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Super-Shuffle Product and Cut-Box Coproduct on (0,1)-Matrices

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Abstract

In 2014, Vargas first defined a super-shuffle product and a cut-box coproduct on permutations. In 2020, Aval, Bergeron and Machacek introduced the super-shuffle product and the cut-box coproduct on labeled simple graphs. In this paper, we generalize the super-shuffle product and the cut-box coproduct from labeled simple graphs to (0,1)-matrices. Then we prove that the vector space spanned by (0,1)-matrices with the super-shuffle product is a graded algebra and with the cut-box coproduct is a graded coalgebra.

Keywords

(0,1)-Matrix, Super-Shuffle Product, Cut-Box Coproduct, Graded Algebra, Graded Coalgebra

1. Introduction

In 1941, Hopf [1] first put forward the concept of both algebra structure and coalgebra structure in the study of cohomology algebra $H^*(G,K)$ of Lie group *G*. After that, more and more interesting questions about algebras and coalgebras have attracted many mathematicians to work and study on them continuously. Among those questions, it is a hot topic how to construct algebras and coalgebras on combinatorial objects.

In 2014, Vargas [2] defined a super-shuffle product $\Cup\!\!\!\Cup$ and a coproduct $\Delta_\diamond$, called cut-box coproduct by Liu and Li [3] on permutations. In 2005, Aguiar and Sottile introduced the global descents of permutations in symmetric groups [4]. On this basis, Zhao and Li derived another shuffle product and deconcatenation coproduct from the classical one on permutations. Then they proved the vector space spanned by permutations with the shuffle product that is a graded algebra and with the deconcatenation coproduct that is a graded coalgebra [5] in 2020.

In the same year, Aval, Bergeron and Machacek introduced the super-shuffle product and the cut-box coproduct on labeled simple graphs without proof [6]. In 2023, Dong [7] proved the vector space spanned by labeled graphs with the super-shuffle product is a graded algebra and with the cut-box coproduct is a graded coalgebra.

In fact, matrices are related to permutations and graphs closely. A (0,1)-matrix is a matrix whose entries are all 0 or 1, also called a binary matrix. It is widely used in graph theory [8] [9], combinatorics [10], linear programming [11] [12] [13] and computer science [14]. In this paper, we first generalize the super-shuffle product and the cut-box coproduct from labeled simple graphs to (0,1)-matrices, then we prove that the vector space with the super-shuffle product that is a graded algebra and with the cut-box coproduct that is a graded coalgebra.

This paper is organized as follows. We start by recalling some notations on (0,1)-matrices and defining the vector space $\mathcal{M}$ spanned by (0,1)-matrices in Section 2. In Section 3, we define the cut-box coproduct Δ on $\mathcal{M}$ and prove $\mathcal{M}$ with coproduct Δ that is a graded coalgebra. In Section 4, we define the super-shuffle product $*$ on $\mathcal{M}$ and prove $\mathcal{M}$ with product $*$ that is a graded algebra. Lastly, we summarize our main conclusions in Section 5.

2. Basic Definitions

An $s\times n$ matrix $A=\left(a_{ij}\right)_{s\times n}$ is called a (0,1)-*matrix* if

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \end{pmatrix}_{s\times n},$$

where a_{ij} is either 0 or 1. In particular, the *empty matrix* is the matrix with no entries, denoted by ε.

Define

$$[n]=\begin{cases}\{1,2,\cdots,n\}, & n>0,\\ \varnothing, & n=0,\end{cases}$$

and

$$[i,j]=\begin{cases}\{i,i+1,\cdots,j\}, & i\le j,\\ \varnothing, & i>j.\end{cases}$$

Let $I=\{i_1,i_2,\cdots,i_k\}\subseteq[s]$ and $J=\{j_1,j_2,\cdots,j_q\}\subseteq[n]$, where $i_1<i_2<\cdots<i_k\le s$ and $j_1<j_2<\cdots<j_q\le n$. For an $s\times n$ (0,1)-matrix *A*, the *restriction* of *A* on $I\times J$ is the submatrix formed by the entries, in the same relative positions, in both rows indexed by *I* and columns indexed by *J*, denoted by $A_{I\times J}$. In particular, if $I=[s]$ and $J=[n]$, $A_{I\times J}=A$ and if *I* or *J* is empty, $A_{I\times J}=\varepsilon$. For convenience, let A_I denote $A_{I\times I}$ and call A_I the *restriction* of *A* on *I*.

Example 1. The matrix

$$A=\begin{pmatrix}0&1&0&0&0&0&1\\0&0&0&0&1&0&0\\0&1&0&1&0&0&0\\0&0&0&0&0&1&0\end{pmatrix}$$

is a 4 × 7 (0,1)-matrix. We have

$$A_{\{1,2\}\times\{1,2,7\}}=\begin{pmatrix}0&1&1\\0&0&0\end{pmatrix}$$

and

$$A_{[3]}=\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}.$$

Let $M_n=\{A\,|\,A=(a_{ij})\text{ is an }n\times n\ (0,1)\text{-matrix}\}$ and $\mathcal{M}_n$ be the vector space spanned by M_n over field $\mathbb{K}$, for any non-negative integer *n*. For example,

$$M_2=\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}.$$

In particular, $M_0=\{\varepsilon\}$ and $\mathcal{M}_0=\mathbb{K}M_0$. Denote

$$M=\bigcup_{n=0}^{\infty}M_n\quad\text{and}\quad\mathcal{M}=\bigoplus_{n=0}^{\infty}\mathcal{M}_n.$$

If *A* and *B* are both non-empty matrices, then we denote $A\lozenge B=\begin{bmatrix}A&O\\O&B\end{bmatrix}$, where *O*'s are zero matrices. In particular, $\varepsilon\lozenge A=A\lozenge\varepsilon=A$ for any (0,1)-matrix *A*.

Example 2. For $A=\begin{pmatrix}0&0\\1&1\end{pmatrix}$ and $B=\begin{pmatrix}1&0&0\\1&0&1\\0&1&1\end{pmatrix}$, we have

$$A\lozenge B=\begin{pmatrix}0&0&0&0&0\\1&1&0&0&0\\0&0&1&0&0\\0&0&1&0&1\\0&0&0&1&1\end{pmatrix}.$$

For *A* in M_n, we call *i* a *spilt* of *A*, if

$$A_{[i]}\lozenge A_{[n]\backslash[i]}=A,$$

where $0\le i\le n$. By the definition, 0 and *n* are always splits of a (0,1)-matrix in M_n when $n\ge1$, called *trivial splits*. Obviously, $A_{[i]\times([n]\backslash[i])}=A_{([n]\backslash[i])\times[i]}=\varepsilon$ when *i* is a trivial spilt of *A*; $A_{[i]\times([n]\backslash[i])}=O_{i\times(n-i)}$ and $A_{([n]\backslash[i])\times[i]}=O_{(n-i)\times i}$ when *i* is a non-trivial spilt of *A*. We call *A* *indecomposible* if it is non-empty and only has trivial splits.

For *A* in M_n, $n\ge1$, suppose that $\{i_0,i_1,\cdots,i_s\}$ is the set of all splits of *A*, where $0=i_0<i_1<\cdots<i_s=n$. We call $A_{[i_{k-1}+1,i_k]}$ an *atom* of A, $1\le k\le s$. Obviously, there is no non-trivial split of $A_{[i_{k-1}+1,i_k]}$ for $1\le k\le s$. Let

$$A_k=A_{[i_{k-1}+1,i_k]},$$

for $1\le k\le s$. We define the *decomposition* of A by

$$A=A_1\lozenge A_2\lozenge\cdots\lozenge A_s.$$

In particular, when A is indecomposable or empty, its decomposition is itself.

Example 3. 1) The set of splits of

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}$$

is $\{0,1,3,6\}$ and its decomposition is

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}=[1]\lozenge\begin{bmatrix}1&1\\0&1\end{bmatrix}\lozenge\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Its atoms are

$$[1],\ \begin{bmatrix}1&1\\0&1\end{bmatrix}\text{ and }\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}.$$

2) The set of splits of $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$ is $\{0,3\}$, so it is indecomposable. Its decomposition is itself, and so is its atom.

3. Cut-Box Coproduct and Coalgebra

In this section, we define the cut-box coproduct on the vector space $\mathcal{M}$. Then we prove the space with the cut-box coproduct is a graded coalgebra.

Define the *cut-box coproduct* Δ on $\mathcal{M}$ by

$$\Delta(A)=\sum_{j=0}^{s}A_1\lozenge\cdots\lozenge A_j\otimes A_{j+1}\lozenge\cdots\lozenge A_s$$

for non-empty matrix A in M_n with decomposition $A=A_1\lozenge A_2\lozenge\cdots\lozenge A_s$, where $A_1\lozenge\cdots\lozenge A_0=A_{s+1}\lozenge\cdots\lozenge A_s=\varepsilon$. In particular, define $\Delta(\varepsilon)=\varepsilon\otimes\varepsilon$.

Define the *counit* ν from $\mathcal{M}$ to $\mathbb{K}$ by

$$\nu(A)=\begin{cases}1, & A=\varepsilon,\\0, & \text{otherwise},\end{cases}$$

for A in M.

Example 4. From Example 3 and the definition of the cut-box coproduct, we have

$$\Delta\left(\begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}\right)=\Delta\left([1]\lozenge\begin{bmatrix}1&1\\0&1\end{bmatrix}\lozenge\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}\right)$$

$$=\varepsilon\otimes[1]\lozenge\begin{bmatrix}1&1\\0&1\end{bmatrix}\lozenge\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}+[1]\otimes\begin{bmatrix}1&1\\0&1\end{bmatrix}\lozenge\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$+[1]\lozenge\begin{bmatrix}1&1\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}+[1]\lozenge\begin{bmatrix}1&1\\0&1\end{bmatrix}\lozenge\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}\otimes\varepsilon$$

$$=\varepsilon\otimes\begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}+[1]\otimes\begin{bmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&1\\0&0&0&1&0\\0&0&1&0&0\end{bmatrix}$$

$$+\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\otimes\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix}+\begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&1&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}\otimes\varepsilon$$

and

$$\Delta\left(\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\right)=\varepsilon\otimes\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}+\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\otimes\varepsilon.$$

Theorem 1. $(\mathcal{M},\Delta,\nu)$ is a graded coalgebra.

Proof. It is easy to verify that ν is a counit. Obviously,

$$(\mathrm{id}\otimes\Delta)\circ\Delta(\varepsilon)=\varepsilon\otimes\varepsilon\otimes\varepsilon=(\Delta\otimes\mathrm{id})\circ\Delta(\varepsilon).$$

Suppose A in M_n with $n\geq 1$ and its decomposition is

$$A=A_1\lozenge A_2\lozenge\cdots\lozenge A_s.$$

Then

$$\begin{aligned}&(\mathrm{id}\otimes\Delta)\circ\Delta(A)\\&=(\mathrm{id}\otimes\Delta)\circ\Delta(A_1\lozenge A_2\lozenge\cdots\lozenge A_s)\\&=(\mathrm{id}\otimes\Delta)\left(\sum_{j=0}^{s}A_1\lozenge\cdots\lozenge A_j\otimes A_{j+1}\lozenge\cdots\lozenge A_s\right)\\&=\sum_{j=0}^{s}A_1\lozenge\cdots\lozenge A_j\otimes\left(\sum_{k=j}^{s}A_{j+1}\lozenge\cdots\lozenge A_k\otimes A_{k+1}\lozenge\cdots\lozenge A_s\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\le j\le k\le s} A_1 \lozenge \cdots \lozenge A_j \otimes A_{j+1} \lozenge \cdots \lozenge A_k \otimes A_{k+1} \lozenge \cdots \lozenge A_s \\
&= \sum_{k=0}^{s}\left(\sum_{j=0}^{k} A_1 \lozenge \cdots \lozenge A_j \otimes A_{j+1} \lozenge \cdots \lozenge A_k \right) \otimes A_{k+1} \lozenge \cdots \lozenge A_s \\
&= (\Delta \otimes \mathrm{id})\left(\sum_{k=0}^{s} A_1 \lozenge \cdots \lozenge A_k \otimes A_{k+1} \lozenge \cdots \lozenge A_s \right) \\
&= (\Delta \otimes \mathrm{id}) \circ \Delta(A),
\end{aligned}
$$

where $A_{j+1} \lozenge \cdots \lozenge A_j = \varepsilon$ for $0 \le j \le s$. So Δ satisfies the coassociative law.

Obviously, by the definitions of Δ and ν, we have $\Delta(\mathcal{M}_n) \subseteq \bigoplus \mathcal{M}_i \otimes \mathcal{M}_{n-i}$ and $\nu(\mathcal{M}_n) = 0$ for $n > 0$. Hence $(\mathcal{M}, \Delta, \nu)$ is a graded coalgebra.

□

4. Super-Shuffle Product and Algebra

In this section, we define the super-shuffle product on the vector space $\mathcal{M}$. Then we prove the space with the super-shuffle product is a graded algebra.

Define the *super-shuffle product* $*$ on $\mathcal{M}$ by

$$
A * B = \sum_{\substack{C \in M_{m+n} \\ I,\, J:\, I \cup J = [m+n] \\ C_I = A,\, C_J = B}} C \tag{1}
$$

for A in M_m and B in M_n, where C traverses all matrices in M_{m+n} with the restriction on *I* is *A*, on *J* is *B*, on $I \times J$ and $J \times I$ are arbitrary (0,1)-matrices. Obviously, the product $*$ is commutative and $\varepsilon * A = A * \varepsilon = A$, for any A in *M*. Define the *unit* μ from $\mathbb{K}$ to $\mathcal{M}$ by $\mu(1) = \varepsilon$.

Example 5. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B = [1]$ we have

$$
\begin{aligned}
A * B = {} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
& + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
& + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
& + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$+\begin{bmatrix}1&0&1\\1&1&0\\1&1&0\end{bmatrix}+\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\\1&0&0\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&0\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\0&1&1\\1&1&0\end{bmatrix}$$

$$+\begin{bmatrix}1&0&1\\1&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&0\\0&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&1\\0&1&1\\0&1&0\end{bmatrix}$$

$$+\begin{bmatrix}1&0&0\\1&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&0\\1&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&0\\0&1&1\\1&1&0\end{bmatrix}$$

$$+\begin{bmatrix}1&0&1\\1&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\1&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\0&1&1\\1&1&0\end{bmatrix}$$

$$+\begin{bmatrix}1&1&0\\1&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&0&1\\1&1&1\\1&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}.$$

Here, we color the entries of C in $A*B$ restricted to A red and to B blue, respectively. Although $\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}$ and $\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}$ are same matrices, we consider they are different. Then each term in $A*B$ is unique.

Let $W=\{i_1,i_2,\cdots,i_n\}$ be a set of positive integers where $i_1<i_2<\cdots<i_n$. Define a mapping st_W from W to $[|W|]$ by $\mathrm{st}_W(i_a)=a$ for $1\le a\le n$, and call it the *standardization* of W [6]. For a subset T of W, denote $\mathrm{st}_W(T)=\{\mathrm{st}_W(i)\mid i\in T\}$. Obviously, st_W is a 1-1 mapping from the set of subsets of W to the set of subsets of $[|W|]$. Therefore, for any subset H of $[|W|]$, there must exist a unique subset P of W such that $H=\mathrm{st}_W(P)$.

Remark 1. ([15]) Let W be a set of positive integers and P be a subset of W. Then there exists a unique subset H in $[|W|]$ such that

$$\mathrm{st}_P(i)=\mathrm{st}_H(\mathrm{st}_W(i)),$$

for any i in P. Actually, $H=\mathrm{st}_W(P)$.

Example 6. For $W=\{3,5,7,8,9\}$ and $P=\{3,7,8\}$, $\mathrm{st}_P(3)=1$, $\mathrm{st}_P(7)=2$, $\mathrm{st}_P(8)=3$, $\mathrm{st}_W(3)=1$, $\mathrm{st}_W(7)=3$ and $\mathrm{st}_W(8)=4$. Take $H=\mathrm{st}_W(P)=\mathrm{st}_W(\{3,7,8\})=\{1,3,4\}$. Furthermore, $\mathrm{st}_H(\mathrm{st}_W(3))=1$, $\mathrm{st}_H(\mathrm{st}_W(7))=2$, $\mathrm{st}_H(\mathrm{st}_W(8))=3$.

Next, in order to prove $(\mathcal{M},*,\mu)$ is a graded algebra, we give one lemma.

Lemma 2. Assume $A=(a_{ij})_{n\times n}$ is a (0,1)-matrix, $W\subseteq[n]$ and $H\subseteq[|W|]$. Then there exists a subset P of W such that $A_P=(A_W)_H$.

Proof. By the definition of st_W, there must exist a subset P of W such that $\mathrm{st}_W(P)=H$. Next, we prove $A_P=(A_W)_H$.

Let A_P be $B=(b_{ij})$, A_W be $C=(c_{ij})$ and $(A_W)_H$ be $D=(d_{ij})$. Obviously, B and D are both $|P|\times|P|$ (0,1)-matrices. We just need to show that $b_{ij}=d_{ij}$ for each i,j in $[|P|]$. For b_{ij} in $B=A_P$, there must exist i' and j' in P such that $\mathrm{st}_P(i')=i$, $\mathrm{st}_P(j')=j$ and $a_{i'j'}=b_{ij}$. Since P is a subset of W, there must exist i'' and j'' in $[|W|]$ such that $\mathrm{st}_W(i')=i''$, $\mathrm{st}_W(j')=j''$ and $a_{i'j'}=c_{i''j''}$. On the other hand, we have $\mathrm{st}_W(P)=H$ and st_W is a 1-1 mapping from the set of subsets of W to the set of subsets of $[|W|]$, therefore i'' and j'' are in H. Then there must exist i''' and j''' in $[|H|]$ such that $\mathrm{st}_H(i'')=i'''$, $\mathrm{st}_H(j'')=j'''$ and $c_{i''j''}=d_{i'''j'''}$. Hence, $b_{ij}=d_{i'''j'''}$. By Remark 1, we have

$$i=\mathrm{st}_P(i')=\mathrm{st}_H(\mathrm{st}_W(i'))=\mathrm{st}_H(i'')=i'''$$

and

$$j=\mathrm{st}_P(j')=\mathrm{st}_H(\mathrm{st}_W(j'))=\mathrm{st}_H(j'')=j'''.$$

Thus, for each i,j in $[|P|]$, $b_{ij}=d_{ij}$, *i.e.*, $A_P=(A_W)_H$. □

Theorem 3. $(\mathcal{M},*,\mu)$ is a graded algebra.

Proof. It is easy to verify that μ is a unit. For A in M_h, B in M_k and C in M_l, we have

$$A*B=\sum_{\substack{X\in M_{h+k}\\ H,K:H\cup K=[h+k]\\ X_H=A,\,X_K=B}} X.$$

Then for any term Y in $(A*B)*C$, there exist two disjoint subsets W and L of $[h+k+l]$ with $|W|=h+k$ and $|L|=l$ such that Y_W is a term in $A*B$ and $Y_L=C$. It means

$$(A*B)*C=\sum_{\substack{X\in M_{h+k}\\ H,K:H\cup K=[h+k]\\ X_H=A,\,X_K=B}}\ \sum_{\substack{Y\in M_{h+k+l}\\ W,L:W\cup L=[h+k+l]\\ Y_W=X,\,Y_L=C}} Y. \tag{2}$$

For a fixed W in $[h+k+l]$ with cardinality $h+k$, there exist two disjoint subsets H and K of $[h+k]$ with $|H|=h$ and $|K|=k$ such that

$$(Y_W)_H=A \text{ and } (Y_W)_K=B.$$

Since H is a subset of $[|W|]=[h+k]$, due to the Lemma 2, there exists a subset P of W corresponding to H with $|P|=h$ such that $H=\mathrm{st}_W(P)$ and

$$Y_P=(Y_W)_H=A.$$

Similarly, there exists a subset Q of W with $|Q|=k$ corresponding to K such that $K=\mathrm{st}_W(Q)$ and

$$Y_Q=(Y_W)_K=B.$$

In (2), for a fixed subset W in $[h+k+l]$ with cardinality $h+k$, H traverses all subsets with cardinality h in $[h+k]$, since Y_W traverses all terms in $A*B$. Meanwhile, P traverses all subsets with cardinality h in W. Therefore, P traverses all subsets with cardinality h in $[h+k+l]$ when W traverses all subsets with cardinality $h+k$ in $[h+k+l]$. Similarly, Q traverses all subsets with cardinality k in $[h+k+l]$ when W traverses all subsets with cardinality $h+k$ in

$\left[h+k+l\right]$ from $W = P \cup Q$. Thus (2) can be rewritten as

$$\left(A*B\right)*C = \sum_{\substack{Y \in M_{h+k+l} \\ P,Q,L:\, P\cup Q\cup L=\left[h+k+l\right] \\ Y_P=A,\, Y_Q=B,\, Y_L=C}} Y. \tag{3}$$

Similarly, $A*\left(B*C\right)$ can be rewritten as (3). Hence, $*$ satisfies the associative law and $\left(\mathcal{M},*,\mu\right)$ is an algebra.

By the definitions of the product $*$ and μ, we have $\mathcal{M}_r * \mathcal{M}_s \subseteq \mathcal{M}_{r+s}$ and $\mu\left(\mathbb{K}\right) \subseteq \mathcal{M}_0$. So $\left(\mathcal{M},*,\mu\right)$ is a graded algebra. □

5. Conclusion and Suggestion

Let $\mathcal{M}$ be the vector space spanned by (0,1)-matrices. Firstly, we introduce splits and the decomposition of a (0,1)-matrix. Then we define the cut-box coproduct Δ and the super-shuffle product $*$ on $\mathcal{M}$. We prove the cut-box coproduct Δ satisfies coassociativity and the super-shuffle product $*$ satisfies associativity, *i.e.*, $\left(\mathcal{M},\Delta,\nu\right)$ is a graded coalgebra and $\left(\mathcal{M},*,\mu\right)$ is a graded algebra.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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