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An Extended Riccati Equation Method to Find New Solitary Wave Solutions of the Burgers-Fisher Equation

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Abstract

In this paper, our objective is to explore novel solitary wave solutions of the Burgers-Fisher equation, which characterizes the interplay between diffusion and reaction phenomena. Understanding this equation is crucial for addressing challenges in fluid, chemical kinetics and population dynamics. We tackle this task by employing the Riccati equation and employing various function transformations to solve the Burgers-Fisher equation. By adopting different coefficients in the Riccati equation, we obtain a wide range of exact solutions, many of which have not been previously documented. These abundant solitary wave solutions serve as valuable tools for comprehending the Burgers-Fisher equation and contribute to expanding our knowledge in this field.

Keywords

Solitary Wave, Soliton, Burger-Fisher Equation, Riccati Equation, Nonlinear Evolution Equation

1. Introduction

Currently, the investigation into solitary waves and solitons is a prominent topic in the field of nonlinear physics, encompassing diverse areas such as chemical kinetics, population dynamics, plasma, optics, and biology [1]-[6]. By means of experimental evidence and physical mechanisms, the existence of solitons has been comprehended. Mathematical models based on reasonable assumptions have been established to describe the governing laws of numerous physical phenomena. In these studies, nonlinear evolution equations play a crucial role, and the discovery of exact solutions, particularly solitary wave solutions, is essential for understanding nonlinear problems and their characteristics. These solutions have practical implications in fields such as chemical kinetics and population dynamics.

Throughout the centuries, numerous scientists have dedicated their efforts to obtaining exact solutions for nonlinear evolution equations. Several effective and powerful methods have been proposed in previous works, including the tanh-sech method, extended tanh-coth method [7] [8], F-expansion method [9] [10], Jacobi elliptic function expansion method [11] [12], auxiliary equation method [13] [14] [15] [16], and others. However, while some of these methods have achieved success, not all of them are suitable for obtaining exact solutions for the Burgers-Fisher equation [16], which describes the interaction between diffusion and reaction processes. This equation arises in chemical kinetics and population dynamics, involving phenomena such as the nonlinear evolution of one- dimensional conventional neutron populations in nuclear reactions. In Ref. [17], the extended tanh-coth method was employed to solve the Burgers-Fisher equation, leading to the discovery of numerous new solitary wave solutions.

This paper proposes the utilization of the Riccati equation as an auxiliary equation to solve the Burgers-Fisher equation, offering the potential for new findings. The Burgers-Fisher equation is a highly nonlinear equation that combines reaction, convection, and diffusion mechanisms. It has extensive applications in fluid dynamics, gas dynamics, chemical dynamics, population dynamics, and other fields. This paper proposes an extended Riccati method, which not only simplifies the solving process of the Burgers-Fisher equation but also reveals new solitary wave solutions to the equation.

The structure of this paper is organized as follows: Section 2 introduces the methodology for constructing abundant exact solutions of the Riccati equation. Section 3 demonstrates the application of this method to derive new solitary wave solutions for the Burgers-Fisher equation. Finally, Section 4 presents the conclusion of the paper.

2. Abundant Exact Solutions of Riccati Equation

The Riccati equation method is very simple but very effective. Hence, it is an ideal method to solve constant coefficient, variable coefficient and high-dimensional nonlinear evolution equations. In the paper, it first comes to our mind that we can use the Racatti equation to solve the problems in the following form:

$$f'(\xi) = p_1 f^2(\xi) + q_1 \tag{1}$$

where p_1 and q_1 are constants and can be determined later. To find out new exact solutions of Equation (1), a new auxiliary function $g(\xi)$ is introduced, which satisfies the following form:

$$\left[g'(\xi)\right]^2 = p_2 g^2(\xi) + q_2 \tag{2}$$

where p_2 and q_2 are constants. Equation (2) has the following hyperbolic function solution:

$$g_1(\xi)=\sinh(\xi),\ (p_2=1, q_2=1) \tag{3}$$

$$g_2(\xi)=\cosh(\xi),\ (p_2=1, q_2=-1) \tag{4}$$

$$g_3(\xi)=\cosh^2(\xi)-\frac{1}{2}=\sinh^2(\xi)+\frac{1}{2},\ (p_2=4, q_2=-1) \tag{5}$$

$$g_4(\xi)=\sqrt{\cosh(\xi)+\varepsilon},\ \left(p_2=\frac{1}{4}, q_2=-\frac{1}{2}\varepsilon, \varepsilon^2=1\right) \tag{6}$$

Then we assume $f(\xi)$ and $g(\xi)$ have the following formal solution:

$$f(\xi)=\frac{g'(\xi)}{g(\xi)+r} \tag{7}$$

where r is a constant. Substituting Equation (7) into Equation (1) and using Equation (2), we can obtain

$$\begin{cases} p_1p_2+q_1=0, \\ p_2r=2q_1r, \\ -q_2=p_1q_2+q_1r^2. \end{cases} \tag{8}$$

Solving this system, we can obtain

$$\begin{cases} r=0, \\ p_1=-1, \\ q_1=p_2, \end{cases} \text{ or } \begin{cases} r=\pm\sqrt{-\frac{q_2}{p_2}}, \\ p_1=-\frac{1}{2}, \\ q_1=\frac{p_2}{2}. \end{cases} \tag{9}$$

So, we have the following exact solutions of Equation (1):

$$f_1(\xi)=\frac{\cosh(\xi)}{\sinh(\xi)},\ (p_1=-1, q_1=1) \tag{10}$$

$$f_2(\xi)=\frac{\sinh(\xi)}{\cosh(\xi)},\ (p_1=-1, q_1=1) \tag{11}$$

$$f_3(\xi)=\frac{2\sinh(\xi)\cosh(\xi)}{\cosh^2(\xi)-\frac{1}{2}},\ (p_1=-1, q_1=4) \tag{12}$$

$$f_4(\xi)=\frac{\sinh(\xi)}{\cosh(\xi)+\varepsilon},\ \left(p_1=-\frac{1}{2}, q_1=\frac{1}{2}, \varepsilon^2=1\right) \tag{13}$$

$$f_5(\xi)=\frac{\cosh(\xi)}{\sinh(\xi)+\varepsilon},\ \left(p_1=-\frac{1}{2}, q_1=\frac{1}{2}, \varepsilon^2=-1\right) \tag{14}$$

$$f_6(\xi)=\frac{\frac{1}{2}\sinh(\xi)}{\cosh(\xi)+\varepsilon\pm\sqrt{2\varepsilon}\sqrt{\cosh(\xi)+\varepsilon}},\ \left(p_1=-\frac{1}{2}, q_1=\frac{1}{8}, \varepsilon^2=1\right) \tag{15}$$

Next, we use the following another formal solution to solve Equation (1):

$$f(\xi)=\frac{g(\xi)g'(\xi)}{g^2(\xi)+r} \tag{16}$$

where $r \neq 0$. Substituting Equation (16) into Equation (1) and using Equation (2), we can obtain

$$\begin{cases} p_1 p_2 + q_1 = 0, \\ -q_2 + 2p_2 r = p_1 q_2 + 2q_1 r, \\ r q_2 = q_1 r^2. \end{cases} \tag{17}$$

Solving this system, we can obtain

$$\begin{cases} r = \dfrac{q_2}{2p_2}, \\ p_1 = -2, \\ q_1 = 2p_2. \end{cases} \tag{18}$$

So, we can have the following exact solutions

$$f_7(\xi) = \frac{2\sinh(\xi)\cosh^3(\xi) - \sinh(\xi)\cosh(\xi)}{\left[\cosh^2(\xi) - \frac{1}{2}\right]^2 - \frac{1}{8}}, \ (p_1 = -2, q_1 = 8) \tag{19}$$

It is easy to know that $h(\xi) = 1/f(\xi)$ can also satisfy Equation (1) in the condition of $p_1' = -q_1$, $q_1' = -p_1$. Equations (10) and (11) are a pair of solutions on this condition. Therefore, the following equations are also the solutions of Equation (1):

$$f_8(\xi) = \frac{\cosh^2(\xi) - \frac{1}{2}}{2\sinh(\xi)\cosh(\xi)}, \ (p_1 = -4, q_1 = 1) \tag{20}$$

$$f_9(\xi) = \frac{\cosh(\xi) + \varepsilon}{\sinh(\xi)}, \ (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}, \varepsilon^2 = 1) \tag{21}$$

$$f_{10}(\xi) = \frac{\sinh(\xi) + \varepsilon}{\cosh(\xi)}, \ (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}, \varepsilon^2 = -1) \tag{22}$$

$$f_{11}(\xi) = \frac{\cosh(\xi) + \varepsilon \pm \sqrt{2\varepsilon}\sqrt{\cosh(\xi) + \varepsilon}}{\frac{1}{2}\sinh(\xi)}, \ (p_1 = -\frac{1}{8}, q_1 = \frac{1}{2}, \varepsilon^2 = 1) \tag{23}$$

$$f_{12}(\xi) = \frac{\left[\cosh^2(\xi) - \frac{1}{2}\right]^2 - \frac{1}{8}}{2\sinh(\xi)\cosh^3(\xi) - \sinh(\xi)\cosh(\xi)}, \ (p_1 = -8, q_1 = 2) \tag{24}$$

$f_6(\xi)$, $f_7(\xi)$, $f_{11}(\xi)$ and $f_{12}(\xi)$ are the new types of exact solutions of Equation (1), which are rarely found in the other documents. Then, we use the Equation (1) and its solutions (10)-(15), (19) and (20)-(24) to solve the Burgers-Fisher equation, and the solving process can be greatly simplified.

3. Application of the Method

The following Burgers-Fisher equation [16] [17] is considered:

$$u_t + u u_x + u_{xx} + u(1-u) = 0 \tag{25}$$

Then suppose Equation (25) has the traveling wave solution:

$$u(x,t)=u(\xi),\ \xi=\mu x+ct \tag{26}$$

where μ and c are travelling wave parameters. Substituting the traveling wave equations into Equation (25), the following equation can be obtained:

$$cu'+\mu uu'+\mu^2u''+u(1-u)=0 \tag{27}$$

We assume that Equation (27) has the following formal solution:

$$u(\xi)=\sum_{i=0}^{n}a_i f^i(\xi)+\sum_{i=1}^{n}b_i f^{-i}(\xi) \tag{28}$$

where a_i and b_i are constants to be determined and $f^i(\xi)$ is the solutions of Equation (1) and n can be determined by the homogeneous balance method. In Equation (27), it is easy to know n = 1, so that the solution can be expressed as:

$$u(\xi)=a_0+a_1 f(\xi)+b_1 f^{-1}(\xi) \tag{29}$$

We bring the above equation into Equation (27) and use Equation (1), resulting in a series of equations a set of algebraic equations about a_0, a_1, a_2, b_1, b_2, μ and c. Then we collect all the terms with the same power of $f(\xi)$, and set each coefficient to zero. Finally, we can obtain:

Case 1

$$a_0=\frac{1}{2},a_1=\pm\frac{1}{2}\sqrt{-\frac{p_1}{q_1}},b_1=0,\mu=\mp\frac{1}{4\sqrt{-p_1q_1}},c=\pm\frac{5}{4\sqrt{-p_1q_1}} \tag{30}$$

Case 2

$$a_0=\frac{1}{2},a_1=0,b_1=\pm\frac{1}{2}\sqrt{-\frac{q_1}{p_1}},\mu=\pm\frac{1}{4\sqrt{-p_1q_1}},c=\mp\frac{5}{4\sqrt{-p_1q_1}} \tag{31}$$

Case 3

$$a_0=\frac{1}{2},a_1=\pm\frac{1}{4}\sqrt{-\frac{p_1}{q_1}},b_1=\mp\frac{1}{4}\sqrt{-\frac{q_1}{p_1}},\mu=\mp\frac{1}{8\sqrt{-p_1q_1}},c=\pm\frac{5}{16\sqrt{-p_1q_1}} \tag{32}$$

According to Case 1, we have the following solitary wave solutions of the Burgers-Fisher equation:

$$u_1(\xi)=\frac{1}{2}\pm\frac{1}{2}\coth(\xi) \tag{33}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{4},c=\pm\frac{5}{4}$. Figure 1(a) shows the three-dimensional diagrams of Equation (33), which represents the travelling wave solution with singularities. Figure 1(b) shows that the amplitude and velocity of this travelling wave remain unchanged during propagation.

$$u_2(\xi)=\frac{1}{2}\pm\frac{1}{2}\tanh(\xi) \tag{34}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{4},c=\pm\frac{5}{4}$. Figure 2(a) shows the three-dimensional

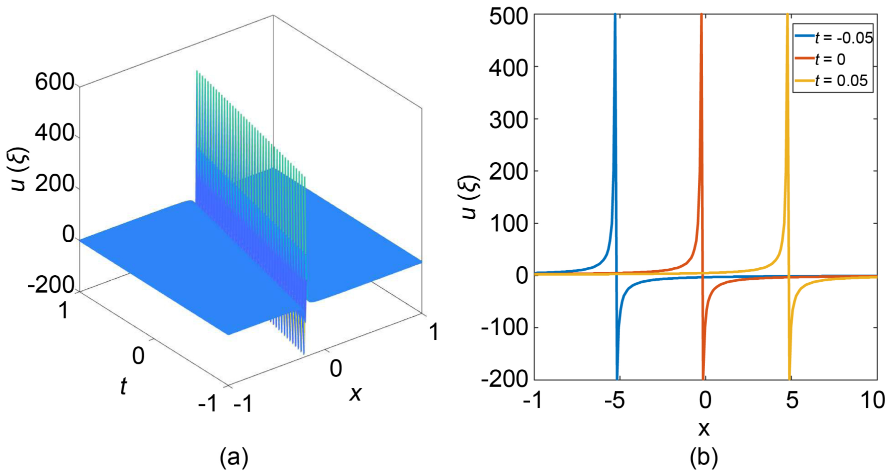


Figure 1. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (33), when ± sign takes − and ∓ sign takes +.

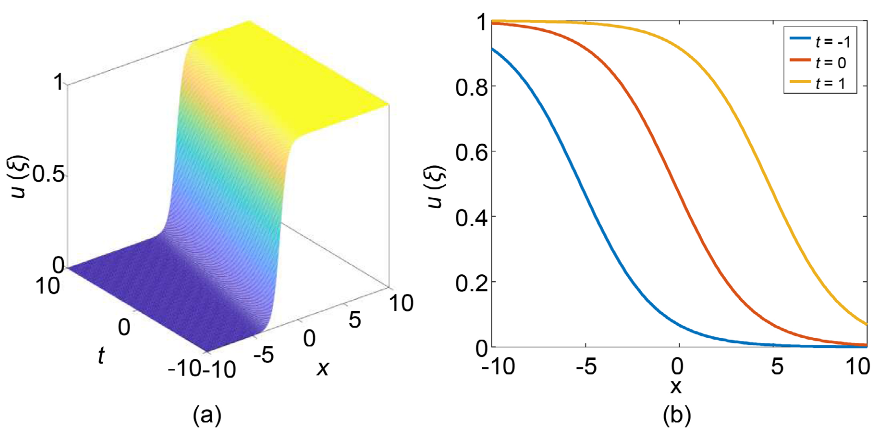


Figure 2. (a) Three dimensional and (b) two dimensional plots represent the anti-kink solitary wave solution of Equation (34), when ± sign takes − and ∓ sign takes +.

diagrams of Equation (34), which represents the ani-kink solitary wave solution. Figure 2(b) shows that the amplitude and velocity of this ani-kink solitary wave remain unchanged during propagation.

$$u_3(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{\sinh(\xi)\cosh(\xi)}{\cosh^2(\xi) - \frac{1}{2}} \tag{35}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{8}, c = \pm\frac{5}{8}$. This solution, like Equation (34), also represents kinked solitary wave solution (see Figure 3).

$$u_4(\xi) = \frac{1}{2} \pm \frac{\sinh(\xi)}{\cosh(\xi) + \varepsilon} \tag{36}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{2}, c = \pm\frac{5}{2}, \varepsilon^2 = 1$. This solution, like Equation (34), also represents kinked solitary wave solution (see Figure 4).

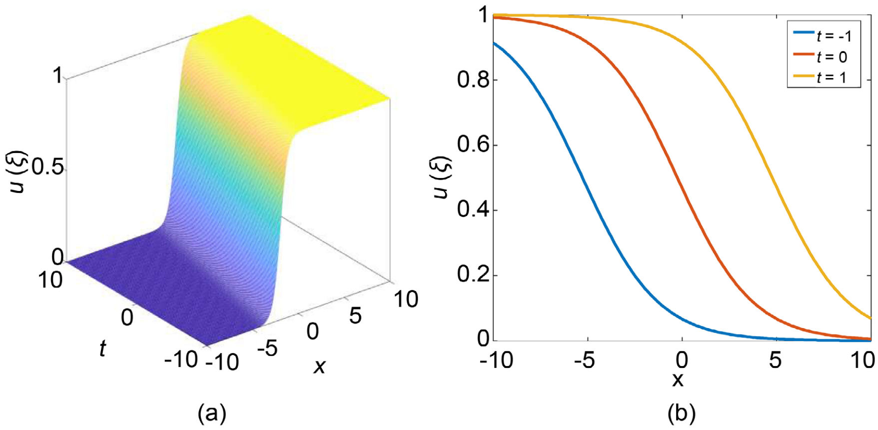


Figure 3. (a) Three dimensional and (b) two dimensional plots represent the ani-kink solitary wave solution of Equation (35), when ± sign takes − and ∓ sign takes +.

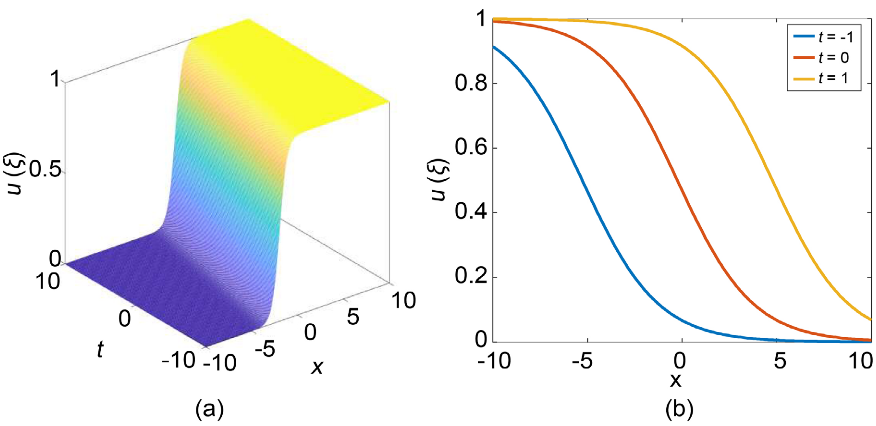


Figure 4. (a) Three dimensional and (b) two dimensional plots represent the ani-kink solitary wave solution of Equation (36), when ± sign takes −, ∓ sign takes + and $\varepsilon = 1$.

$$u_5(\xi) = \frac{1}{2} \pm \frac{\cosh(\xi)}{\sinh(\xi) + \varepsilon} \tag{37}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{2}, c = \pm\frac{5}{2}, \varepsilon^2 = -1$. The solution of Equation (37) represents the traveling wave solutions of Equation (27) in complex space.

$$u_6(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{\sinh(\xi)}{\cosh(\xi) + \varepsilon \pm \sqrt{2\varepsilon}\sqrt{\cosh(\xi) + \varepsilon}} \tag{38}$$

where $\xi = \mu x + ct, \mu = \mp 1, c = \pm 5, \varepsilon^2 = 1$. This solution, like Equation (34), also represents kinked solitary wave solution (see Figure 5).

$$u_7(\xi) = \frac{1}{2} \pm \frac{1}{4}\frac{2\sinh(\xi)\cosh^3(\xi) - \sinh(\xi)\cosh(\xi)}{\left[\cosh^2(\xi) - \frac{1}{2}\right]^2 - \frac{1}{8}} \tag{39}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{16}, c = \pm\frac{5}{16}$. Figure 6(a) shows the three-dimensional diagrams of Equation (39), which represents the kink solitary wave solution.

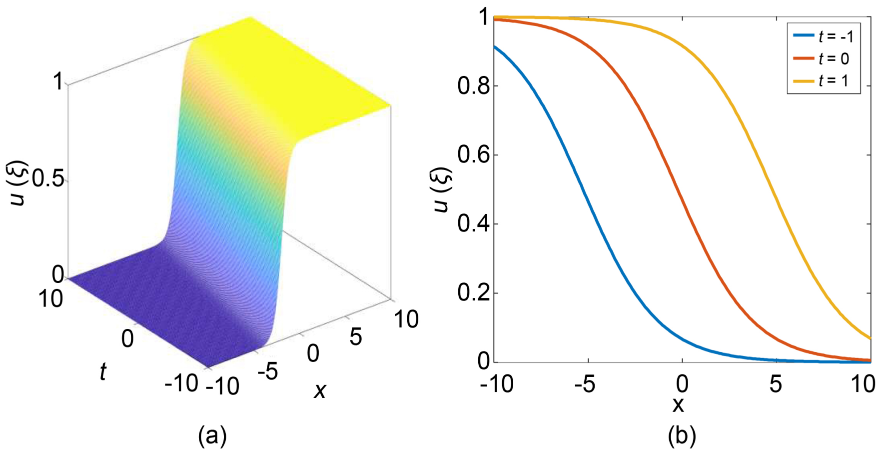


Figure 5. (a) Three dimensional and (b) two dimensional plots represent the ani-kink solitary wave solution of Equation (38), when ± sign takes −, ∓ sign takes + and $\varepsilon = 1$.

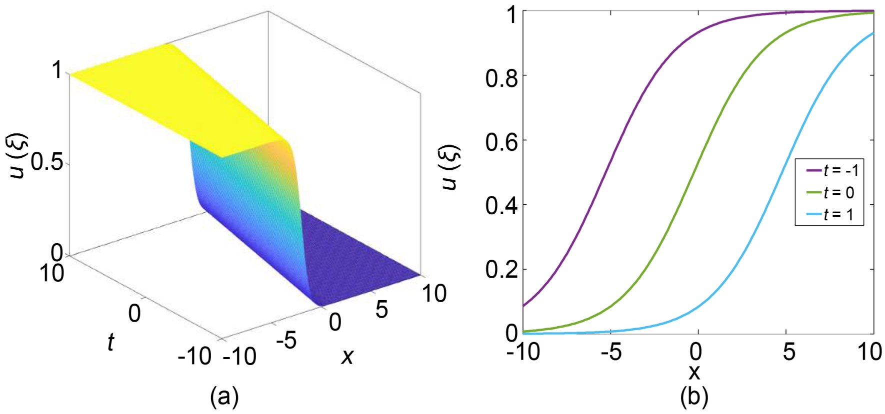


Figure 6. (a) Three dimensional and (b) two dimensional plots represent thekink solitary wave solution of Equation (39), when ± sign takes − and ∓ sign takes +.

Figure 6(b) shows that the amplitude and velocity of this kink solitary wave remain unchanged during propagation.

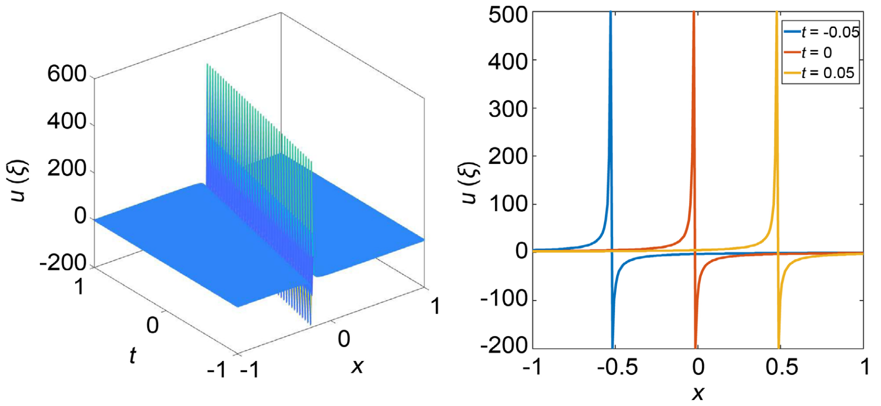
$$u_8(\xi) = \frac{1}{2} \pm \frac{\cosh^2(\xi) - \frac{1}{2}}{2\sinh(\xi)\cosh(\xi)} \tag{40}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{8}, c = \pm\frac{5}{8}$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 7**).

$$u_9(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{\cosh(\xi) + \varepsilon}{\sinh(\xi)} \tag{41}$$

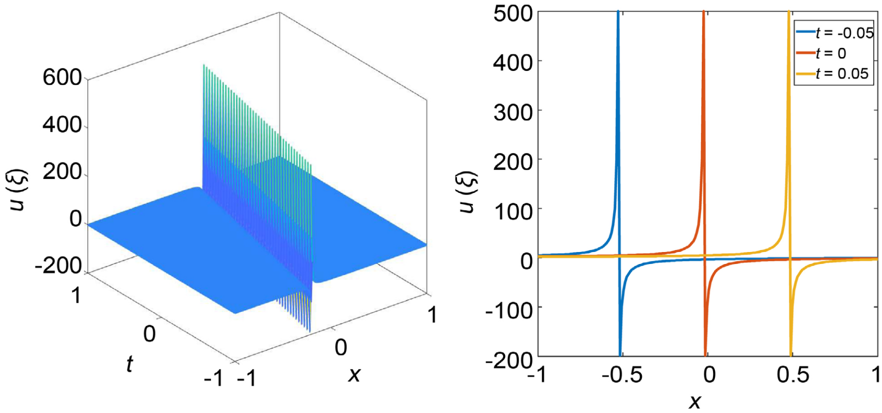
where $\xi = \mu x + ct, \mu = \mp\frac{1}{2}, c = \pm\frac{5}{2}, \varepsilon^2 = 1$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 8**).

$$u_{10}(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{\sinh(\xi) + \varepsilon}{\cosh(\xi)} \tag{42}$$



(a) (b)

Figure 7. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (40), when ± sign takes – and ∓ sign takes +.



(a) (b)

Figure 8. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (41), when ± sign takes –, ∓ sign takes + and $\varepsilon = 1$.

where $\xi = \mu x + ct, \mu = \mp\frac{1}{2}, c = \pm\frac{5}{2}, \varepsilon^2 = -1$. The solution of Equation (42) represents the traveling wave solutions of Equation (27) in complex space.

$$u_{11}(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{\cosh(\xi) + \varepsilon \pm \sqrt{2\varepsilon}\sqrt{\cosh(\xi) + \varepsilon}}{\sinh(\xi)} \tag{43}$$

where $\xi = \mu x + ct, \mu = \mp 1, c = \pm 5, \varepsilon^2 = 1$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 9**).

$$u_{12}(\xi) = \frac{1}{2} \pm \frac{\left[\cosh^2(\xi) - \frac{1}{2}\right]^2 - \frac{1}{8}}{2\sinh(\xi)\cosh^3(\xi) - \sinh(\xi)\cosh(\xi)} \tag{44}$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{16}, c = \pm\frac{5}{16}$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 10**).

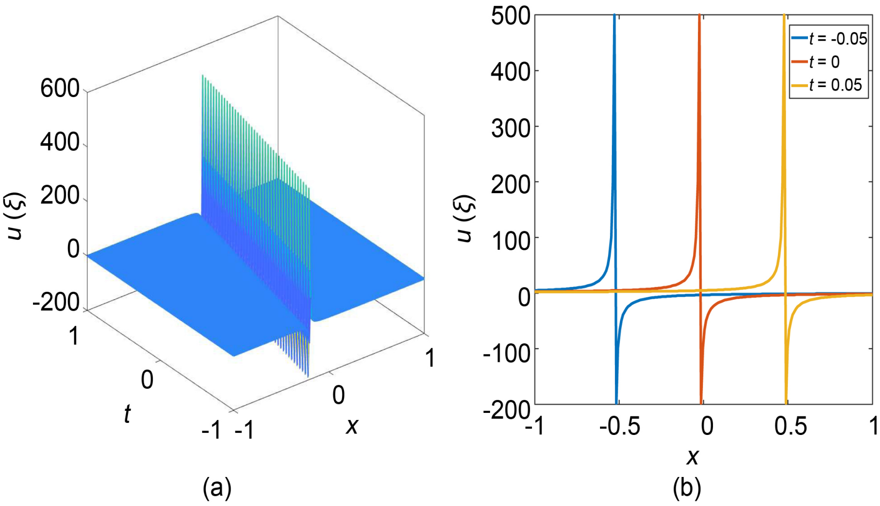


Figure 9. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (43), when ± sign takes −, ∓ sign takes + and $\varepsilon = 1$.

Because the solutions $f_1(\xi) - f_{12}(\xi)$ of Equation (1) contain that corresponding to $h(\xi) = 1/f(\xi)$, the solitary wave solutions of the Burgers-Fisher equation in Case 2 are the same as in Case 1. Corresponding to case 3, we express the solitary wave solution of the Burgers-Fisher equation as

$$u_{13}(\xi) = \frac{1}{2} \pm \frac{1}{4}\coth(\xi) \mp \frac{1}{4}\tanh(\xi) \quad (45)$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{8}, c = \pm\frac{5}{16}$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 11**).

$$u_{14}(\xi) = \frac{1}{2} \pm \frac{1}{4}\frac{\sinh(\xi)\cosh(\xi)}{\cosh^2(\xi) - \frac{1}{2}} \mp \frac{1}{4}\frac{\cosh^2(\xi) - \frac{1}{2}}{\sinh(\xi)\cosh(\xi)} \quad (46)$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{16}, c = \pm\frac{5}{32}$. This solution represents another travelling wave solution with singularities (see **Figure 12**).

$$u_{15}(\xi) = \frac{1}{2} \pm \frac{1}{4}\frac{\sinh(\xi)}{\cosh(\xi) + \varepsilon} \mp \frac{1}{4}\frac{\cosh(\xi) + \varepsilon}{\sinh(\xi)} \quad (47)$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{4}, c = \pm\frac{5}{8}, \varepsilon^2 = 1$. This solution, like Equation (33), represents the travelling wave solution with singularities (see **Figure 13**).

$$u_{16}(\xi) = \frac{1}{2} \pm \frac{1}{4}\frac{\cosh(\xi)}{\sinh(\xi) + \varepsilon} \mp \frac{1}{4}\frac{\sinh(\xi) + \varepsilon}{\cosh(\xi)} \quad (48)$$

where $\xi = \mu x + ct, \mu = \mp\frac{1}{4}, c = \pm\frac{5}{8}, \varepsilon^2 = -1$. The solution of Equation (48) represents the traveling wave solutions of Equation (27) in complex space.

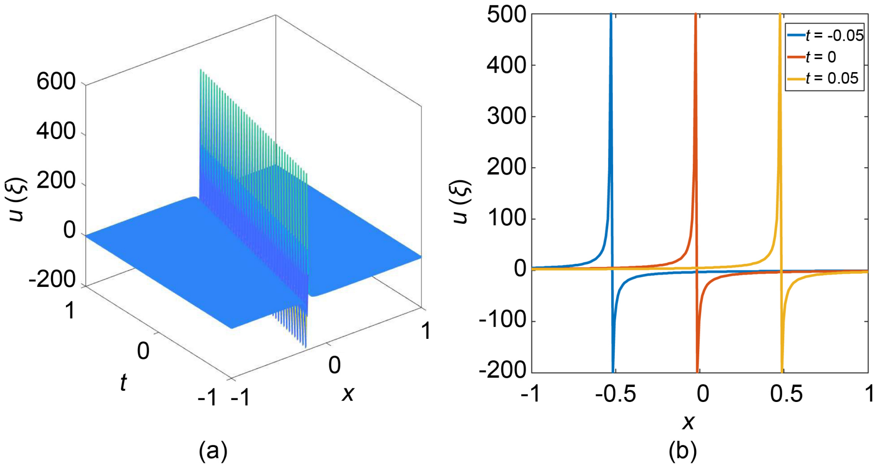


Figure 10. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (44), when ± sign takes − and ∓ sign takes +.

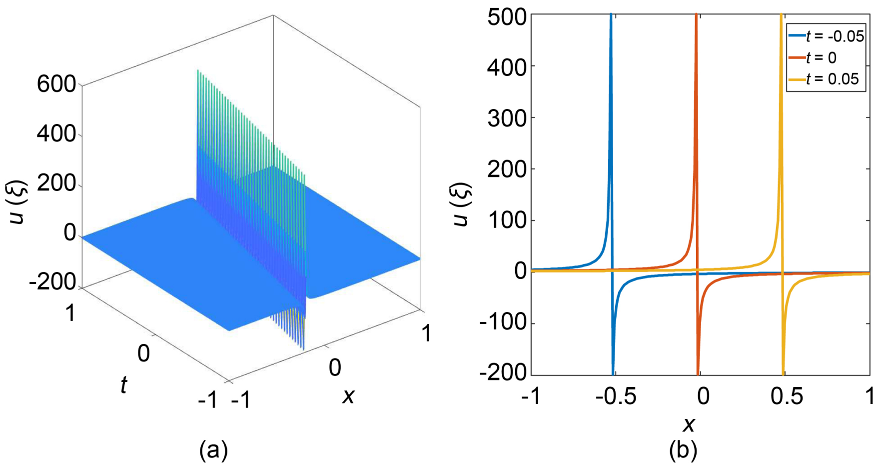


Figure 11. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (45), when ± sign takes − and ∓ sign takes +.

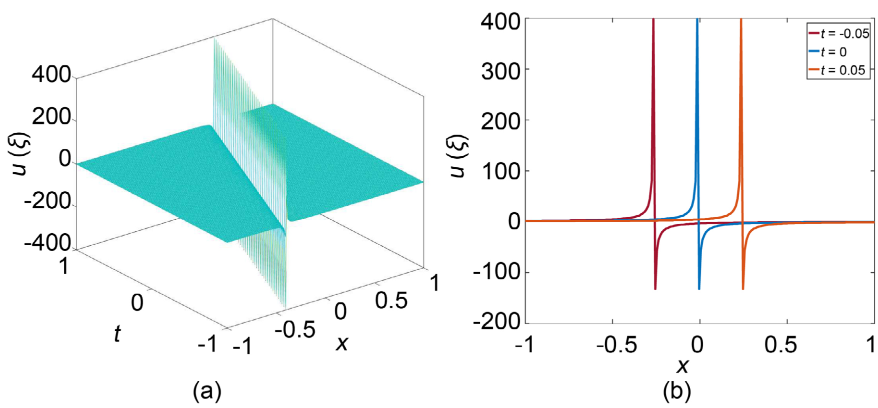


Figure 12. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (46), when ± sign takes − and ∓ sign takes +.

$$u_{17}(\xi)=\frac{1}{2}\pm\frac{1}{4}\frac{\sinh(\xi)}{\cosh(\xi)+\varepsilon\pm\sqrt{2\varepsilon}\sqrt{\cosh(\xi)+\varepsilon}}\mp\frac{1}{4}\frac{\cosh(\xi)+\varepsilon\pm\sqrt{2\varepsilon}\sqrt{\cosh(\xi)+\varepsilon}}{\sinh(\xi)} \tag{49}$$

where $\xi=\mu x+ct, \mu=\mp\frac{1}{2}, c=\pm\frac{5}{4}, \varepsilon^2=1$. This solution, like Equation (33), represents the travelling wave solution with singularities (see Figure 14).

$$u_{18}(\xi)=\frac{1}{2}\pm\frac{1}{8}\frac{2\sinh(\xi)\cosh^3(\xi)-\sinh(\xi)\cosh(\xi)}{\left[\cosh^2(\xi)-\frac{1}{2}\right]^2-\frac{1}{8}}\mp\frac{1}{2}\frac{\left[\cosh^2(\xi)-\frac{1}{2}\right]^2-\frac{1}{8}}{2\sinh(\xi)\cosh^3(\xi)-\sinh(\xi)\cosh(\xi)} \tag{50}$$

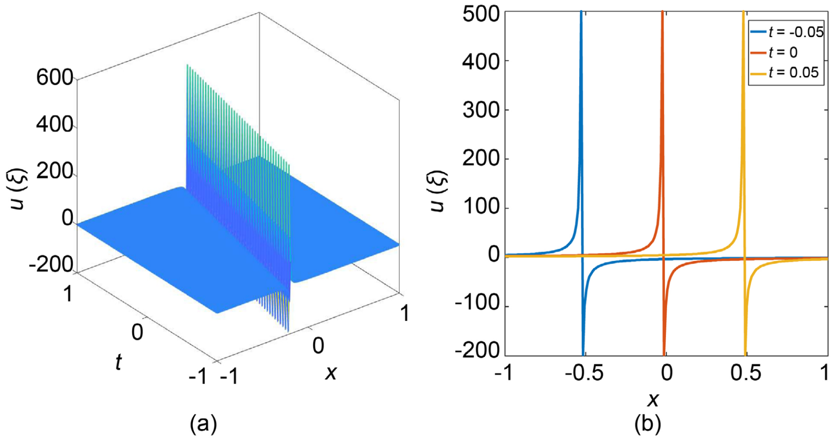


Figure 13. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (47), when ± sign takes −, ∓ sign takes + and $\varepsilon=1$.

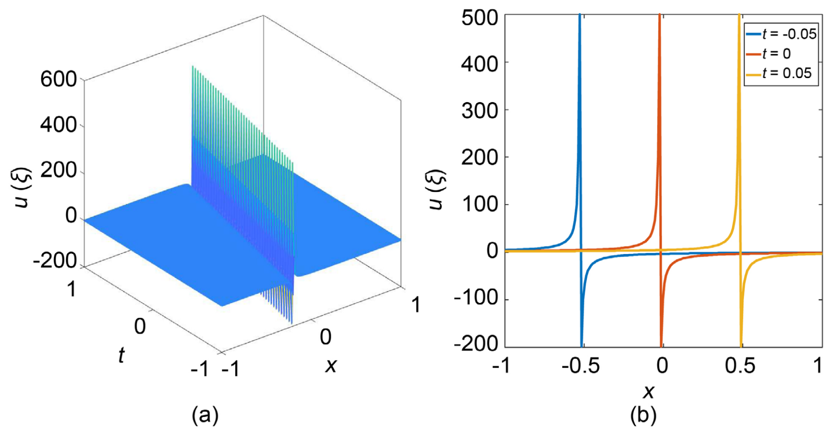


Figure 14. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (49), when ± sign takes −, ∓ sign takes + and $\varepsilon=1$.

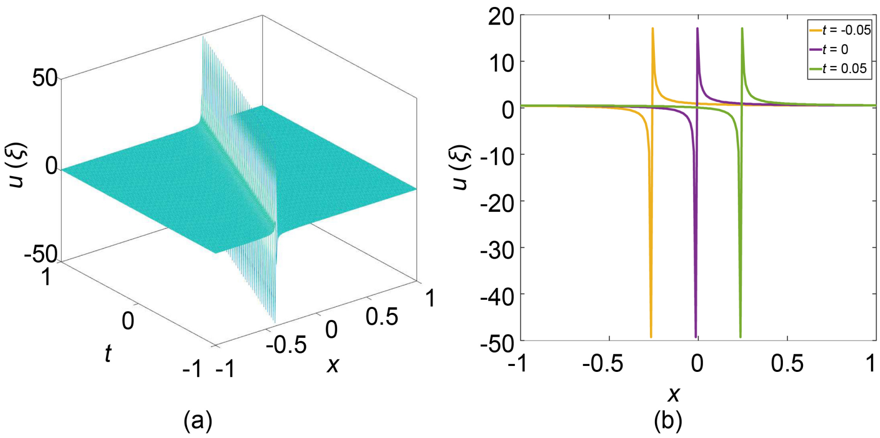


Figure 15. (a) Three dimensional and (b) two dimensional plots represent the travelling wave solution with singularities of Equation (50), when ± sign takes − and ∓ sign takes +.

where $\xi = \mu x + ct, \mu = \mp\frac{1}{2}, c = \pm\frac{5}{4}$. This solution represents the new type of travelling wave solution with singularities (see Figure 15).

4. Conclusion

In this paper, we apply a new method to deal with the Burgers-Fisher equation to find more solitary wave solutions. The rich hyperbolic functions solutions of the Riccati equation are constructed through the hyperbolic functions equation, and then the Riccati equation is used as an auxiliary equation to solve the Burgers-Fisher equation so that many new exact solutions are obtained. With the formal solution of Equation (29), we have constructed abundant solitary wave solutions for the Burgers-Fisher equation. The solitary wave solutions expressed by Equations (38), (39), (43), (44), (49) and (50) are rarely found in other documents. The numerical images show that although the new expressions of many solutions are different, the solitary waves represented by them, including amplitude, wave velocity and space-time width, are the same. These solitary wave solutions exhibit characteristics such as kinks, anti-kinks, and singularities. Further research is needed on the application of these solitary wave solutions in specific research fields such as chemical kinetics and population dynamics. This method can greatly simplify the calculation process, especially suitable for solving more complex nonlinear systems.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Guo, H.D., Xia, T.C. and Hu, B.B. (2020) High-Order Lumps, High-Order Breathers and Hybrid Solutions for an Extended (3 + 1)-Dimensional Jimbo—Miwa Equation in Fluid Dynamics. *Nonlinear Dynamics*, **100**, 601-614.

https://doi.org/10.1007/s11071-020-05514-9

[2] Lan, Z.Z. and Guo, B.L. (2020) Nonlinear Waves Behaviors for a Coupled Generalized Nonlinear Schrodinger-Boussinesq System in a Homogeneous Magnetized Plasma. *Nonlinear Dynamics*, **100**, 3771-3784. https://doi.org/10.1007/s11071-020-05716-1

[3] Biswas, A., Ekici, M., Sonmezoglu, A. and Belic, M.R. (2019) Solitons in Optical Fiber Bragg Gratings with Dispersive Reflectivity by Extended Trial Function Method. *Optik*, **182**, 88-94. https://doi.org/10.1016/j.ijleo.2018.12.156

[4] Seadawy, A.R., Lu, D.C., Nasreen, N. and Nasreen, S. (2019) Structure of Optical Solitons of Resonant Schrodinger Equation with Quadratic Cubic Nonlinearity and Modulation Instability Analysis. *Physica A*: *Statistical Mechanics and Its Applications*, **534**, Article ID: 122155. https://doi.org/10.1016/j.physa.2019.122155

[5] Abdoud, M.A., Owyed, S., Abdel-Aty, A., Raffan, B.M. and Abdel-Khalek, S. (2020) Optical Soliton Solutions for a Space-Time Fractional Perturbed Nonlinear Schrödinger Equation Arising in Quantum Physics. *Results in Physics*, **16**, Article ID: 102895. https://doi.org/10.1016/j.rinp.2019.102895

[6] Peng, W.Q., Tian, S.F. and Zhang, T.T. (2019) Dynamics of the Soliton Waves, Breather Waves, and Rogue Waves to the Cylindrical Kadomtsev-Petviashvili Equation in Pair-Ion-Electron Plasma. *Physics of Fluids*, **31**, Article ID: 102107. https://doi.org/10.1063/1.5116231

[7] Seadawy, A.R. (2014) Stability Analysis for Zakharov-Kuznetsov Equation of Weakly Nonlinear Ion-Acoustic Waves in a Plasma. *Computers & Mathematics with Applications*, **67**, 172-180. https://doi.org/10.1016/j.camwa.2013.11.001

[8] Seadawy, A.R. (2014) Stability Analysis for Two-Dimensional Ion-Acoustic Waves in Quantum Plasmas. *Physics of Plasmas*, **21**, Article ID: 052107. https://doi.org/10.1063/1.4875987

[9] Liu, J.B. and Yang, K.Q. (2004) The Extended F-Expansion Method and Exact Solutions of Nonlinear PDEs. *Chaos*, *Solitons & Fractals*, **22**, 111-121. https://doi.org/10.1016/j.chaos.2003.12.069

[10] Zhang, S. (2007) Application of Exp-Function Method to a KdV Equation with Variable Coefficients. *Physics Letters A*, **365**, 448-453. https://doi.org/10.1016/j.physleta.2007.02.004

[11] Liu, S.K., Fu, Z.T., Liu, S.D. and Zhao, Q. (2001) Jacobi Elliptic Function Expansion Method and Periodic Wave Solutions of Nonlinear Wave Equations. *Physics Letters A*, **289**, 69-74. https://doi.org/10.1016/S0375-9601(01)00580-1

[12] Fu, Z.T., Liu, S.K., Liu, S.D. and Zhao, Q. (2001) New Jacobi Elliptic Function Expansion and New Periodic Solutions of Nonlinear Wave Equations. *Physics Letters A*, **290**, 72-76. https://doi.org/10.1016/S0375-9601(01)00644-2

[13] Wu, G., Han, J., Zhang, W. and Zhang, M. (2007) New Periodic Wave Solutions to Nonlinear Evolution Equations by the Extended Mapping Method. *Physica D*: *Nonlinear Phenomena*, **229**, 116-122. https://doi.org/10.1016/j.physd.2007.03.015

[14] Sirendaoreji and Sun, J. (2003) Auxiliary Equation Method for Solving Nonlinear Partial Differential Equations. *Physics Letters A*, **309**, 387-396. https://doi.org/10.1016/S0375-9601(03)00196-8

[15] Sirendaoreji (2004) New Exact Travelling Wave Solutions for the Kawahara and Modified Kawahara Equations. *Chaos*, *Solitons Fractals*, **19**, 147-150. https://doi.org/10.1016/S0960-0779(03)00102-4

[16] Wazwaz, A.M. (2005) The Tanh Method for Generalized Forms of Nonlinear Heat Conduction and Burgers—Fisher Equations. *Applied Mathematics and Computa-*

tion, **169**, 321-338. https://doi.org/10.1016/j.amc.2004.09.054

[17] Wazwaz, A.M. (2007) The Extended Tanh Method for Abundant Solitary Wave Solutions of Nonlinear Wave Equations. *Applied Mathematics and Computation*, **187**, 1131-1142. https://doi.org/10.1016/j.amc.2006.09.013