and $v_\alpha-f(v_\alpha)\ge v_\beta-f(v_\beta)$, as $f(x)$ and $R_f(x)$ are nondecreasing for all $x\ge 0$ and $\alpha\ge\beta$. Note that $0\le v_\alpha\le f(v_\alpha)$

and $0 \le v_\beta \le f(v_\beta)$ since $0 \le f(x) \le x$ for all $x \ge 0$. Recall the definition of m in (3.2). Equality (12) reduces to

$$Q(f) = m_0 \int_{f(v_\gamma)}^{f(v_\beta)} S_{f(x)}(t)\,dt + s\int_{f(v_\beta)}^{f(v_\alpha)} S_{f(x)}(t)\,dt + m\int_{f(v_\alpha)}^{\infty} S_{f(x)}(t)\,dt \\ -\lambda f(v_\alpha) + (1-\lambda) f(v_\beta) + (2\lambda-1)(1+\gamma) f(v_\gamma) \tag{13}$$

(i) If $m > 0$, then $s > 0$. For the above $f \in \mathfrak{C}$, define

$$L(x) = \begin{cases} \dfrac{f(v_\beta)}{v_\beta} x, & 0 \le x < x \le \dfrac{v_\alpha - f(v_\alpha)}{v_\beta - f(v_\beta)} v_\beta, \\ x - (v_\alpha - f(v_\alpha)), & \dfrac{v_\alpha - f(v_\alpha)}{v_\beta - f(v_\beta)} v_\beta \le x < v_\alpha, \\ \dfrac{f(v_\alpha)}{v_\alpha} x, & x \ge v_\alpha. \end{cases} \tag{14}$$

Denote $\theta_1 = \dfrac{f(v_\beta)}{v_\beta}, \theta_2 = \dfrac{f(v_\alpha)}{v_\alpha}$ and $d_1 = v_\alpha - f(v_\alpha)$, we have

$$L(x) = \begin{cases} \theta_1 x, & 0 \le x < \dfrac{d_1}{1-\theta_1}, \\ x - d_1, & \dfrac{d_1}{1-\theta_1} \le x < v_\alpha, \\ \theta_2 x, & x \ge v_\alpha. \end{cases} \tag{15}$$

where $0 \le \theta_1 \le \theta_2 \le 1, 0 \le d_1 \le v_\alpha$.

The relationship between $f(x)$ and $L(x)$ is illustrated by **Figure A4**. One

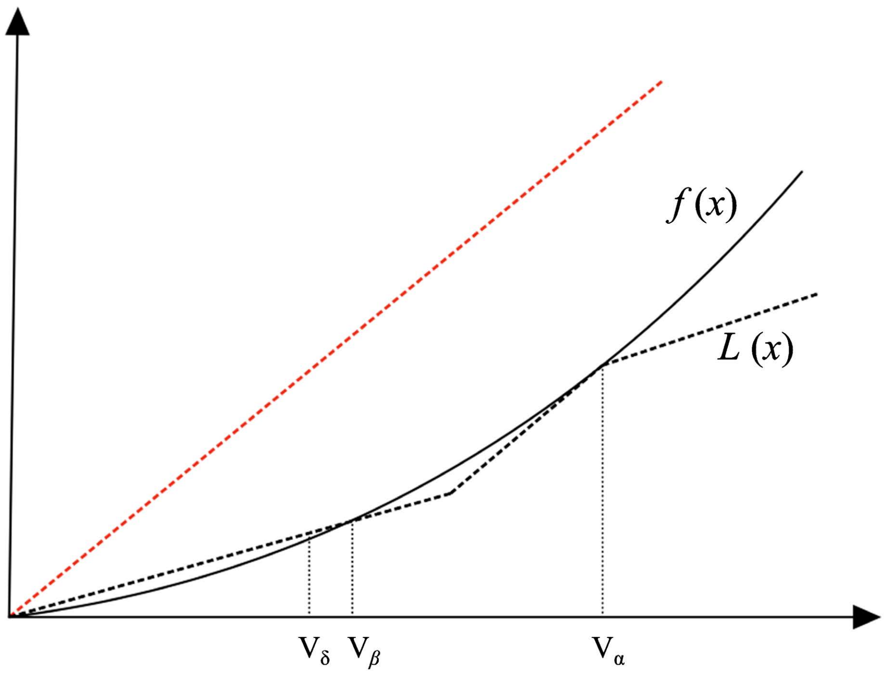

Figure A4. Relationship between $f(x)$ and $L(x)$ in case (i).

can show that $L(x)\in\mathfrak{C}$ and $Q(f)>Q(L)$ for any $f\in\mathfrak{C}$. Indeed, from **Figure A4**, we conclude that for $x\geq 0$, $f(x)>L(x)$. Moreover, since $m\geq 0$, we have

$$m_0\int_\gamma^\beta f(v_t(X))\mathrm{d}t>m_0\int_\gamma^\beta L(v_t(X))\mathrm{d}t$$

$$s\int_\beta^\alpha f(v_t(X))\mathrm{d}t>s\int_\beta^\alpha L(v_t(X))\mathrm{d}t$$

$$m\int_\alpha^1 f(v_t(X))\mathrm{d}t>m\int_\alpha^1 L(v_t(X))\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(L)$, where the inequality is strict if *f* and *L* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (15) in case (i). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}}Q(f)=\min Q(L)$. The expression of $Q(J)$ is as follows.

$$\begin{aligned}Q(L)=&-\frac{\lambda}{1-\alpha}\theta_2\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\left[\theta_1\int_{v_\beta}^{\frac{d_1}{1-\theta_1}}S_X(t)\mathrm{d}t+\int_{\frac{d_1}{1-\theta_1}}^{v_\alpha}S_X(t)\mathrm{d}t+\theta_2\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t\right]\\&+m_0\left[\theta_1\int_{v_\gamma}^{\frac{d_1}{1-\theta_1}}S_X(t)\mathrm{d}t+\int_{\frac{d_1}{1-\theta_1}}^{v_\alpha}S_X(t)\mathrm{d}t+\theta_2\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t\right]-\lambda(v_\alpha-d_1)\\&+(1-\lambda)\theta_1v_\beta+(2\lambda-1)(1+\delta)\theta_1v_\gamma.\end{aligned}\tag{16}$$

Taking the derivative of $Q(L)$ with respect to θ_2, we have

$$\frac{\partial Q(L)}{\partial\theta_2}=m\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t>0.$$

We can imply that $Q(L)$ is decreasing in $\theta_2\in[\theta_1,1]$, thus,

$$\begin{aligned}&Q(L)\geq Q(L_1)\\&=-\frac{\lambda}{1-\alpha}\theta_1\int_{v_\alpha}^\infty S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\theta_1\int_{v_\beta}^\infty S_X(t)\mathrm{d}t+m_0\theta_1\int_{v_\gamma}^\infty S_X(t)\mathrm{d}t\\&\quad-\lambda\theta_1v_\alpha+(1-\lambda)\theta_1v_\beta+(2\lambda-1)(1+\delta)\theta_1v_\gamma.\end{aligned}$$

Taking the derivative of $Q(L_1)$ with respect to θ_1, we have

$$\frac{\partial Q(L_1)}{\partial\theta_1}=c_3.$$

If $c_3>0$, $Q(L_1)$ is increasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=0$ in this situation.

If $c_3<0$, $Q(L_1)$ is decreasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=x$ in this situation.

If $c_3=0$, the optimal ceded function is

$$f^*(x)=\theta_1x.$$

where θ_1 can be any constant in $[0,1]$.

(ii) If $m\leq 0$ and $s>0$, for the above $f\in\mathfrak{C}$, define

$$L(x)=\begin{cases}\frac{f(v_\beta)}{v_\beta}x, & 0\le x<\frac{v_\alpha-f(v_\alpha)}{v_\beta-f(v_\beta)}v_\beta,\\ x-(v_\alpha-f(v_\alpha)), & x\ge\frac{v_\alpha-f(v_\alpha)}{v_\beta-f(v_\beta)}v_\beta.\end{cases}\tag{17}$$

Denote $\theta_1=\frac{f(v_\beta)}{v_\beta}$ and $d_1=v_\alpha-f(v_\alpha)$, we have

$$L(x)=\begin{cases}\theta_1 x, & 0\le x<\frac{d_1}{1-\theta_1},\\ x-(v_\alpha-f(v_\alpha)), & x\ge\frac{d_1}{1-\theta_1}.\end{cases}\tag{18}$$

where $0\le\theta_1\le 1$ and $(1-\theta_1)v_\beta\le d_1\le(1-\theta_1)v_\alpha$.

The relationship between $f(x)$ and $J(x)$ is illustrated by Figure A5. One can show that $L(x)\in\mathfrak{C}$ and $Q(f)>Q(L)$ for any $f\in\mathfrak{C}$. Indeed, from Figure A5, we conclude that for $0\le x\le v_\beta$, $f(x)>L(x)$, and for $x>v_\beta$, $f(x)<L(x)$. Moreover, since $m<0$ and $s<0$, we have

$$m_0\int_\gamma^\beta f(v_t(X))\mathrm{d}t>m_0\int_\gamma^\beta L(v_t(X))\mathrm{d}t$$

$$s\int_\beta^\alpha f(v_t(X))\mathrm{d}t>s\int_\beta^\alpha L(v_t(X))\mathrm{d}t$$

$$m\int_\alpha^1 f(v_t(X))\mathrm{d}t>m\int_\alpha^1 L(v_t(X))\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(L)$, where the inequality is strict if f and L are not identical almost everywhere, which means that

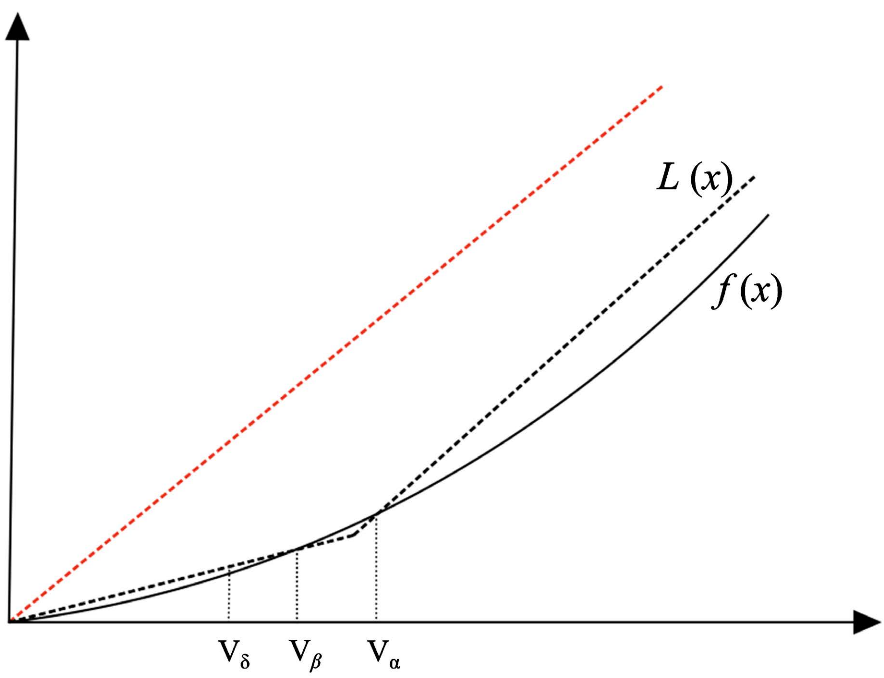

Figure A5. Relationship between $f(x)$ and $L(x)$ in case (ii).

the optimal reinsurance contract can only take the form of (17) in case (ii). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}} Q(f)=\min Q(L)$. The expression of $Q(J)$ is as follows.

$$\begin{aligned}Q(L)=&-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty}S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\left[\theta_1\int_{v_\beta}^{\frac{d_1}{1-\theta_1}}S_X(t)\mathrm{d}t+\int_{\frac{d_1}{1-\theta_1}}^{\infty}S_X(t)\mathrm{d}t\right]\\&+m_0\left[\theta_1\int_{v_\gamma}^{\frac{d_1}{1-\theta_1}}S_X(t)\mathrm{d}t+\int_{\frac{d_1}{1-\theta_1}}^{\infty}S_X(t)\mathrm{d}t\right]-\lambda(v_\alpha-d_1)+(1-\lambda)\theta_1v_\beta\\&+(2\lambda-1)(1+\delta)\theta_1v_\gamma.\end{aligned}\tag{19}$$

Taking the derivative of $Q(J)$ with respect to d_1, we have

$$\frac{\partial Q(L)}{\partial d_1}=-sS_X\left(\frac{d_1}{1-\theta_1}\right)+\lambda$$

and

$$\frac{\partial Q(L)}{\partial d_1}\le 0\Leftrightarrow S_X\left(\frac{d_1}{1-\theta_1}v_\alpha\right)\ge\frac{\lambda}{s}.$$

Case 1 assumes that $s\le\lambda$

We can imply that $Q(L)$ is increasing in $d_1\in\left[(1-\theta_1)v_\alpha,(1-\theta_1)v_\beta\right]$, thus,

$$\begin{aligned}&Q(L)\ge Q(L_1)\\&=-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty}S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t+m_0\left[\theta_1\int_{v_\gamma}^{v_\beta}S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]\\&-\lambda\left(v_\alpha-(1-\theta_1)v_\beta\right)+(1-\lambda)\theta_1v_\beta+(2\lambda-1)(1+\delta)\theta_1v_\gamma\end{aligned}$$

Taking the derivative of $Q(L_1)$ with respect to θ_1, we have

$$\frac{\partial Q(L)}{\partial\theta_1}=c_4.$$

If $c_4>0$, $Q(L_1)$ is increasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=(x-v_\beta)_+$ in this situation.

If $c_4<0$, $Q(L_1)$ is decreasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=x$ in this situation.

If $c_4=0$, the optimal ceded function is

$$f^*(x)=\begin{cases}\theta_1x, & 0\le x<v_\beta,\\ x-(1-\theta_1)v_\beta, & x\ge v_\beta.\end{cases}$$

where θ_1 can be any constant in $[0,1]$.

Case 2 assumes that $s>\lambda$

$$\begin{aligned}\frac{\partial Q(L)}{\partial d_1}\le 0\Leftrightarrow S_X\left(\frac{d_1}{1-\theta_1}v_\alpha\right)\ge\frac{\lambda}{s}\\ \Leftrightarrow v_\Delta\le v_\alpha.\end{aligned}\tag{20}$$

Because $m\le 0$, we can imply that $0\le\Delta\le\alpha$, thus, $Q(L)$ is decreasing in $d_1\in\left[(1-\theta_1)v_\beta,(1-\theta_1)v_\Delta\right]$ and increasing in $d_1\in\left[(1-\theta_1)v_\Delta,(1-\theta_1)v_\alpha\right]$. $Q(L)$ obtains its minimum at $d_1=(1-\theta_1)v_\Delta$. Thus,

$$\begin{aligned} Q(L) &\ge Q(L_1) \\ &= -\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty} S_X(t)\mathrm{d}t + \frac{1-\lambda}{1-\beta}\left[\theta_1\int_{v_\beta}^{v_\Delta} S_X(t)\mathrm{d}t + \int_{v_\Delta}^{\infty} S_X(t)\mathrm{d}t\right] \\ &\quad + m_0\left[\theta_1\int_{v_\gamma}^{v_\Delta} S_X(t)\mathrm{d}t + \int_{v_\Delta}^{\infty} S_X(t)\mathrm{d}t\right] - \lambda\left(v_\alpha - (1-\theta_1)v_\Delta\right) \\ &\quad + (1-\lambda)\theta_1 v_\beta + (2\lambda-1)(1+\delta)\theta_1 v_\gamma. \end{aligned}$$

Taking the derivative of $Q(L_1)$ with respect to θ_1, we have

$$\frac{\partial Q(L_1)}{\partial \theta_1} = c_5$$

If $c_5 > 0$, $Q(L_1)$ is increasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = (x - v_\Delta)_+$ in this situation.

If $c_5 < 0$, $Q(L_1)$ is decreasing in $\theta_1 \in [0,1]$. In other words, the optimal ceded function is $f^*(x) = x$ in this situation.

If $c_5 = 0$, the optimal ceded function is

$$f^*(x) = \begin{cases} \theta_1 x, & 0 \le x < v_\Delta, \\ x - (1-\theta_1)v_\Delta, & x \ge v_\Delta. \end{cases}$$

where θ_1 can be any constant in $[0,1]$.

(iii) If $m \le 0$ and $s \le 0$, for the above $f \in \mathfrak{C}$, define

$$L(x) = \begin{cases} \dfrac{f(v_\beta)}{v_\beta}x, & 0 \le x < v_\beta, \\ x - \left(v_\beta - f(v_\beta)\right), & x \ge v_\beta. \end{cases} \tag{21}$$

Denote $\theta_1 = \dfrac{f(v_\beta)}{v_\beta}$ and $d_1 = v_\beta - f(v_\beta)$, we have

$$L(x) = \begin{cases} \theta_1 x, & 0 \le x < v_\beta, \\ x - d_1, & x \ge v_\beta. \end{cases} \tag{22}$$

where $0 \le \theta_1 \le 1$ and $0 \le d_1 \le v_\beta$.

The relationship between $f(x)$ and $L(x)$ is illustrated by **Figure A6**. One can show that $J(x) \in \mathfrak{C}$ and $Q(f) > Q(L)$ for any $f \in \mathfrak{C}$. Indeed, from **Figure A6**, we conclude that for $x \ge 0$, $f(x) < L(x)$. Moreover, since $m < 0$ and $s \le 0$, we have

$$m_0\int_\gamma^\beta f(v_t(X))\mathrm{d}t > m_0\int_\gamma^\beta L(v_t(X))\mathrm{d}t$$

$$s\int_\beta^\alpha f(v_t(X))\mathrm{d}t > s\int_\beta^\alpha L(v_t(X))\mathrm{d}t$$

$$m\int_\alpha^1 f(v_t(X))\mathrm{d}t > m\int_\alpha^1 L(v_t(X))\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f) > Q(L)$, where the inequality is strict if *f* and *L* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (22) in case (iii). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}} Q(f) = \min Q(L)$. The expression of

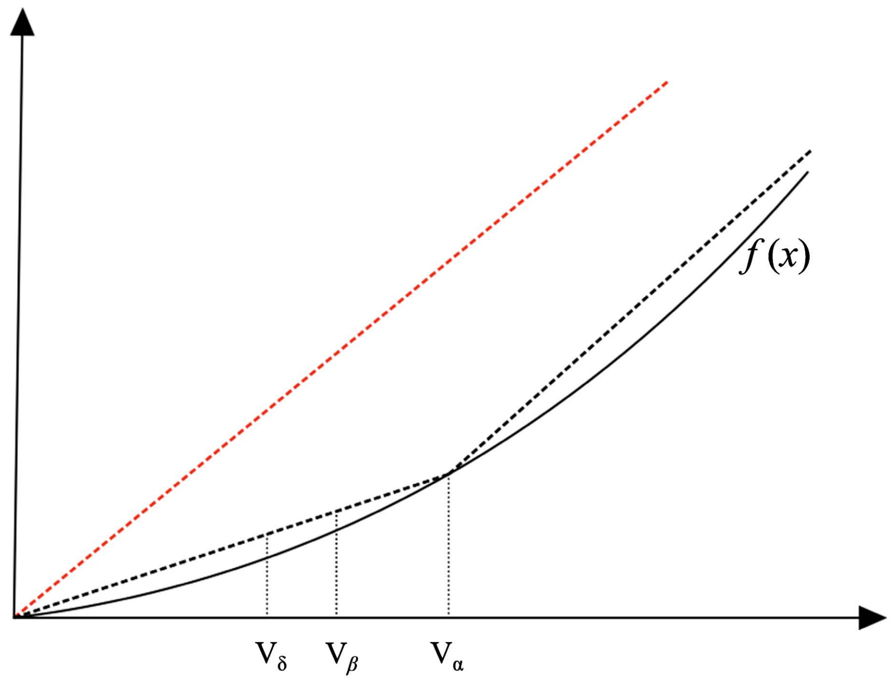

Figure A6. Relationship between $f(x)$ and $J(x)$ in case (iii).

$Q(J)$ is as follows.

$$Q(L)=-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty}S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t+m_0\left[\theta_1\int_{v_\gamma}^{v_\beta}S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]$$
$$-\lambda(v_\alpha-d_1)+(1-\lambda)\theta_1 v_\beta+(2\lambda-1)(1+\delta)\theta_1 v_\gamma. \quad (23)$$

Taking the derivative of $Q(L)$ with respect to d_1, we have

$$\frac{\partial Q(L)}{\partial d_1}=\lambda>0$$

We can imply that $Q(L)$ is increasing in $d_1\in[0,v_\beta]$, thus,

$$Q(L)\geq Q(L_1)$$
$$=-\frac{\lambda}{1-\alpha}\int_{v_\alpha}^{\infty}S_X(t)\mathrm{d}t+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t+m_0\left[\theta_1\int_{v_\gamma}^{v_\beta}S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]$$
$$-\lambda v_\alpha+(1-\lambda)v_\beta+(2\lambda-1)(1+\delta)\theta_1 v_\gamma.$$

Taking the derivative of $Q(L_1)$ with respect to θ_1, we have

$$\frac{\partial Q(L_1)}{\partial\theta_1}=m_0\int_{v_\gamma}^{v_\beta}S_X(t)\mathrm{d}t+(2\lambda-1)(1+\delta)v_\gamma<0.$$

$Q(L_1)$ is decreasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=x$ in this situation. The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.4

Under the condition of $0\leq\lambda<\frac{1}{2}$ and $0<\alpha\leq\beta<1$, recall the definition of m_0, m, and s, equality (2.7) reduces to

$$Q(f) = m_0\int_{\gamma}^{\alpha} f\left(v_t(X)\right)\mathrm{d}t + n\int_{\alpha}^{\beta} f\left(v_t(X)\right)\mathrm{d}t + m\int_{\beta}^{1} f\left(v_t(X)\right)\mathrm{d}t \quad (24)$$

Clearly $f(v_\alpha) \le f(v_\beta)$ and $v_\alpha - f(v_\alpha) \le v_\beta - f(v_\beta)$, as $f(x)$ and $R_f(x)$ are nondecreasing for all $x \ge 0$ and $\alpha \le \beta$. Note that $0 \le v_\alpha \le f(v_\alpha)$ and $0 \le v_\beta \le f(v_\beta)$ since $0 \le f(x) \le x$ for all $x \ge 0$. Recall the definition of m in (3.2). Equality (24) reduces to

$$\begin{aligned} Q(f) = {} & m_0\int_{f(v_\gamma)}^{f(v_\beta)} S_{f(X)}(t)\mathrm{d}t + n\int_{f(v_\alpha)}^{f(v_\beta)} S_{f(X)}(t)\mathrm{d}t + m\int_{f(v_\beta)}^{\infty} S_{f(X)}(t)\mathrm{d}t \\ & -(2\lambda-1) f(v_\alpha) + (1-\lambda)\left[f(v_\beta) - f(v_\alpha)\right] + (2\lambda-1)(1+\gamma) f(v_\gamma) \end{aligned} \quad (25)$$

(i) If $m > 0$, for the above $f \in \mathfrak{C}$, define

$$L(x) = \begin{cases} \dfrac{f(v_\alpha)}{v_\alpha}x, & 0 \le x < v_\alpha, \\ x - (1-\theta_1)v_\alpha, & v_\alpha \le x < v_\beta, \\ \dfrac{f(v_\beta)}{v_\beta}x, & x \ge v_\beta. \end{cases} \quad (26)$$

Denote $\theta_1 = \dfrac{f(v_\alpha)}{v_\alpha}, \theta_2 = \dfrac{f(v_\beta)}{v_\beta}$ and $d_1 = (1-\theta_1)v_\alpha$, we have

$$L(x) = \begin{cases} \theta_1 x, & 0 \le x < v_\alpha, \\ x - (1-\theta_1)v_\alpha, & v_\alpha \le x < v_\beta, \\ \theta_2 x, & x \ge v_\beta. \end{cases} \quad (27)$$

where $0 \le \theta_1 \le \theta_2 \le 1$ and $0 \le d_1 \le v_\alpha$.

The relationship between $f(x)$ and $L(x)$ is illustrated by Figure A7. One

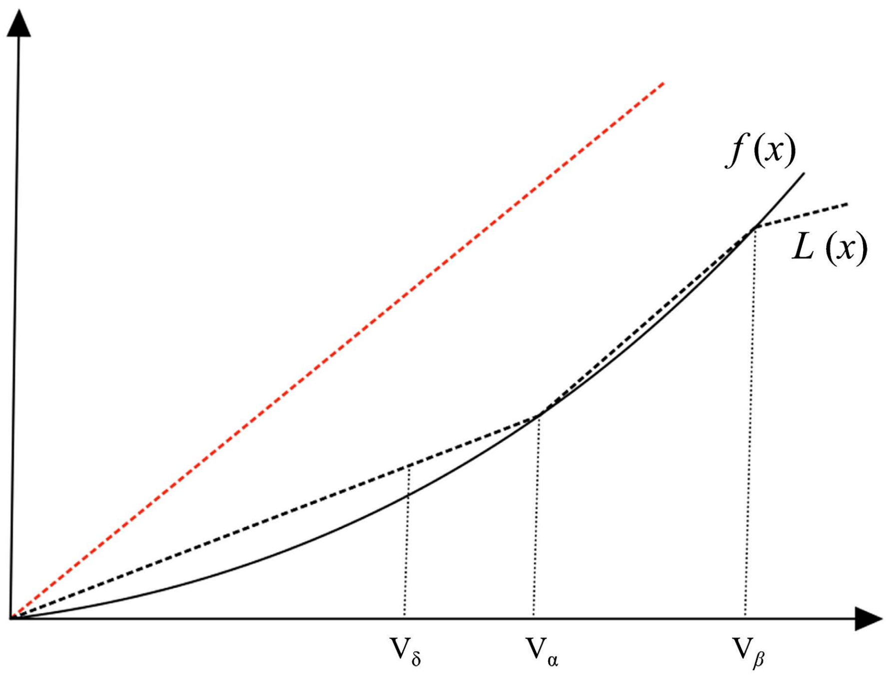

Figure A7. Relationship between $f(x)$ and $L(x)$ in case (i).

can show that $L(x)\in\mathfrak{C}$ and $Q(f)>Q(L)$ for any $f\in\mathfrak{C}$. Indeed, from Figure A7, we conclude that for $0\le x\le v_\alpha$ and $x\ge v_\beta$, $f(x)<L(x)$. Moreover, since $m>0$, we have

$$m_0\int_\gamma^\beta f\left(v_t(X)\right)\mathrm{d}t>m_0\int_\gamma^\beta L\left(v_t(X)\right)\mathrm{d}t$$

$$s\int_\beta^\alpha f\left(v_t(X)\right)\mathrm{d}t>s\int_\beta^\alpha L\left(v_t(X)\right)\mathrm{d}t$$

$$m\int_\alpha^1 f\left(v_t(X)\right)\mathrm{d}t>m\int_\alpha^1 L\left(v_t(X)\right)\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(L)$, where the inequality is strict if *f* and *L* are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (27) in case (i). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}}Q(f)=\min Q(L)$. The expression of $Q(J)$ is as follows.

$$Q(L)=-\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{v_\beta}S_X(t)\mathrm{d}t+\theta_1\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]+\frac{1-\lambda}{1-\beta}\theta_2\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t \tag{28}$$

$$+m_0\left[\theta_1\int_{v_\gamma}^{v_\alpha}S_X(t)\mathrm{d}t+\int_{v_\alpha}^{v_\beta}S_X(t)\mathrm{d}t+\theta_2\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]-\lambda\theta_1v_\alpha$$
$$+(1-\lambda)\theta_2v_\beta+(2\lambda-1)(1+\delta)\theta_1v_\gamma. \tag{29}$$

Taking the derivative of $Q(L)$ with respect to θ_2, we have

$$\frac{\partial Q(L)}{\partial\theta_2}=m\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t+(1-\lambda)v_\beta>0$$

We can imply that $Q(L)$ is increasing in $\theta_2\in[\theta_1,1]$, thus,

$$\begin{aligned}&Q(L)\ge Q(L_1)\\&=-\frac{\lambda}{1-\alpha}\left[\int_{v_\alpha}^{v_\beta}S_X(t)\mathrm{d}t+\theta_1\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]+\frac{1-\lambda}{1-\beta}\theta_1\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\\&\quad+m_0\left[\theta_1\int_{v_\gamma}^{v_\alpha}S_X(t)\mathrm{d}t+\int_{v_\alpha}^{v_\beta}S_X(t)\mathrm{d}t+\theta_1\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]\\&\quad-\lambda\theta_1v_\alpha+(1-\lambda)\theta_1v_\beta+(2\lambda-1)(1+\delta)\theta_1v_\gamma.\end{aligned}$$

Taking the derivative of $Q(L_1)$ with respect to θ_1, we have

$$\frac{\partial Q(L_1)}{\partial\theta_1}=c_6.$$

If $c_6>0$, $Q(L_1)$ is increasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=0$ in this situation.

If $c_6<0$, $Q(L_1)$ is decreasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=x$ in this situation.

If $c_6=0$, the optimal ceded function is

$$f^*(x)=\theta_1x.$$

where θ_1 can be any constant in $[0,1]$.

(ii) If $m<0$, for the above $f\in\mathfrak{C}$, define

$$L(x)=\begin{cases}\dfrac{f(v_\beta)}{v_\beta}x, & 0\le x<v_\beta,\\ x-(1-\theta_1)v_\beta, & x\ge v_\beta.\end{cases} \tag{30}$$

Denote $\theta_1=\dfrac{f(v_\beta)}{v_\beta}$ and $d_1=(1-\theta_1)v_\beta$, we have

$$L(x)=\begin{cases}\theta_1 x, & 0\le x<v_\beta,\\ x-d_1, & x\ge v_\beta.\end{cases} \tag{31}$$

where $0\le\theta_1\le 1$ and $0\le d_1\le v_\alpha$.

The relationship between $f(x)$ and $L(x)$ is illustrated by **Figure A8**. One can show that $L(x)\in\mathfrak{C}$ and $Q(f)>Q(L)$ for any $f\in\mathfrak{C}$. Indeed, from **Figure A8**, we conclude that for $x\ge 0$, $f(x)<L(x)$. Moreover, since $m<0$ and $s<0$, we have

$$m_0\int_\gamma^\alpha f(v_t(X))\mathrm{d}t>m_0\int_\gamma^\alpha L(v_t(X))\mathrm{d}t$$

$$n\int_\alpha^\beta f(v_t(X))\mathrm{d}t>n\int_\alpha^\beta L(v_t(X))\mathrm{d}t$$

$$m\int_\beta^1 f(v_t(X))\mathrm{d}t>m\int_\beta^1 L(v_t(X))\mathrm{d}t$$

Hence, it follows immediately from (24) that $Q(f)>Q(L)$, where the inequality is strict if f and L are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (17) in case (ii). The equivalence of (24) implies that $\min_{f\in\mathfrak{C}}Q(f)=\min Q(L)$. The expression of $Q(J)$ is as follows.

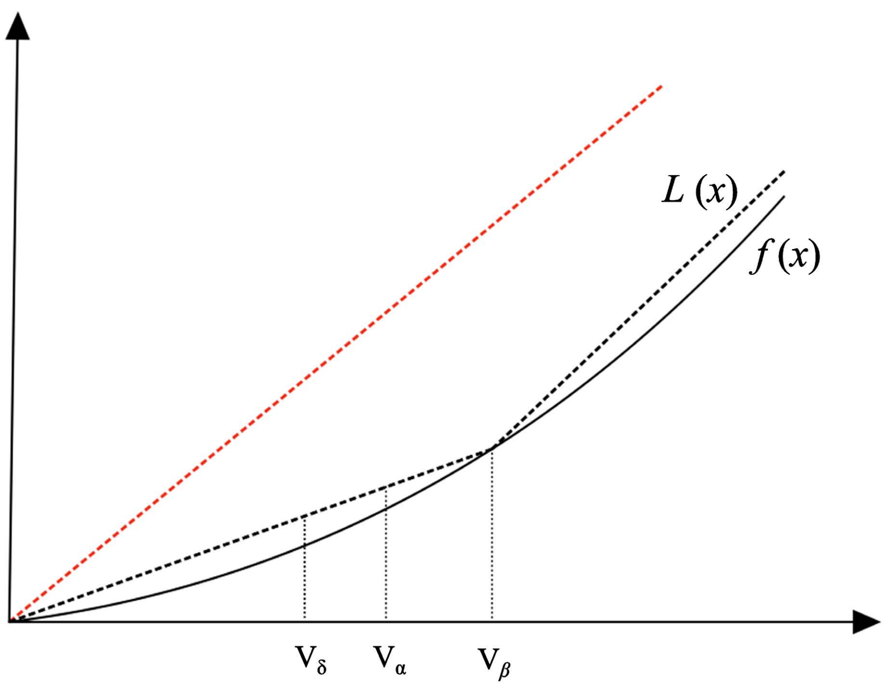

Figure A8. Relationship between $f(x)$ and $L(x)$ in case (ii).

$$\begin{aligned}Q(L)=&-\frac{\lambda}{1-\alpha}\left[\theta_1\int_{v_\alpha}^{v_\beta}S_X(t)\mathrm{d}t+\int_{v_\beta}^{v_\infty}S_X(t)\mathrm{d}t\right]+\frac{1-\lambda}{1-\beta}\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\\&+m_0\left[\theta_1\int_{v_\gamma}^{v_\beta}S_X(t)\mathrm{d}t+\int_{v_\beta}^{\infty}S_X(t)\mathrm{d}t\right]-\lambda\theta_1 v_\alpha+(1-\lambda)\theta_1 v_\beta\\&+(2\lambda-1)(1+\delta)\theta_1 v_\gamma.\end{aligned}\tag{32}$$

Taking the derivative of $Q(L)$ with respect to θ_1, we have

$$\frac{\partial Q(L)}{\partial\theta_1}=c_7.$$

If $c_7>0$, $Q(L_1)$ is increasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=(x-v_\beta)_+$ in this situation.

If $c_7<0$, $Q(L_1)$ is decreasing in $\theta_1\in[0,1]$. In other words, the optimal ceded function is $f^*(x)=x$ in this situation.

If $c_7=0$, the optimal ceded function is

$$f^*(x)=\begin{cases}\theta_1 x, & 0\le x<v_\beta,\\ x-(1-\theta_1)v_\beta, & x\ge v_\beta.\end{cases}$$

where θ_1 can be any constant in $[0,1]$. The proof of Theorem 3.4 is completed. □