Open Journal of Applied Sciences, 2023, 13, 120-135
https://www.scirp.org/journal/ojapps
ISSN Online: 2165-3925
ISSN Print: 2165-3917

Hopf Algebra of Labeled Simple Graphs

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How to cite this paper: Dong, J.M. and Li, H.L. (2023) Hopf Algebra of Labeled Simple Graphs. *Open Journal of Applied Sciences*, **13**, 120-135.
https://doi.org/10.4236/ojapps.2023.131011

Received: December 31, 2022
Accepted: January 28, 2023
Published: January 31, 2023

Abstract

A lot of combinatorial objects have a natural bialgebra structure. In this paper, we prove that the vector space spanned by labeled simple graphs is a bialgebra with the conjunction product and the unshuffle coproduct. In fact, it is a Hopf algebra since it is graded connected. The main conclusions are that the vector space spanned by labeled simple graphs arising from the unshuffle coproduct is a Hopf algebra and that there is a Hopf homomorphism from permutations to label simple graphs.

Keywords

Hopf Algebra, Labeled Simple Graph, Conjunction Product, Unshuffle Coproduct, Compatibility

1. Introduction

The basic structure of Hopf algebra was first proposed by Hopf to study algebraic topology and the properties of algebraic groups in 1941 [1]. In 1960's, Milnor, Moore, Chase and Sweedler systematically developed the theory of Hopf algebra and gave the explicit definition, basic properties and common symbols of Hopf algebra [2] [3] [4]. In 1979, Joni and Rota constructed Hopf algebras on polynomials and on puzzles [5]. After that, Hopf algebra has been used to study a lot of objects, such as posets [6], symmetric functions [7] [8], quantum groups [9] [10], Clifford algebras [11] [12] and Lie superalgebras [13] [14].

Graphs are important combinatorial objects, on which there are rich Hopf algebra structures. In 1994, Schmitt studied incidence of Hopf algebras and gave Hopf algebras on many objects, such as Hopf algebras on permutations, matroids and graphs [15]. Schmitt also studied invariants of graphs through the properties of a Hopf algebra on graphs in 1995 [16]. Later, Connes and Kreimer established connections between quantum physics and Hopf algebras on rooted

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trees and on rooted forests [17] [18].

Permutations are related to graphs closely. In 1995, Malvenuto and Reutenauer first gave a classical Hopf algebra on permutations by shuffle product $ш$ [19]. On this basis, Aguiar and Sottile considered the concept of global descents on permutations in 2005 [20]. In 2004, Novelli, Thibon and Thiéry studied Hopf algebras with bases labeled by graphs and hypergraphs, which are graded by the number of edges [21]. In 2010, Forcey, Lauve and Sottile studied Hopf algebras on binary trees [22]. In 2014, Vargas defined the super-shuffle product $\underline{ш}$ and the cut-box coproduct $\Delta_{\Diamond}$ on permutations by global ascents of permutations [23]. In 2016, Giraudo and Vialette defined the unshuffle coproduct Δ^{*} on permutations [24]. In 2020, Zhao and Li derived a new Hopf algebra on permutations with another shuffle product $\underline{ш}_G^*$ from the classical one [25]. In the same year, Guo, Thibon and Yu introduced the Hopf algebra of signed permutations and established its relationship with the Hopf algebras of permutations and weak quasi-symmetric functions [26]. In 2021, Liu and Li proved that the vector space spanned by permutations arising from the super-shuffle product $\underline{ш}$ and the cut-box coproduct $\Delta_{\Diamond}$ is a Hopf algebra [27]. It is well-known that permutations are elements of symmetric groups, which are widely used in various fields, such as algebraic number theory [28] and substochastic matrices [29] [30] [31] [32].

In 2020, Aval, Bergeron and Machacek gave a Hopf algebra on labeled simple graphs with the conjunction product and the unshuffle coproduct without a proof. Here the Hopf algebra is graded by the number of vertices. They also introduced a mapping, named $\mathfrak{g}$ in this paper, from permutations to labeled simple graphs and claimed that it is a Hopf homomorphism [33]. In this paper, we will prove these conclusions.

This paper is organized as follows. We review basic concepts of Hopf algebra in Section 2. In Section 3, we define the vector space $\mathcal{H}$ spanned by labeled simple graphs and give the definitions of the conjunction product $\Diamond$ and the unshuffle coproduct Δ_* on the vector space. In Section 4, we prove $(\mathcal{H}, \Diamond, \mu)$ is a graded algebra, $(\mathcal{H}, \Delta_*, \nu)$ is a graded coalgebra and the compatibility between the conjunction product and the unshuffle coproduct. So $(\mathcal{H}, \Diamond, \mu, \Delta_*, \nu)$ is a Hopf algebra from that $\mathcal{H}$ is graded connected. In Section 5, we recall a mapping $\mathfrak{g}$ from permutations to labeled simple graphs and prove it is a Hopf homomorphism. Lastly, we summarize our main conclusions in Section 6.

In this paper, we not only prove the conclusions but also provide a lot of examples to help people to understand the operation rules and the Hopf structure on labled simple graphs. Furthermore, the Hopf homomorphism from permutations to labled simple graphs idicates the closed relations between them.

2. Hopf Algebra

Firstly, we introduce some basic definitions of Hopf algebra. For more details see [2] [3]. Let $\mathbb{K}$ be a commutative ring and A, B be $\mathbb{K}$-modules. Denote $\mathrm{id}: A \to A$ to be the identity mapping of A. The notation $A \xrightarrow{\cong} B$ means that A is isomorphic to B as $\mathbb{K}$-modules.

If there are linear mappings m from $A \otimes A$ to A and μ from $\mathbb{K}$ to A such that the diagrams in **Figure 1** are commutative, then we say that (A, m, μ) is an *algebra*, m is a *product* and μ is a *unit*.

If there are linear mappings Δ from A to $A \otimes A$ and ν from A to $\mathbb{K}$ such that the diagrams in **Figure 2** are commutative, then we say that (A, Δ, ν) is a *coalgebra*, Δ is a *coproduct* and ν is a *counit*.

We call (A, m, μ, Δ, ν) a *bialgebra*, if (A, m, μ) is an algebra, (A, Δ, ν) is a coalgebra and A satisfies *compatibility*, *i.e.*,

$$\Delta\left(m(x \otimes y)\right) = m\left(\Delta(x) \otimes \Delta(y)\right)$$

and

$$\nu\left(m(x \otimes y)\right) = m\left(\nu(x) \otimes \nu(y)\right),$$

for any x and y in A.

The bialgebra (A, m, μ, Δ, ν) is a *Hopf algebra*, if there is a linear mapping S from A to A satisfies

$$m \circ (\mathrm{id} \otimes S) \circ \Delta(x) = \mu\left(\nu(x)\right) = m \circ (S \otimes \mathrm{id}) \circ \Delta(x)$$

for any x in A, *i.e.*, the diagram in **Figure 3** is commutative. We call S an *antipode*.

The vector space A is *graded* if $A = \bigoplus_{n \geq 0} A_n$ and A is *connected* if $\mathbb{K}A_0 \cong \mathbb{K}$. The algebra (A, m, μ) is a *graded algebra* if A is a graded vector space, m satisfies $m\left(A_i \otimes A_j\right) \subseteq A_{i+j}$ for $i, j \geq 0$ and μ satisfies $\mu(\mathbb{K}) \subseteq A_0$. Similarly, the coalgebra (A, Δ, ν) is a *graded coalgebra* if Δ satisfies $\Delta(A_n) \subseteq \bigoplus_{i \geq 0} A_i \otimes A_{n-i}$ and ν satisfies $\nu(A_n) = 0$ for any $n \geq 1$. The bialgebra (A, m, μ, Δ, ν) is a *graded bialgebra* when (A, m, μ) is a graded algebra and (A, Δ, ν) is a graded coalgebra. In fact, a graded connected bialgebra is always a Hopf algebra ([34], Proposition 1.4.14.).

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[d, "m"] & \\
\mathbb{K} \otimes A \arrow[ur, "\mu \otimes \mathrm{id}"] \arrow[r, "\cong"] & A & A \otimes \mathbb{K} \arrow[ul, "\mathrm{id} \otimes \mu"'] \arrow[l, "\cong"']
\end{tikzcd}$$

Figure 1. Associative law and unitary property.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta \otimes \mathrm{id}"] \\
A \otimes A \arrow[r, "\mathrm{id} \otimes \Delta"] & A \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dl, "\mathrm{id} \otimes \nu"'] \arrow[dr, "\nu \otimes \mathrm{id}"] & \\
\mathbb{K} \otimes A & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, "\Delta"] & A \otimes \mathbb{K}
\end{tikzcd}$$

Figure 2. Coassociative law and counitary property.

$$\begin{tikzcd}
 & A\otimes A \arrow[rr, "\mathrm{id}\otimes S"] & & A\otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\nu"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "\mu"] & & A \\
 & A\otimes A \arrow[rr, "S\otimes \mathrm{id}"'] & & A\otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Figure 3. Antipode.

3. Basic Definitions

3.1. Labeled Simple Graph

A *graph* can generally be represented as $\Gamma=(V,E)$, where V is the vertex set and E is the edge set. We call a grpah $\Gamma=(V,E)$ a *labeled simple graph* if it does not have cycles and multiple edges, V is a set of positive integers, also denoted by $V(\Gamma)$, and E is the set of all edges of Γ, also denoted by $E(\Gamma)$. Obviously, $E\subseteq V\times V$ and if $(i_1,i_2)\in E$, then $i_1\neq i_2$ and $(i_2,i_1)\notin E$ since simple graphs do not have cycles and multiple edges. In particular, Γ is the *empty graph* when V is empty, denoted by ε.

Let $\Gamma=(V,E)$ and $I\subseteq V$. Define the *restriction of* Γ *on* I by $\Gamma_I=(I,E_I)$, where $E_I=\{(i,j)\mid i,j\in I,(i,j)\in E\}$, and we call Γ_I a *subgraph of* Γ. If $\Gamma_1=(V_1,E_1)$, $\Gamma_2=(V_2,E_2)$ and $V_1\cap V_2=\varnothing$, then the *union graph* of Γ_1 and Γ_2 is defined by $\Gamma_1\cup\Gamma_2=(V_1\cup V_2,E_1\cup E_2)$. For more details, see [35].

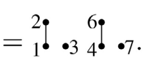
Define

$$[n]=\begin{cases}\{1,2,\cdots,n\}, & n>0,\\ \varnothing, & n=0,\end{cases}$$

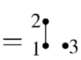
and

$$[i,j]=\begin{cases}\{i,i+1,\cdots,j\}, & i\leq j,\\ \varnothing, & i>j.\end{cases}$$

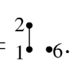
Example 1. The graph $\Gamma=(\{1,2,3,4,6,7\},\{(1,2),(4,6)\})$ is a labeled simple graph and can be represented by

$\Gamma =$ .

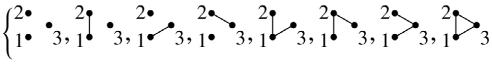
We have

$\Gamma_{[3]} = ([3],\{(1,2)\}) =$ 

and

$\Gamma_{\{1,2,6\}} = (\{1,2,6\},\{(1,2)\}) =$ .

Let $H_n=\{\Gamma\mid\Gamma=([n],E)\text{ is a labeled simple graph}\}$ and $\mathcal{H}_n$ be the linear space spanned by H_n over field $\mathbb{K}$, for any non-negative integer *n*. For example,

$$H_3 = \Big\{$$  $$\Big\}.$$

In particular, $H_0 = \{\varepsilon\}$ and $\mathcal{H}_0 = \mathbb{K}H_0$. Denote

$$H = \bigcup_{n=0}^{\infty} H_n \text{ and } \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Let $I = \{i_1, i_2, \cdots, i_n\}$ be a set of positive integers where $i_1 < i_2 < \cdots < i_n$ and denote $|I| = n$, the cardinality of *I*" at the end of the first sentence. Define a mapping st_I from I to $[|I|]$ by $\mathrm{st}_I(i_a) = a$ for $1 \le a \le n$, and call it the *standardization* of *I*. For x, y in *I*, $\mathrm{st}_I(x) < \mathrm{st}_I(y)$ if and only if $x < y$. Sometimes, we omit the subscript of the standardization when the set is obvious. Let *T* be a subset of *I*, then $\mathrm{st}_I(T) = \{\mathrm{st}_I(x) \mid x \in T\}$.

For any labeled simple graph $\Gamma = (V, E)$, define the *standard form* of Γ by

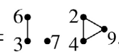
$$\mathrm{st}(\Gamma) = (\mathrm{st}_V(V), \mathrm{st}_V(E)) = (\mathrm{st}(V), \mathrm{st}(E)),$$

where $\mathrm{st}(V) = [|V|]$ and $\mathrm{st}(E)$ satisfies

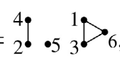
$$(\mathrm{st}(v_1), \mathrm{st}(v_2)) \in \mathrm{st}(E) \Leftrightarrow (v_1, v_2) \in E,$$

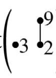
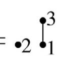
for v_1 and v_2 in *V*. In particular, $\mathrm{st}(\varepsilon) = \varepsilon$. That means, the standardization maintains the edge relationships of the vertices in Γ. For any labeled simple graph $\Gamma = (V, E)$, there is a graph in H_n which is the standard form of Γ, where $n = |V|$. In addition, for a non-negative integer *n*, let $\Gamma^{\uparrow n}$ be the graph by raising each vertex in Γ by *n* and maintaining its edge relationships. Similarly, let $\Gamma^{\downarrow n}$ be the graph by reducing each vertex in Γ by *n* and maintaining its edge relationships.

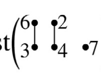
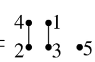
Example 2. For labeled simple graph

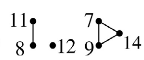
$$\Gamma =$$  ,

we have

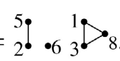
$$\mathrm{st}(\Gamma) =$$  ,

$$\mathrm{st}(\Gamma_{\{2,3,9\}}) = \mathrm{st}\Big($$  $$\Big) =$$  ,

$$\mathrm{st}(\Gamma_{\{2,3,4,6,7\}}) = \mathrm{st}\Big($$  $$\Big) =$$  ,

$$\Gamma^{\uparrow 5} =$$ 

and

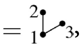
$$\Gamma^{\downarrow 1} =$$  .

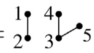
3.2. Conjunction Product and Unshuffle Coproduct

Define the *conjunction product* $\lozenge$ on $\mathcal{H}$ [33] by

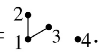
$$\Gamma_1 \Diamond \Gamma_2 = \Gamma_1 \cup \Gamma_2^{\uparrow m},$$

for Γ_1 in H_m and Γ_2 in H_n, and the *unit* μ from $\mathbb{K}$ to $\mathcal{H}$ by $\mu(1) = \varepsilon$.

Example 3. For $\Gamma_1 =$  and $\Gamma_2 =$ , the conjunction product of Γ_1 and Γ_2 is

$\Gamma_1 \diamond \Gamma_2 =$ 

and the conjunction product of Γ_2 and Γ_3 is

$\Gamma_2 \diamond \Gamma_3 =$ 

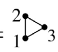
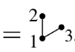
Define the *unshuffle coproduct* Δ_* on $\mathcal{H}$ [33] by

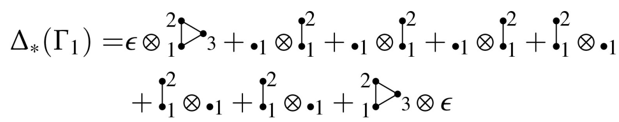
$$\Delta_*(\Gamma) = \sum_{I \subseteq [n]} \mathrm{st}(\Gamma_I) \otimes \mathrm{st}\left(\Gamma_{[n] \setminus I}\right),$$

for $\Gamma = ([n], E)$ in H_n, and the *counit* ν from $\mathcal{H}$ to $\mathbb{K}$ by

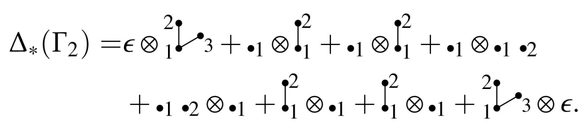
$$\nu(\Gamma) = \begin{cases} 1, & \Gamma = \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\Delta_*(\varepsilon) = \varepsilon \otimes \varepsilon$.

Example 4. For $\Gamma_1 =$  and $\Gamma_2 =$ , the unshuffle coproduct of Γ_1 is



and the unshuffle coproduct of Γ_2 is



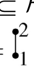
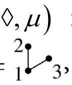
4. Main Theorem

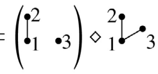
Theorem 2. *The vector space $\mathcal{H}$ with the conjunction product $\Diamond$ and the unit μ is a graded algebra.*

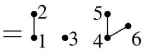
Proof. For any Γ_1 in H_m, Γ_2 in H_n and Γ_3 in H_k, we have

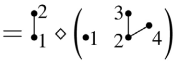
$$\begin{aligned}(\Gamma_1 \Diamond \Gamma_2) \Diamond \Gamma_3 &= \left(\Gamma_1 \cup \Gamma_2^{\uparrow m}\right) \Diamond \Gamma_3 \\ &= \left(\Gamma_1 \cup \Gamma_2^{\uparrow m}\right) \cup \Gamma_3^{\uparrow m+n} \\ &= \Gamma_1 \cup \Gamma_2^{\uparrow m} \cup \Gamma_3^{\uparrow m+n} \\ &= \Gamma_1 \cup \left(\Gamma_2 \cup \Gamma_3^{\uparrow n}\right)^{\uparrow m} \\ &= \Gamma_1 \Diamond (\Gamma_2 \Diamond \Gamma_3).\end{aligned}$$

So, $\Diamond$ is associative. It is easy to prove that the μ is a unit. Then $(\mathcal{H}, \Diamond, \mu)$ is an algebra. Obviously, by the definitions of $\Diamond$ and μ, we have $\mathcal{H}_i \Diamond \mathcal{H}_j \subseteq \mathcal{H}_{i+j}$ for $i, j \geq 0$ and $\mu(\mathbb{K}) \subseteq \mathcal{H}_0$. So $(\mathcal{H}, \Diamond, \mu)$ is a graded algebra. □

Example 5. For $\Gamma_1 =$  and $\Gamma_3 =$ , we have

$$(\Gamma_1 \diamond \Gamma_2) \diamond \Gamma_3 =$$ 

$=$ 

$=$ 

$$= \Gamma_1 \diamond (\Gamma_2 \diamond \Gamma_3).$$

Lemma 1. *Assume I is a set of positive integers, $J \subseteq I$ and $K = \mathrm{st}_I(J)$. Then*

$$\mathrm{st}_K(\mathrm{st}_I(i)) = \mathrm{st}_J(i),$$

for any i in J. □

Proof. Denote $I = \{i_1, i_2, \cdots, i_n\}$ and $J = \{i_{j_1}, i_{j_2}, \cdots, i_{j_t}\}$, where $t \le n$. Suppose $i_1 < i_2 < \cdots < i_n$ and $1 \le j_1 < j_2 < \cdots < j_t \le n$. Obviously, $K = \mathrm{st}_I(J) = \{j_1, j_2, \cdots, j_t\}$ and $\mathrm{st}_K(j_m) = m$ for any $i_{j_m} \in J$. Then

$$\mathrm{st}_K(\mathrm{st}_I(i_{j_s})) = s = \mathrm{st}_J(i_{j_s}),$$

for any $1 \le s \le t$. □

Example 6. If $I = \{3,5,7,8,9\}$ and $J = \{3,7,8\}$, then $\mathrm{st}_J(3) = 1$, $\mathrm{st}_J(7) = 2$ and $\mathrm{st}_J(8) = 3$. On the other hand, $\mathrm{st}_I(3) = 1$, $\mathrm{st}_I(7) = 3$, $\mathrm{st}_I(8) = 4$ and $K = \mathrm{st}_I(\{3,7,8\}) = \{1,3,4\}$. So $\mathrm{st}_K(\mathrm{st}_I(3)) = 1$, $\mathrm{st}_K(\mathrm{st}_I(7)) = 2$, $\mathrm{st}_K(\mathrm{st}_I(8)) = 3$.

Lemma 2. *Assume $\Gamma = (V, E)$ is a labeled simple graph, $J \subseteq I \subseteq V$ and $K = \mathrm{st}_I(J)$. Then*

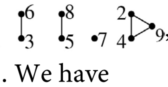
$$\mathrm{st}(\mathrm{st}(\Gamma_I)_K) = \mathrm{st}(\Gamma_J).$$

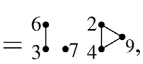
Proof. For convenience, we denote $\mathrm{st}(\mathrm{st}(\Gamma_I)_K)$ is Γ_1 and $\mathrm{st}(\Gamma_J)$ is Γ_2. Obviously,

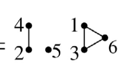
$$V(\Gamma_1) = |K| = |J| = V(\Gamma_2).$$

We just need to show that their edges are the same. For (i,j) in $E(\Gamma_1)$, there must exist i'' and j'' in K such that $\mathrm{st}_K(i'') = i$ and $\mathrm{st}_K(j'') = j$. Meanwhile, there must exist i' and j' in J such that $\mathrm{st}_I(i') = i''$ and $\mathrm{st}_I(j') = j''$, *i.e.*, $\mathrm{st}_K(\mathrm{st}_I(i')) = i$ and $\mathrm{st}_K(\mathrm{st}_I(j')) = j$. Then (i', j') in $E(\Gamma)$. By the definition of $\mathrm{st}(\Gamma_J)$, $(\mathrm{st}_J(i'), \mathrm{st}_J(j'))$ is in $E(\Gamma_2)$. By Lemma 1, $\mathrm{st}_J(i') = i$ and $\mathrm{st}_J(j') = j$, then $(i,j) \in E(\Gamma_2)$. Similarly, we can prove if $(i,j) \in E(\Gamma_2)$ then $(i,j) \in E(\Gamma_1)$. So

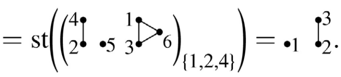
$$\mathrm{st}(\mathrm{st}(\Gamma_I)_K) = \mathrm{st}(\Gamma_J).$$ □

Example 7. Let $\Gamma =$ , $I = \{2,3,4,6,7,9\}$ and $J = \{2,3,6\}$ then $K = \mathrm{st}_I(J) = \{1,2,4\}$. We have

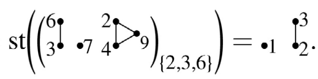
$\Gamma_I =$ ,

$\mathrm{st}(\Gamma_I) =$ 

and

$$\mathrm{st}(\mathrm{st}(\Gamma_I)_K) = $$ 

On the other hand,

$$\mathrm{st}((\Gamma_I)_J) = $$ 

Theorem 2. *The vector space* $\mathcal{H}$ *with the unshuffle coproduct* Δ_* *and the counit* ν *is a graded coalgebra.*

Proof. Obviously, the empty graph ε satisfies

$$(\Delta_* \otimes \mathrm{id}) \circ \Delta_*(\varepsilon) = \varepsilon \otimes \varepsilon \otimes \varepsilon = (\mathrm{id} \otimes \Delta_*) \circ \Delta_*(\varepsilon).$$

For any $\Gamma = ([n], E)$ in H_n where $n \geq 1$, we have

$$(\Delta_* \otimes \mathrm{id}) \circ \Delta_*(\Gamma) = (\Delta_* \otimes \mathrm{id}) \left(\sum_{I \subseteq [n]} \mathrm{st}(\Gamma_I) \otimes \mathrm{st}\left(\Gamma_{[n] \setminus I}\right) \right). \tag{1}$$

Denote $\mathrm{st}(\Gamma_I)$ by Θ, then

$$\Delta_*(\Theta) = \sum_{K \subseteq [|I|]} \mathrm{st}(\Theta_K) \otimes \mathrm{st}\left(\Theta_{[|I|] \setminus K}\right). \tag{2}$$

Denote J as a subset in I such that $\mathrm{st}_I(J) = K$, then $\mathrm{st}_I(I \setminus J) = [|I|] \setminus K$. By Lemma 2,

$$\mathrm{st}\left(\mathrm{st}(\Gamma_I)_K\right) = \mathrm{st}(\Gamma_J)$$

and

$$\mathrm{st}\left(\mathrm{st}(\Gamma_I)_{[|I|] \setminus K}\right) = \mathrm{st}(\Gamma_{I \setminus J}).$$

Since K in (2) traverses all subsets of $[|I|]$, the corresponding J also traverses all subsets of I. Then (2) can be rewritten as

$$\sum_{J \subseteq I} \mathrm{st}(\Gamma_J) \otimes \mathrm{st}(\Gamma_{I \setminus J}). \tag{3}$$

Then (1) can be rewritten as

$$\sum_{I \subseteq [n], J \subseteq I} \mathrm{st}(\Gamma_J) \otimes \mathrm{st}(\Gamma_{I \setminus J}) \otimes \mathrm{st}\left(\Gamma_{[n] \setminus I}\right). \tag{4}$$

By the arbitrariness of I and J, (4) can be rewritten as

$$\sum_{\substack{I, J, K \subseteq [n] \\ I \cup J \cup K = [n] \\ |I| + |J| + |K| = n}} \mathrm{st}(\Gamma_I) \otimes \mathrm{st}(\Gamma_J) \otimes \mathrm{st}(\Gamma_K). \tag{5}$$

Similarly, we can get $(\mathrm{id} \otimes \Delta_*) \circ \Delta_*(\Gamma)$ is also equal to (5). Then Δ_* satisfies coassociativity. It is easy to prove that ν is a counit. So, $(\mathcal{H}, \Delta_*, \nu)$ is a coalgebra. Obviously, by the definition of Δ_* and ν, we have $\Delta_*(\mathcal{H}_n) \subseteq \bigoplus_{0 \leq i \leq n} \mathcal{H}_i \otimes \mathcal{H}_{n-i}$ and $\mu(\mathcal{H}_n) = 0$ for $n > 0$. So $(\mathcal{H}, \Delta_*, \nu)$ is a graded coalgebra. □

Next we prove the compatibility between the conjunction product and the unshuffle coproduct.

Lemma 3. *Assume* $\Gamma_1 = ([m], E_1), \Gamma_2 = ([n], E_2)$, $I \subset [m]$ *and*

$J \subseteq [m+1, m+n]$. *Then*

$$\mathrm{st}\left(\left(\Gamma_1 \cup \Gamma_2^{\uparrow m}\right)_{I\cup J}\right) = \mathrm{st}\left((\Gamma_1)_I\right) \cup \mathrm{st}\left(\left(\Gamma_2^{\uparrow m}\right)_J\right)^{\uparrow |I|}.$$

Proof. Denote $\Theta = \Gamma_1 \cup \Gamma_2^{\uparrow m}$. Since $I \subset [m], J \subseteq [m+1, m+n]$, there are no edges between Θ_I and Θ_J in Θ, *i.e.*, $\Theta_{I\cup J} = \Theta_I \cup \Theta_J$. Then, $\mathrm{st}(\Theta_{I\cup J}) = \mathrm{st}(\Theta_I \cup \Theta_J)$.

Since $\max\{I\} < \min\{J\}$, $\mathrm{st}_{I\cup J}(I) = [|I|]$ and $\mathrm{st}_{I\cup J}(J) = [|I|+1, |I|+|J|]$. By Lemma 2,

$$\mathrm{st}\left(\mathrm{st}(\Theta_{I\cup J})_{[|I|]}\right) = \mathrm{st}(\Theta_I)$$

and

$$\mathrm{st}\left(\mathrm{st}(\Theta_{I\cup J})_{[|I|+1, |I|+|J|]}\right) = \mathrm{st}(\Theta_J).$$

Since there are no edges between Θ_I and Θ_J, there are no edges between $\mathrm{st}(\Theta_{I\cup J})_{[|I|]}$ and $\mathrm{st}(\Theta_{I\cup J})_{[|I|+1, |I|+|J|]}$. So, $\mathrm{st}(\Theta_{I\cup J}) = \mathrm{st}(\Theta_I) \cup \mathrm{st}(\Theta_J)^{\uparrow |I|}$.

Since $\Theta = \Gamma_1 \cup \Gamma_2^{\uparrow m}$, $I \subseteq [m]$ and $J \subseteq [m+1, m+n]$, we have $\Theta_I = (\Gamma_1)_I$ and $\Theta_J = \left(\Gamma_2^{\uparrow m}\right)_J$. Then

$$\mathrm{st}\left(\left(\Gamma_1 \cup \Gamma_2^{\uparrow m}\right)_{I\cup J}\right) = \mathrm{st}\left((\Gamma_1)_I\right) \cup \mathrm{st}\left(\left(\Gamma_2^{\uparrow m}\right)_J\right)^{\uparrow |I|}.$$ □

Corollary 1. *Assume* $\Gamma_1 = ([m], E_1), \Gamma_2 = ([n], E_2)$, $I \subset [m]$ *and* $J \subseteq [m+1, m+n]$. *Then*

$$\mathrm{st}\left((\Gamma_1 \lozenge \Gamma_2)_{I\cup J}\right) = \mathrm{st}\left((\Gamma_1)_I\right) \lozenge \mathrm{st}\left((\Gamma_2)_{J'}\right), \tag{6}$$

where $J' = \mathrm{st}_{[m+1, m+n]}(J)$.

Proof. By the definition of $\lozenge$ and Lemma 3,

$$\mathrm{st}\left((\Gamma_1 \lozenge \Gamma_2)_{I\cup J}\right) = \mathrm{st}\left((\Gamma_1)_I\right) \lozenge \mathrm{st}\left(\left(\Gamma_2^{\uparrow m}\right)_J\right). \tag{7}$$

Obviously, by the definition of J' and Lemma 2, we have

$$\mathrm{st}\left(\left(\Gamma_2^{\uparrow m}\right)_J\right) = \mathrm{st}\left(\mathrm{st}\left(\Gamma_2^{\uparrow m}\right)_{J'}\right) = \mathrm{st}\left((\Gamma_2)_{J'}\right).$$

Then (7) can be rewritten as (6). □

Theorem 3. $(\mathcal{H}, \lozenge, \mu, \Delta_*, \nu)$ is a bialgebra.

Proof. Obviously, the counit μ is an algebra homomorphism. We only need to prove the Δ_* is an algebra homomorphism, *i.e.*,

$$\Delta_*(\Gamma_1 \lozenge \Gamma_2) = \Delta_*(\Gamma_1) \lozenge \Delta_*(\Gamma_2), \tag{8}$$

for any Γ_1 and Γ_2 in *H*. If $\Gamma_1 = \varepsilon$ or $\Gamma_2 = \varepsilon$ then (8) holds. If $\Gamma_1 = ([m], E_1)$ and $\Gamma_2 = ([n], E_2)$ are non-empty, we denote $\Theta = \Gamma_1 \lozenge \Gamma_2 = \Gamma_1 \cup \Gamma_2^{\uparrow m}$ and

$$\Delta_*(\Theta) = \sum_{I \subseteq [m+n]} \mathrm{st}(\Theta_I) \otimes \mathrm{st}\left(\Theta_{[m+n]\setminus I}\right). \tag{9}$$

Denote $I_{11} = I \cap [m]$, $I_{12} = I \cap [m+1, m+n]$, $I_{21} = ([m+n] \setminus I) \cap [m]$ and $I_{22} = ([m+n] \setminus I) \cap [m+1, m+n]$. Furthermore, denote $\mathrm{st}_{[m+1, m+n]}(I_{12})$ by J_{12}

and denote $\mathrm{st}_{[m+1,m+n]}(I_{22})$ by J_{22}. We have $I_{11} \subseteq [m]$, $I_{12} \subseteq [m+1,m+n]$ and $J_{12} = \mathrm{st}_{[m+1,m+n]}(I_{12})$. By Corollary 1,

$$\mathrm{st}(\Theta_I) = \mathrm{st}(\Theta_{I_{11} \cup I_{12}}) = \mathrm{st}((\Gamma_1)_{I_{11}}) \lozenge \mathrm{st}((\Gamma_2)_{J_{12}}).$$

Similarly,

$$\mathrm{st}(\Theta_{[m+n] \setminus I}) = \mathrm{st}(\Theta_{I_{21} \cup I_{22}}) = \mathrm{st}((\Gamma_1)_{I_{21}}) \lozenge \mathrm{st}((\Gamma_2)_{J_{22}}).$$

Then (9) can be rewritten as

$$\sum_{I \subseteq [m+n]} \mathrm{st}((\Gamma_1)_{I_{11}}) \lozenge \mathrm{st}((\Gamma_2)_{J_{12}}) \otimes \mathrm{st}((\Gamma_1)_{I_{21}}) \lozenge \mathrm{st}((\Gamma_2)_{J_{22}}). \tag{10}$$

Obviously, when I traverses all subsets of $[m+n]$, I_{12} and I_{21} traverse all disjoint subsets of $[m]$, I_{12} and I_{22} traverse all disjoint subsets of $[m+1,m+n]$, and meanwhile J_{12} and J_{22} travese all disjoint subsets of $[n]$.

Then we rewrite (10) as

$$\sum_{\substack{I_{11} \cap I_{21} = \varnothing \\ I_{11} \cup I_{21} = [m]}} \mathrm{st}((\Gamma_1)_{I_{11}}) \otimes \mathrm{st}((\Gamma_1)_{I_{21}}) \lozenge \sum_{\substack{J_{12} \cap J_{22} = \varnothing \\ J_{12} \cup J_{22} = [n]}} \mathrm{st}((\Gamma_1)_{J_{12}}) \otimes \mathrm{st}((\Gamma_2)_{J_{22}}). \tag{11}$$

By the definition of Δ_*, (11) is equal to

$$\Delta_*(\Gamma_1) \lozenge \Delta_*(\Gamma_2).$$

Therefore,

$$\Delta_*(\Gamma_1 \lozenge \Gamma_2) = \Delta_*(\Gamma_1) \lozenge \Delta_*(\Gamma_2).$$

So $(\mathcal{H}, \lozenge, \mu, \Delta_*, \nu)$ is a bialgebra. □

Corollary 2. $(\mathcal{H}, \lozenge, \mu, \Delta_*, \nu)$ is a Hopf algebra.

Proof. By Theorem 1, Theorem 2 and Theorem 3, $(\mathcal{H}, \lozenge, \mu, \Delta_*, \nu)$ is a graded connected bialgebra. So it is a Hopf algebra. □

Example 8. For $\Gamma_1 = \Big\vert^{\bullet_2}_{\bullet_1}$ and $\Gamma_2 = {}^{\bullet_2}_{\bullet_1}$, we have

$$\begin{aligned}
&\Delta_*\big(\Big\vert^{\bullet_2}_{\bullet_1} \diamond {}^{\bullet_2}_{\bullet_1}\big) \\
=&\Delta_*\big(\Big\vert^{\bullet_2}_{\bullet_1}\ {}^{\bullet_4}_{\bullet_3}\big) \\
=&\epsilon \otimes \Big\vert^{\bullet_2}_{\bullet_1}\ {}^{\bullet_4}_{\bullet_3} + \bullet_1 \otimes \bullet_1\ {}^{\bullet_3}_{\bullet_2} + \bullet_1 \otimes \bullet_1\ {}^{\bullet_3}_{\bullet_2} + \bullet_1 \otimes \Big\vert^{\bullet_2}_{\bullet_1} \bullet_3 + \bullet_1 \otimes \Big\vert^{\bullet_2}_{\bullet_1} \bullet_3 \\
&+ \Big\vert^{\bullet_2}_{\bullet_1} \otimes {}^{\bullet_2}_{\bullet_1} + \bullet_1 \bullet_2 \otimes \bullet_1 \bullet_2 + \bullet_1 \bullet_2 \otimes \bullet_1 \bullet_2 + \bullet_1 \bullet_2 \otimes \bullet_1 \bullet_2 \\
&+ \bullet_1 \bullet_2 \otimes \bullet_1 \bullet_2 + {}^{\bullet_2}_{\bullet_1} \otimes \Big\vert^{\bullet_2}_{\bullet_1} + \Big\vert^{\bullet_2}_{\bullet_1} \bullet_3 \otimes \bullet_1 + \Big\vert^{\bullet_2}_{\bullet_1} \bullet_3 \otimes \bullet_1 + \bullet_1\ {}^{\bullet_3}_{\bullet_2} \otimes \bullet_1 \\
&+ \bullet_1\ {}^{\bullet_3}_{\bullet_2} \otimes \bullet_1 + \Big\vert^{\bullet_2}_{\bullet_1}\ {}^{\bullet_4}_{\bullet_3} \otimes \epsilon \\
=&(\epsilon \diamond \epsilon) \otimes \left(\Big\vert^{\bullet_2}_{\bullet_1} \diamond {}^{\bullet_4}_{\bullet_3}\right) + (\bullet_1 \diamond \epsilon) \otimes \left(\bullet_1 \diamond {}^{\bullet_2}_{\bullet_1}\right) + (\bullet_1 \diamond \epsilon) \otimes \left(\bullet_1 \diamond {}^{\bullet_2}_{\bullet_1}\right) \\
&+ (\epsilon \diamond \bullet_1) \otimes \left(\Big\vert^{\bullet_2}_{\bullet_1} \diamond \bullet_1\right) + (\epsilon \diamond \bullet_1) \otimes \left(\Big\vert^{\bullet_2}_{\bullet_1} \diamond \bullet_1\right) + \left(\Big\vert^{\bullet_2}_{\bullet_1} \diamond \epsilon\right) \otimes \left(\epsilon \otimes {}^{\bullet_2}_{\bullet_1}\right) \\
&+ (\bullet_1 \diamond \bullet_1) \otimes (\bullet_1 \diamond \bullet_1) + (\bullet_1 \diamond \bullet_1) \otimes (\bullet_1 \diamond \bullet_1) \\
&+ (\bullet_1 \diamond \bullet_1) \otimes (\bullet_1 \diamond \bullet_1) + (\bullet_1 \diamond \bullet_1) \otimes (\bullet_1 \diamond \bullet_1)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\epsilon\diamond \begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\diamond\epsilon\right)+\left(\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\diamond\bullet_{1}\right)\otimes(\epsilon\diamond\bullet_{1})+\left(\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\diamond\bullet_{1}\right)\otimes(\epsilon\diamond\bullet_{1})\\
&+\left(\bullet_{1}\diamond\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\right)\otimes(\bullet_{1}\diamond\epsilon)+\left(\bullet_{1}\diamond\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\right)\otimes(\bullet_{1}\diamond\epsilon)+\left(\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\diamond\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\right)\otimes(\epsilon\otimes\epsilon)\\
&=\left(\epsilon\otimes\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}+\bullet_{1}\otimes\bullet_{1}+\bullet_{1}\otimes\bullet_{1}+\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\otimes\epsilon\right)\diamond\left(\epsilon\otimes\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}+\bullet_{1}\otimes\bullet_{1}+\bullet_{1}\otimes\bullet_{1}+\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\otimes\epsilon\right)\\
&=\Delta_{*}\left(\begin{smallmatrix}\bullet_{2}\\ \vert\\ \bullet_{1}\end{smallmatrix}\right)\diamond\Delta_{*}\left(\begin{smallmatrix}\bullet_{2}\\ \bullet_{1}\end{smallmatrix}\right).
\end{aligned}
$$

5. Hopf Homomorphism from Permutations to Labeled Simple Graphs

In 2020, Aval, Bergeron and Machacek gave a Hopf homomorphism from permutations to labeled simple graphs without a proof [33]. Next, we prove this mapping is a Hopf homomorphism.

Firstly, we review some basic concepts as follows. For more details, see [27] [33].

Denote S_n as the set of all permutations of degree *n*. For any permutation α in S_n, denote $\alpha=\alpha_1\alpha_2\cdots\alpha_n$, where $\alpha(i)=\alpha_i$ for $1\le i\le n$. For example, $S_3=\{123,132,213,231,312,321\}$. In particular, $S_0=\{\varepsilon_p\}$, where ε_p is the empty permutation. If $I=\{\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s}\}\subseteq[n]$ and $i_1<i_2<\cdots<i_s$, then we define the *restriction* of α on *I* by $\alpha_I=\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_s}$, and call it a *subsequence* of α.

More generally, for a set $B=\{\beta_1,\beta_2,\cdots,\beta_n\}$ of positive integers, we call $\beta=\beta_1\beta_2\cdots\beta_n$ a *sequence* on *B* with *length n*. Define the *standard form* of β by $\mathrm{st}(\beta)=\mathrm{st}_B(\beta_1)\mathrm{st}_B(\beta_2)\cdots\mathrm{st}_B(\beta_n)$. In fact, any permutation is a sequence. Define $\mathcal{S}_n$ to be the vector space spanned by S_n over field $\mathbb{K}$. Denote

$$S=\bigcup_{n=1}^{\infty}S_n \quad\text{and}\quad \mathcal{S}=\bigoplus_{n=1}^{\infty}\mathcal{S}_n.$$

Next, we review definitions of the conjunction product $\bullet$ and the unshuffle coproduct Δ^* on permutations [24] [33].

Define the *conjunction product* $\bullet$ on $\mathcal{S}$ by

$$\alpha\bullet\beta=\alpha_1\alpha_2\cdots\alpha_m(\beta_1+m)(\beta_2+m)\cdots(\beta_n+m),$$

and the *unshuffle coproduct* on $\mathcal{S}$ by

$$\Delta^*(\alpha)=\sum_{I\subseteq[m]}\mathrm{st}(\alpha_I)\otimes\mathrm{st}\left(\alpha_{[m]\setminus I}\right),$$

for $\alpha=\alpha_1\alpha_2\cdots\alpha_m$ in S_m and $\beta=\beta_1\beta_2\cdots\beta_n$ in S_n.

Obviously, we have $\alpha\bullet\varepsilon_p=\varepsilon_p\bullet\alpha=\alpha$ for any $\alpha\in S$ and $\Delta(\varepsilon_p)=\varepsilon_p\otimes\varepsilon_p$. Define the *unit* μ_p by $\mu_p(1)=\varepsilon_p$ and the *counit* ν_p by

$$\nu_p(\alpha)=\begin{cases}1 & \alpha=\varepsilon_p,\\ 0 & \text{otherwise},\end{cases}$$

for α in *S*. Then $(\mathcal{S},\bullet,\mu_p,\Delta^*,\nu_p)$ is a Hopf algebra [33].

Example 9. For $\alpha=312$ and $\beta=4123$, we have

$$\beta_{\{1,3,4\}} = 413, \quad \beta_{\{2,3,4\}} = 423,$$

$$\mathrm{st}(413) = 312, \quad \mathrm{st}(423) = 312,$$

$$\alpha \bullet \beta = 3127456, \quad \beta \bullet \alpha = 4123756$$

and

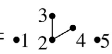
$$\Delta^*(\alpha) = \varepsilon_p \otimes 312 + 1 \otimes 12 + 1 \otimes + 1 \otimes 21 + 21 \otimes 1 + 21 \otimes 1 + 12 \otimes 1 + 312 \otimes \varepsilon_p.$$

For $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ in S_n, we call (α_i, α_j) an *inversion* of α if $i < j$ and $\alpha_i > \alpha_j$. Define a linear mapping $\mathfrak{g}$ from $\mathcal{S}$ to $\mathcal{H}$ by $\mathfrak{g}(\alpha) = ([n], E_\alpha)$, where

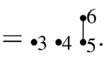
$$E_\alpha = \{(\alpha_i, \alpha_j) \mid i < j \text{ and } \alpha_i > \alpha_j\}, \tag{12}$$

for $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ in S_n. In particular, $\mathfrak{g}(\varepsilon_p) = \varepsilon$. Obviously, each edge of $\mathfrak{g}(\alpha)$ connects an inversion of α. In fact, we can define the mapping $\mathfrak{g}$ sending a sequence on B to a labeled simple graph on B by (12).

Example 10. For $\alpha = 13425$ and $\beta = 3465$,

$$\mathfrak{g}(\alpha) = ([5], \{(3,2), (4,2)\}) =$$ 

and

$$\mathfrak{g}(\beta) = (\{3,4,5,6\}, \{(6,5)\}) =$$ .

Next, we prove $\mathfrak{g}$ is a Hopf homomorphism from $\mathcal{S}$ to $\mathcal{H}$, that means it is an algebra homomorphism and a coalgebra homomorphism.

Lemma 4. *The mapping* $\mathfrak{g}$ *is an algebra homomorphism.*

Proof. Suppose $\alpha = \alpha_1 \alpha_2 \cdots \alpha_m$ in S_m and $\beta = \beta_1 \beta_2 \cdots \beta_n$ in S_n. If α or β is an empty permutation, then obviously

$$\mathfrak{g}(\alpha \bullet \beta) = \mathfrak{g}(\alpha) \lozenge \mathfrak{g}(\beta).$$

If they are non-empty permutations, then

$$\sigma := \alpha \bullet \beta = \alpha_1 \alpha_2 \cdots \alpha_m (\beta_1 + m)(\beta_2 + m) \cdots (\beta_n + m).$$

Since

$$\max_{1 \le i \le m} \{\alpha_i\} < \min_{1 \le j \le n} \{\beta_j + m\},$$

there are no edges between $\{\alpha_1, \cdots, \alpha_m\}$ and $\{\beta_1 + m, \cdots, \beta_n + m\}$. If we denote $\mathfrak{g}(\sigma) = ([m+n], E_\sigma)$, then

$$\begin{aligned} E_\sigma = &\{(\alpha_i, \alpha_j) \mid i < j \text{ and } \alpha_i > \alpha_j\} \\ &\cup \{(\beta_i + m, \beta_j + m) \mid i < j \text{ and } \beta_i + m > \beta_j + m\} \end{aligned}$$

Denote $E_1 = \{(\alpha_i, \alpha_j) \mid i < j \text{ and } \alpha_i > \alpha_j\}$ and $E_2 = \{(\beta_i + m, \beta_j + m) \mid i < j \text{ and } \beta_i + m > \beta_j + m\}$, then

$$([m+n], E_\sigma) = ([m], E_1) \cup ([m+1, m+n], E_2).$$

Obviously, $\mathfrak{g}(\alpha) = ([m], E_1)$ by the definition of $\mathfrak{g}$. On the other hand, $\mathfrak{g}(\beta) = ([n], E_\beta)$, where

$$E_\beta=\left\{\left(\beta_i,\beta_j\right)\mid i<j\text{ and }\beta_i>\beta_j\right\}.$$

Then $\mathfrak{g}(\beta)^{\uparrow m}=\left([m+1,m+n],E_2'\right)$, where

$$E_2'=\left\{\left(\beta_i+m,\beta_j+m\right)\mid i<j\text{ and }\beta_i>\beta_j\right\},$$

which is equal to E_2. So $\mathfrak{g}(\beta)^{\uparrow m}=\left([m+1,m+n],E_2\right)$. Then

$$\mathfrak{g}(\alpha\bullet\beta)=\mathfrak{g}(\alpha)\cup\mathfrak{g}(\beta)^{\uparrow m}=\mathfrak{g}(\alpha)\Diamond\mathfrak{g}(\beta).$$

So $\mathfrak{g}$ is an algebra homomorphism. □

Lemma 5. *Assume* α *is a non-empty permutation in* S_n *and I is a subset of* $[n]$. *Then*

$$\mathfrak{g}\left(\operatorname{st}\left(\alpha_I\right)\right)=\operatorname{st}\left(\mathfrak{g}(\alpha)_I\right).$$

Proof. Let $\alpha=\alpha_1\alpha_2\cdots\alpha_n\in S_n$, $I=\left\{\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s}\right\}$ and $i_1<i_2<\cdots<i_s$. Then $\alpha_I=\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_s}$ and

$$\operatorname{st}\left(\alpha_I\right)=\operatorname{st}_I\left(\alpha_{i_1}\right)\operatorname{st}\left(\alpha_{i_2}\right)\cdots\operatorname{st}\left(\alpha_{i_s}\right).$$

Denote $\gamma_k=\operatorname{st}_I\left(\alpha_{i_k}\right)$ for $1\le k\le s$, then $\operatorname{st}\left(\alpha_I\right)=\gamma_1\gamma_2\cdots\gamma_s$. If $\mathfrak{g}\left(\operatorname{st}\left(\alpha_I\right)\right)=\left(\left[|I|\right],E_1\right)$, then

$$\begin{aligned}E_1&=\left[\left(\gamma_j,\gamma_k\right)\mid j<k\text{ and }\gamma_j>\gamma_k\right]\\&=\left\{\left(\operatorname{st}_I\left(\alpha_{i_j}\right),\operatorname{st}_I\left(\alpha_{i_k}\right)\right)\mid i_j<i_k\text{ and }\operatorname{st}_I\left(\alpha_{i_j}\right)>\operatorname{st}_I\left(\alpha_{i_k}\right)\right\}.\end{aligned}$$

On the other hand, $\mathfrak{g}(\alpha)=\left([n],E_2\right)$, where

$$E_2=\left\{\left(\alpha_j,\alpha_k\right)\mid j<k\text{ and }\alpha_j>\alpha_k\right\}.$$

If we have $\mathfrak{g}(\alpha)_I=\left(I,E_3\right)$ then

$$E_3=\left(E_2\right)_I=\left\{\left(\alpha_{i_j},\alpha_{i_k}\right)\mid i_j<i_k\text{ and }\alpha_{i_j}>\alpha_{i_k}\right\}.$$

And $\operatorname{st}\left(\mathfrak{g}(\alpha)_I\right)=\operatorname{st}\left(I,E_3\right)=\left(\left[|I|\right],\operatorname{st}\left(E_3\right)\right)$, where

$$\begin{aligned}\operatorname{st}\left(E_3\right)&=\left\{\left(\operatorname{st}_I\left(\alpha_{i_j}\right),\operatorname{st}_I\left(\alpha_{i_k}\right)\right)\mid i_j<i_k\text{ and }\alpha_{i_j}>\alpha_{i_k}\right\}\\&=\left\{\left(\operatorname{st}_I\left(\alpha_{i_j}\right),\operatorname{st}_I\left(\alpha_{i_k}\right)\right)\mid i_j<i_k\text{ and }\operatorname{st}_I\left(\alpha_{i_j}\right)>\operatorname{st}_I\left(\alpha_{i_k}\right)\right\}\\&=E_1.\end{aligned}$$

So $\mathfrak{g}\left(\operatorname{st}\left(\alpha_I\right)\right)=\operatorname{st}\left(\mathfrak{g}(\alpha)_I\right)$. □

Lemma 6. *The mapping* $\mathfrak{g}$ *is a coalgebra homomorphism.*

Proof. Obviously, for empty permutation ε_p, we have

$$\mathfrak{g}\left(\Delta^*\left(\varepsilon_p\right)\right)=\varepsilon\otimes\varepsilon=\Delta_*\left(\mathfrak{g}\left(\varepsilon_p\right)\right).$$

For any non-empty permutation $\alpha\in S_n$, by Lemma 5 we have

$$\begin{aligned}\mathfrak{g}\left(\Delta^*(\alpha)\right)&=\mathfrak{g}\left(\sum_{I\subseteq[n]}\operatorname{st}\left(\alpha_I\right)\otimes\operatorname{st}\left(\alpha_{[n]\setminus I}\right)\right)\\&=\sum_{I\subseteq[n]}\mathfrak{g}\left(\operatorname{st}\left(\alpha_I\right)\right)\otimes\mathfrak{g}\left(\operatorname{st}\left(\alpha_{[n]\setminus I}\right)\right)\end{aligned}$$

$$= \sum_{I \subseteq [n]} \mathrm{st}\left(\mathfrak{g}(\alpha)_I\right) \otimes \mathrm{st}\left(\mathfrak{g}(\alpha)_{[n]\setminus I}\right)$$
$$= \Delta_*\left(\mathfrak{g}(\alpha)\right).$$

So $\mathfrak{g}$ is a coalgebra homomorphism. □

Corollary 3. The mapping $\mathfrak{g}$ is a Hopf homomorphism.

Proof. By Lemma 4 and Lemma 6, $\mathfrak{g}$ is a Hopf homomorphism. □

6. Conclusion

Let $\mathcal{H}$ be the vector space spanned by labeled simple graphs. Firstly, we give the definitions of the conjunction product $\Diamond$ and the unshuffle coproduct Δ_* on $\mathcal{H}$. Then we prove the conjunction product $\Diamond$ satisfies associativity and the unshuffle coproduct Δ_* satisfies coassociativity, *i.e.*, $(\mathcal{H}, \Diamond, \mu)$ is an algebra and $(\mathcal{H}, \Delta_*, \nu)$ is a coalgebra. We prove the compatibility between $\Diamond$ and Δ_*, and $(\mathcal{H}, \Diamond, \mu, \Delta_*, \nu)$ is a graded connected bialgebra. So $(\mathcal{H}, \Diamond, \mu, \Delta_*, \nu)$ is Hopf algebra. Lastly, let $\mathcal{S}$ be the vector space spanned by permutations. We recall a mapping $\mathfrak{g}$ from $\mathcal{S}$ to $\mathcal{H}$ and prove it is a Hopf homomorphism. In the future, we will study the duality of the Hopf algebra $(\mathcal{H}, \Diamond, \mu, \Delta_*, \nu)$.

Acknowledgements

This work is supported by National Natural Science Foundation of China (Nos. 11701339 and 12071265).

Conflicts of Interest

The author declares that there are no conflicts of interest in this paper.

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