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A Family of Inertial Manifolds for a Class of Asymmetrically Coupled Generalized Higher-Order Kirchhoff Equations

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Abstract

In this paper, we study the inertial manifolds for a class of asymmetrically coupled generalized Higher-order Kirchhoff equations. Under appropriate assumptions, we firstly exist Hadamard's graph transformation method to structure a graph norm of a Lipschitz continuous function, then we prove the existence of a family of inertial manifolds by showing that the spectral gap condition is true.

Keywords

Inertial Manifold, Hadamard's Graph Transformation Method, Lipschitz Continuous, Spectral Gap Condition

1. Introduction

In this paper, we study the inertial manifolds for a class of asymmetrically coupled generalized Higher-order Kirchhoff equations:

$$u_{tt} + M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^m v\right\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(u_t, v) = f_1(x), \tag{1}$$

$$v_{tt} + M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} v + \beta(-\Delta)^{2m} v_t + g(u, v_t) = f_2(x), \tag{2}$$

the boundary conditions:

$$\frac{\partial^i u}{\partial n^i} = 0, i = 0,1,2,\cdots,m-1, x \in \partial\Omega, t > 0, \tag{3}$$

$$\frac{\partial^j v}{\partial v^j} = 0, j = 0,1,2,\cdots,2m-1, x \in \partial\Omega, t > 0, \tag{4}$$

the initial conditions:

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, \quad (5)$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $u_0(x),u_1(x)$ is a known function, $g(u,v)$, $f_i(x),i=1,2$ are nonlinear source term and the external force interference terms, $m>1,\beta$ is real number.

Recently, the existence of inertial manifolds for Kirchhoff-type equation has been favored by many scholars. Many scholars have done a lot of research on this kind of problems and obtained good results [1] [2] [3].

Lin Chen, Wei Wang and Guoguang Lin [1] studied higher-order Kirchhoff-type equation with nonlinear strong dissipation in n dimensional space:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla u\|^2\right)(-\Delta)^m u+g(u)=f(x),x\in\Omega,t>0,m>1,$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t>0,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),$$

for the above equation, they made some suitable assumptions about $\phi(s)$ and $g(u)$ to get the existence of exponential attractors and inertial manifolds.

Guoguang Lin, Ming Zhang [2] studied the initial boundary value problem for a class of Kirchhoff-type coupled equations:

$$u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u-\beta\Delta u_t+g_1(u,v)=f_1(x),$$

$$v_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta v-\beta\Delta v_t+g_2(u,v)=f_2(x),$$

they used the one order coupled evolution equation which is equivalent to Kirchhoff-type coupled Equations. Then by using the graph norm in X, they get the existence of the inertial manifolds.

Lin Guoguang, Yang Lujiao [3] studied the existence of exponential attractors and a family of inertial manifolds for a class of generalized Kirchhoff-type equation with damping term:

$$u_{tt}+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u)=f(x),$$

by using Hadamard's graph transformation method, they proved the spectral interval condition is true; then they obtained the existence of a family of the inertial manifolds for the equation.

For more significant research results about the existence of inertial manifolds for Kirchhoff-type equations, please refer to the literature [4]-[17].

This paper is organized as follows. In Section 2, we present the preliminaries and some lemmas. In Section 3, the inertial manifold is obtained.

2. Preliminaries

The following symbols and assumptions are introduced for the convenience of the statement:

$$V_0=L^2(\Omega),V_{m+k}=H^{m+k}(\Omega)\cap H_0^1(\Omega),V_{2m+2k}=H^{2m+2k}(\Omega)\cap H_0^1(\Omega),$$
$$V_k=H^k(\Omega)\cap H_0^1(\Omega),V_{2k}=H^{2k}(\Omega)\cap H_0^1(\Omega),E_0=V_m\times V_0\times V_{2m}\times V_0,$$
$$E_k=V_{m+k}\times V_k\times V_{2m+2k}\times V_{2k},k=0,1,2,\cdots,m.$$

In order to obtain our results, we consider system (1)-(5) under some assumptions on $M(s)$ and $g(u,v)$. Precisely, we state the general assumptions:

(H1) $g(u_t,v), g(u,v_t) \in C^1(\Omega)$,

(H2) $\varepsilon \le m_0 \le M(s) \le m_1$.

Definition 1 [6] Assuming $S = (S(t))_{t\ge 0}$ is a solution semigroup on Banach space E_k, subset $\mu_k \subset E_k$ is said to be a family of inertial manifolds, if they satisfy the following three properties:

1) μ_k is a finite-dimensional Lipschitz manifold;

2) μ_k is positively invariant, *i.e.*, $S(t)\mu_k \subseteq \mu_k, t > 0$;

3) μ_k attracts exponentially all orbits of solution , that is, for any $x \in E_k$, there are constants $\eta > 0, C > 0$ such that

$$dist(S(t)x, \mu_k) \le Ce^{-\eta t}, t \ge 0, \tag{6}$$

Definition 2 [6] Let $A: X \to X$ be an operator and assume that $F \in C_b(X, X)$ satisfies the Lipschitz condition:

$$\|F(U) - F(V)\|_X \le l_F \|U - V\|_X, \tag{7}$$

If the point spectrum of the operator A can be divided into the following two parts σ_1 and σ_2, where σ_1 is finite

$$\Lambda_1 = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma_1\}, \Lambda_2 = \inf\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma_2\}, \tag{8}$$

$$X_i = span\{\omega_j \mid \lambda_j \in \sigma_i\}, i = 1, 2. \tag{9}$$

Then

$$\Lambda_2 - \Lambda_1 > 4l_F, \tag{10}$$

and the orthogonal decomposition

$$X = X_1 \oplus X_2, \tag{11}$$

holds with continuous orthogonal projections $P_1: X \to X_1$ and $P_2: X \to X_2$.

Lemma 1 [6] Let the eigenvalues $\mu_j^{\pm}, j \ge 1$ be arranged in nondecreasing order, for all $m \in N$, there is $N \ge m$ such that μ_N^- and μ_{N+1}^- are consecutive.

3. A Family of Inertia Manifolds

Equations (1)-(5) are equivalent to the following one-order evolution equation:

$$U_t + AU = F(U), \quad U \in E_k \tag{12}$$

where $U = (u, p, v, q)$, $p = u_t$, $q = v_t$, and

$$A = \begin{pmatrix} 0 & -I & 0 & 0 \\ M(r)(-\Delta)^m & \beta(-\Delta)^m & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & M(r)(-\Delta)^{2m} & \beta(-\Delta)^{2m} \end{pmatrix},$$

$$F(U) = \begin{pmatrix} 0 \\ f_1(x) - g(u_t, v) \\ 0 \\ f_2(x) - g(u, v_t) \end{pmatrix}.$$

We consider the usual graph norm in E_k, as follows

$$\begin{aligned}(U,U)_{E_k} &= \left(M(s)\cdot\nabla^{m+k}u,\nabla^{m+k}\overline{a}\right)+\left(\nabla^{k}p,\nabla^{k}\overline{b}\right)\\ &\quad+\left(M(s)\cdot\nabla^{2m+2k}v,\nabla^{2m+2k}\overline{c}\right)+\left(\nabla^{2k}q,\nabla^{2k}\overline{d}\right),\end{aligned}\tag{13}$$

where $U=(u,p,v,q)^{\mathrm{T}}$, $V=(a,b,c,d)^{\mathrm{T}}$, $s=\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2$, $\overline{a},\overline{b},\overline{c},\overline{d}$ denote the conjugation of a,b,c,d respectively. Evidently, the operator A is monotone, and we obtain

$$\begin{aligned}(AU,U)_{E_k} &= -\left(M(s)\nabla^{m+k}p,\nabla^{m+k}\overline{u}\right)+\left(M(s)\nabla^{m+k}u,\nabla^{m+k}\overline{p}\right)\\ &\quad+\beta\left(\nabla^{m+k}p,\nabla^{m+k}\overline{p}\right)-\left(M(s)\nabla^{2m+2k}q,\nabla^{2m+2k}\overline{v}\right)\\ &\quad+\left(M(s)\nabla^{2m+2k}v,\nabla^{2m+2k}\overline{q}\right)+\beta\left(\nabla^{2m+2k}q,\nabla^{2m+2k}\overline{q}\right)\\ &= \beta\left(\left\|\nabla^{m+k}p\right\|^2+\left\|\nabla^{2m+2k}q\right\|^2\right)\ge 0,\end{aligned}\tag{14}$$

so, $(AU,U)_{E_k}$ is a nonnegative and real number.

In order to determine the eigenvalues of *A*, we consider the eigenvalues equation:

$$AU=\lambda U,\quad U=(u,p,v,q)^{\mathrm{T}}\in E_k,\tag{15}$$

that is

$$\begin{cases}-p=\lambda u,\\ M(s)(-\Delta)^m u+\beta(-\Delta)^m p=\lambda p,\\ -q=\lambda v,\\ M(s)(-\Delta)^{2m} v+\beta(-\Delta)^{2m} q=\lambda q,\end{cases}\tag{16}$$

combined with (16), we obtain

$$\begin{cases}\lambda^2 u+M(s)(-\Delta)^m u-\beta\lambda(-\Delta)^m u=0,\\ \lambda^2 v+M(s)(-\Delta)^{2m} v-\beta\lambda(-\Delta)^{2m} v=0.\end{cases}\tag{17}$$

where $u\big|_{\partial\Omega}=(-\Delta)^m u\big|_{\partial\Omega}=v\big|_{\partial\Omega}=(-\Delta)^{2m} v\big|_{\partial\Omega}=0$.

Taking $(-\Delta)^k u$ and $(-\Delta)^{2k} v$ inner product with the Equations (17), we have

$$\begin{cases}\lambda^2\left\|\nabla^k u\right\|^2+M(s)\left\|\nabla^{m+k}u\right\|^2-\beta\lambda\left\|\nabla^{m+k}u\right\|^2=0,\\ \lambda^2\left\|\nabla^{2k} v\right\|^2+M(s)\left\|\nabla^{2m+2k}v\right\|^2-\beta\lambda\left\|\nabla^{2m+2k}v\right\|^2=0.\end{cases}\tag{18}$$

adding them together,

$$\begin{aligned}&\lambda^2\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^{2k} v\right\|^2\right)+M(s)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)\\ &-\beta\lambda\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)=0,\end{aligned}\tag{19}$$

(19) is regard as a quadratic equation with one unknown about λ, so we get

$$\lambda_j^{\pm}=\frac{\beta\mu_j\pm\sqrt{\beta^2\mu_j^2-4M(s)\cdot\mu_j}}{2},\tag{20}$$

where μ_j is the eigenvalue of $\begin{pmatrix} \Delta^m & 0 \\ 0 & \Delta^{2m} \end{pmatrix}$, and μ_j is non-derogatory, for $\forall j \geq 1$, we have

$$\left\|\nabla^k u_j\right\|^2 + \left\|\nabla^{2k} v_j\right\|^2 = 1,\ \left\|\nabla^{m+k} u_j\right\|^2 + \left\|\nabla^{2m+2k} v_j\right\|^2 = \mu_j,$$

$$\left\|\nabla^{-m-k} u_j\right\|^2 + \left\|\nabla^{-2m-2k} v_j\right\|^2 = \frac{1}{\mu_j}.$$

If $\mu_j \geq \frac{4}{\beta^2} M(s)$, we can get the eigenvalues of A are all positive and real numbers. The corresponding eigenfunction is as follows

$$U_j^{\pm} = \left(u_j, -\lambda_j^{\pm} u_j, v_j, -\lambda_j^{\pm} v_j\right). \tag{21}$$

Lemma 2 $g(u_t, v): V_k \times V_{2m+2k} \to V_k \times V_{2m+2k}$, $g(u, v_t): V_{m+k} \times V_{2k} \to V_{m+k} \times V_{2k}$ is uniformly bounded and globally Lipschitz continuous.

Proof. $\forall (u_t, v), (\bar{u}_t, \bar{v}) \in V_k \times V_{2m+2k} \to V_k \times V_{2m+2k}$, by (H1), we have

$$\begin{aligned}
&\left\|g(\bar{u}_t, \bar{v}) - g(u_t, v)\right\|_{V_k \times V_{2m+2k}} \\
&= \left\|g_{u_t}\left(\bar{u}_t + \theta(\bar{u}_t - u_t), \bar{v} + \theta(\bar{v} - v)\right)(\bar{u}_t - u)\right. \\
&\quad \left. + g_v\left(\bar{u}_t + \theta(\bar{u}_t - u_t), \bar{v} + \theta(\bar{v} - v)\right)(\bar{v} - v)\right\|_{V_k \times V_{2m+2k}} \\
&\leq l\left\|\bar{u}_t - u_t\right\|_{V_k} + l\left\|\bar{v} - v\right\|_{V_{2m+2k}} \\
&\leq l\left(\left\|\bar{p} - p\right\|_{V_k} + \left\|\bar{v} - v\right\|_{V_{2m+2k}}\right);
\end{aligned} \tag{22}$$

Similarly, we have $\left\|g(\bar{u}, \bar{v}_t) - g(u, v_t)\right\|_{V_{m+k} \times V_{2k}} \leq l\left(\left\|\bar{u} - u\right\|_{V_{m+k}} + \left\|\bar{q} - q\right\|_{V_{2k}}\right)$, where $\theta \in (0,1)$, l is Lipschitz coefficient of g.

Theorem 1 When $\mu_j \geq \frac{4}{\beta^2} m_1$, l is Lipschitz constant of g, there is a large enough $N_1 \in N$ so that $N \geq N_1$ has

$$\frac{\beta}{2}\left(\mu_{N+1} - \mu_N\right) - \sqrt{\frac{2\beta}{3}\mu_1 - \frac{3}{\beta} m_1}\, \frac{\mu_{N+1} - \mu_N}{2} \geq \frac{4l}{\sqrt{\frac{2\beta}{3}\mu_1 - \frac{3}{\beta} m_1}} + 1 \tag{23}$$

then operator A satisfies the spectral interval condition of Definition 2.

Proof. when $\mu_k \geq \frac{4}{\beta^2} m_1$, the eigenvalues of A are all positive and real numbers, meanwhile $\{\lambda_k^-\}_{k\geq 1}$ and $\{\lambda_k^+\}_{k\geq 1}$ are increasing order.

Next, we divided the whole process of proof into four steps.

Step 1 By Lemma 1, since $\lambda_k^{\pm}$ is nondecreasing order, so there exists N, such that λ_N^- and λ_{N+1}^- are continuous adjacent values, Then the eigenvalues of A are separate as

$$\sigma_1 = \left\{\lambda_i^-, \lambda_j^+ \,\middle|\, \max\left(\lambda_i^-, \lambda_j^+\right) \leq \lambda_N^-\right\}, \sigma_2 = \left\{\lambda_i^-, \lambda_j^{\pm} \,\middle|\, \lambda_i^- \leq \lambda_N^- \leq \min\left\{\lambda_i^-, \lambda_j^{\pm}\right\}\right\}. \tag{24}$$

Step 2 The corresponding E_k is decomposed into

$$E_{k_1} = Span\left\{U_i^-, U_j^\pm \middle| \lambda_i^-, \lambda_j^\pm \in \sigma_1\right\}, E_{k_2} = Span\left\{U_i^-, U_j^\pm \middle| \lambda_i^-, \lambda_j^\pm \in \sigma_2\right\}, \tag{25}$$

We aim at madding two orthogonal subspaces of E_k and verifying the spectral gap condition (11) is true when $\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$, Therefore, we further decompose $E_{k_2} = E_C \oplus E_R$, where

$$E_C = Span\left\{U_i^- \middle| \lambda_i^- \le \lambda_N^- < \lambda_i^+\right\}, E_R = Span\left\{U_j^\pm \middle| \lambda_N^- < \lambda_j^\pm\right\}, \tag{26}$$

Set $E_N = E_{k_1} \oplus E_C$, in order to verify the E_{k_1} and E_{k_2} are orthogonal, we need to introduce two functions $\Phi : E_N \to R$, $\Psi : E_R \to R$.

$$\begin{aligned}\Phi(U,V) &= \beta\left(\nabla^{m+k}u, \nabla^{m+k}\overline{a}\right) - \frac{3}{\beta}M(s)\left(\nabla^k u, \nabla^k \overline{a}\right) + \left(\nabla^{-m-k}\overline{b}, \nabla^{m+k}u\right) \\ &\quad + \left(\nabla^{-m-k}\overline{p}, \nabla^{m+k}a\right) + \frac{3}{\beta}\left(\nabla^{-m-k}p, \nabla^{-m-k}\overline{b}\right) + \beta\left(\nabla^{2m+2k}v, \nabla^{2m+2k}\overline{c}\right) \\ &\quad - \frac{3}{\beta}M(s)\left(\nabla^{2k}v, \nabla^{2k}\overline{c}\right) + \left(\nabla^{-2m-2k}\overline{d}, \nabla^{2m+2k}v\right) \\ &\quad + \left(\nabla^{-2m-2k}\overline{q}, \nabla^{2m+2k}c\right) + \frac{3}{\beta}\left(\nabla^{-2m-2k}q, \nabla^{-2m-2k}\overline{d}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}\Psi(U,V) &= \beta\left(\nabla^{m+k}u, \nabla^{m+k}\overline{a}\right) - \left(\nabla^k \overline{c}, \nabla^{m+k}u\right) + \left(\nabla^k \overline{p}, \nabla^{m+k}a\right) \\ &\quad + \beta\mu_1\left(\nabla^k p, \nabla^k \overline{c}\right) + \beta\left(\nabla^{2m+2k}v, \nabla^{2m+2k}\overline{b}\right) - \left(\nabla^{-2k}\overline{d}, \nabla^{2m+2k}v\right) \\ &\quad + \left(\nabla^{2k}\overline{q}, \nabla^{2m+2k}b\right) + \beta\mu_1\left(\nabla^{2k}q, \nabla^{2k}\overline{d}\right),\end{aligned} \tag{28}$$

where $U = (u,p,v,q)^{\mathrm{T}}, V = (a,b,c,d)^{\mathrm{T}} \in E_k$ are defined before.

Let $U = (u,p,v,q)^{\mathrm{T}} \in E_N$, by (H2), then

$$\begin{aligned}\Phi(U,U) &= \beta\left(\nabla^{m+k}u, \nabla^{m+k}\overline{u}\right) - \frac{3}{\beta}M(s)\left(\nabla^k u, \nabla^k \overline{u}\right) + \left(\nabla^{-m-k}\overline{p}, \nabla^{m+k}u\right) \\ &\quad + \left(\nabla^{-m-k}\overline{p}, \nabla^{m+k}u\right) + \frac{3}{\beta}\left(\nabla^{-m-k}p, \nabla^{-m-k}\overline{p}\right) + \beta\left(\nabla^{2m+2k}v, \nabla^{2m+2k}\overline{v}\right) \\ &\quad - \frac{3}{\beta}M(s)\left(\nabla^{2k}v, \nabla^{2k}\overline{v}\right) + \left(\nabla^{-2m-2k}\overline{q}, \nabla^{2m+2k}v\right) \\ &\quad + \left(\nabla^{-2m-2k}\overline{q}, \nabla^{2m+2k}v\right) + \frac{3}{\beta}\left(\nabla^{-2m-2k}q, \nabla^{-2m-2k}\overline{q}\right), \\ &\ge \beta\left(\left\|\nabla^{m+k}u\right\|^2 + \left\|\nabla^{2m+2k}v\right\|^2\right) - \frac{3}{\beta}M(s)\left(\left\|\nabla^k u\right\|^2 + \left\|\nabla^{2k}v\right\|^2\right) \\ &\quad - \frac{3}{\beta}\left(\left\|\nabla^{-m-k}p\right\|^2 + \left\|\nabla^{-2m-2k}q\right\|^2\right) - \frac{\beta}{3}\left(\left\|\nabla^{m+k}u\right\|^2 + \left\|\nabla^{2m+2k}v\right\|^2\right) \\ &\quad + \frac{3}{\beta}\left(\left\|\nabla^{-m-k}p\right\|^2 + \left\|\nabla^{-2m-2k}q\right\|^2\right) \\ &\ge \left(\frac{2\beta}{3}\mu_1 - \frac{3}{\beta}m_1\right)\left(\left\|\nabla^k u\right\|^2 + \left\|\nabla^{2k}v\right\|^2\right),\end{aligned} \tag{29}$$

since for $\forall j, m_1 \le \beta^2 \mu_j$, we have $\Phi(U,U) \ge 0$, for $\forall U \in E_N$, then Φ is positive definite.

Similarly, for $U \in E_R$, we have

$$\begin{aligned}\Psi(U,U)&=\beta\left(\nabla^{m+k}u,\nabla^{m+k}\bar{u}\right)-\left(\nabla^{k}\bar{p},\nabla^{m+k}u\right)+\left(\nabla^{k}\bar{p},\nabla^{m+k}u\right)\\&\quad+\beta\mu_1\left(\nabla^{k}p,\nabla^{k}\bar{p}\right)+\beta\left(\nabla^{2m+2k}v,\nabla^{2m+2k}\bar{v}\right)-\left(\nabla^{2k}\bar{q},\nabla^{2m+2k}v\right)\\&\quad+\left(\nabla^{2k}\bar{q},\nabla^{2m+2k}v\right)+\beta\mu_1\left(\nabla^{2k}q,\nabla^{2k}\bar{q}\right)\\&\geq\beta\mu_1\left(\left\|\nabla^{k}u\right\|^2+\left\|\nabla^{k}p\right\|^2+\left\|\nabla^{2k}v\right\|^2+\left\|\nabla^{2k}q\right\|^2\right),\end{aligned}\tag{30}$$

so, for $\forall U\in E_R,\Psi(U,U)\geq 0$, the Ψ is also positive definite.

Next, we need to define a scale product in E_k

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi(P_N U,P_N V)+\Psi(P_R U,P_R V).\tag{31}$$

where P_N and P_R are projection $E_k\to E_N$, $E_k\to E_R$ respectively, for convenience, we rewrite (31) as follows

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi(U,V)+\Psi(U,V).\tag{32}$$

We will proof that two subspaces E_{k_1} and E_{k_2} in (25) are orthogonal; in fact, we only need to show E_N and E_C are orthogonal, that is

$$\langle\langle U_j^-,U_j^+\rangle\rangle_{E_k}=0,\left(U_j^-\in E_N,U_j^+\in E_C\right).\tag{33}$$

by (27), (32), we have

$$\begin{aligned}\langle\langle U_j^-,U_j^+\rangle\rangle_{E_k}&=\Phi\left(U_j^-,U_j^+\right)\\&=\beta\left(\nabla^{m+k}u_j,\nabla^{m+k}\bar{u}_j\right)-\frac{3}{\beta}M(s)\left(\nabla^{k}u_j,\nabla^{k}\bar{u}_j\right)\\&\quad-\lambda_j^+\left(\nabla^{-m-k}\bar{u}_j,\nabla^{m+k}u_j\right)-\lambda_j^-\left(\nabla^{-m-k}\bar{u}_j,\nabla^{m+k}u_j\right)\\&\quad+\frac{3}{\beta}\lambda_j^-\lambda_j^+\left(\nabla^{-m-k}u_j,\nabla^{-m-k}\bar{u}_j\right)+\beta\left(\nabla^{2m+2k}v_j,\nabla^{2m+2k}\bar{v}_j\right)\\&\quad-\frac{3}{\beta}M(s)\left(\nabla^{2k}v_j,\nabla^{2k}\bar{v}_j\right)-\lambda_j^+\left(\nabla^{-2m-2k}\bar{v}_j,\nabla^{2m+2k}v_j\right)\\&\quad-\lambda_j^-\left(\nabla^{-2m-2k}\bar{v}_j,\nabla^{2m+2k}v_j\right)+\frac{3}{\beta}\lambda_j^-\lambda_j^+\left(\nabla^{-2m-2k}v_j,\nabla^{-2m-2k}\bar{v}_j\right),\\&=\beta\left(\left\|\nabla^{m+k}u_j\right\|^2+\left\|\nabla^{2m+2k}v_j\right\|^2\right)-\frac{3}{\beta}M(s)\left(\left\|\nabla^{k}u_j\right\|^2+\left\|\nabla^{2k}v_j\right\|^2\right)\\&\quad-\left(\lambda_j^-+\lambda_j^+\right)\left(\left\|u_j\right\|^2+\left\|v_j\right\|^2\right)+\frac{3}{\beta}\lambda_j^-\lambda_j^+\left(\left\|\nabla^{-m-k}u_j\right\|^2+\left\|\nabla^{-2m-2k}v_j\right\|^2\right)\\&=\beta\mu_j-\frac{3}{\beta}M(s)-\left(\lambda_j^-+\lambda_j^+\right)+\frac{3}{\beta}\lambda_j^-\lambda_j^+\cdot\frac{1}{\mu_j}.\end{aligned}\tag{34}$$

Through Equation (19), we can get $\lambda_j^++\lambda_j^-=\beta\mu_j$, $\lambda_j^+\lambda_j^-=M\mu_j$, therefore

$$\langle\langle U_j^-,U_j^+\rangle\rangle_{E_k}=0.\tag{35}$$

Step 3 Further, we estimate the Lipschitz constant l_F of F

$$F(U)=\begin{pmatrix}0\\f_1(x)-g(u_t,v)\\0\\f_2(x)-g(u,v_t)\end{pmatrix},\tag{36}$$

from (27), (28), for $\forall U=(u,p,v,q)^{\mathrm{T}}\in E_k$, we have

$$\begin{aligned}\|U\|_{E_k}^2&=\Phi(P_1U,P_1U)+\Psi(P_2U,P_2U)\\&\geq\left(\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1\right)\left(\left\|\nabla^kP_1u\right\|^2+\left\|\nabla^{2k}P_1v\right\|^2\right)\\&\quad+\beta\mu_1\left(\left\|\nabla^kP_2u\right\|^2+\left\|\nabla^kP_2p\right\|^2+\left\|\nabla^{2k}P_2v\right\|^2+\left\|\nabla^{2k}P_2q\right\|^2\right)\\&\geq\left(\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1\right)\left(\left\|\nabla^ku\right\|^2+\left\|\nabla^kp\right\|^2+\left\|\nabla^{2k}v\right\|^2+\left\|\nabla^{2k}q\right\|^2\right).\end{aligned}\tag{37}$$

By lemma 1, where $U=(u,p,v,q)^{\mathrm{T}}$, $V=(\bar{u},\bar{p},\bar{v},\bar{q})^{\mathrm{T}}\in E_k$, we can get

$$\begin{aligned}&\|F(U)-F(V)\|_{E_k}\\&=\|g(\bar{u}_t,\bar{v})-g(u_t,v)\|_{V_k\times V_{2m+2k}}+\|g(\bar{u},\bar{v}_t)-g(u,v_t)\|_{V_{m+k}\times V_{2k}}\\&\leq l\left(\|\bar{p}-p\|_{V_k}+\|\bar{v}-v\|_{V_{2m+2k}}\right)+l\left(\|\bar{u}-u\|_{V_{m+k}}+\|\bar{q}-q\|_{V_{2k}}\right)\\&\leq\frac{l}{\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}}\|U-V\|_{E_k},\end{aligned}\tag{38}$$

so, we obtain

$$l_F\leq\frac{l}{\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}}.\tag{39}$$

Step 4 Now, we will show the spectral gap condition (10) holds.

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^{-}-\lambda_N^{-}=\frac{\beta}{2}(\mu_{N+1}-\mu_N)+\frac{1}{2}\left(\sqrt{R(N)}-\sqrt{R(N+1)}\right),\tag{40}$$

where $R(N)=\beta^2\mu_N^2-4M(s)\mu_N$.

Let

$$\lim_{N\to\infty}\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}(\mu_{N+1}-\mu_N)=0.\tag{41}$$

letting

$$R_1(N)=\sqrt{\frac{\beta^2\mu_N-4M(s)}{\left(\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1\right)^2\mu_N}},\tag{42}$$

we can compute

$$\begin{aligned}&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}(\mu_{N+1}-\mu_N)\\&=\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}\left(\mu_{N+1}\left(1-R_1(N+1)\right)-\mu_N\left(1-R_1(N)\right)\right),\end{aligned}\tag{43}$$

$$\lim_{N\to\infty}\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}\left(\mu_{N+1}\left(1-R_1(N+1)\right)-\mu_N\left(1-R_1(N)\right)\right)=0.\tag{44}$$

then, we can get

$$\begin{aligned}\Lambda_2-\Lambda_1 &\ge \frac{\beta}{2}\left(\mu_{N+1}-\mu_N\right)-\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}\,\frac{\mu_{N+1}-\mu_N}{2}-1 \\ &\ge \frac{4l}{\sqrt{\frac{2\beta}{3}\mu_1-\frac{3}{\beta}m_1}}\ge 4l_F.\end{aligned} \tag{45}$$

Theorem 1 is proved.

Theorem 2 Under the condition of Theorem 1, the problem (1)-(5) exist an inertial manifold μ_k in E_k,

$$\mu_k = graph(\Phi)=\left\{\xi_k+\Phi(\xi_k)\middle|\xi_k\in E_{k_1}\right\}, \tag{46}$$

where $\Phi: E_{k_1}\to E_{k_2}$ is a Lipschitz continuous function.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chen, L., Wang, W. and Lin, G.G. (2016) Exponential Attractors and Inertial Manifolds for the Higher-Order Nonlinear Kirchhof-Type Equation. *International Journal of Modern Communication Technologies and Research*, **4**, 6-12.

[2] Lin, G.G. and Zhang, M. (2018) The Estimates of the Upper Bounds of Hausdorff Dimensions for the Global Attractor for a Class of Nonlinear Coupled Kirchhoff-Type Equations. *Advances in Pure Mathematics*, **8**, 1-10. https://doi.org/10.4236/apm.2018.81001

[3] Lin, G.G. and Yang, L.J. (2021) A Family of the Exponential Attractors and the Inertial Manifolds for a Class of Generalized Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **9**, 2399-2413. https://doi.org/10.4236/jamp.2021.910152

[4] Constantin, P., Foias, C., Nicolaenko, B., *et al.* (1989) Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations. Springer-Verlag, New York.

[5] Fabes, E., Luskin, M. and Sell, G.R. (1991) Construction of Inertial Manifolds by Elliptic Regularization. *Journal of Differential Equations*, **89**, 355-387. https://doi.org/10.1016/0022-0396(91)90125-S

[6] Lin, G.G. (2011) Nonlinear Evolution Equations. Yunnan University Press, Kunming.

[7] Lin, G.G. and Liu, X.M. (2022) The Family of Exponential Attractors and Inertial Manifolds for a Generalized Nonlinear Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **10**, 172-189. https://doi.org/10.4236/jamp.2022.101013

[8] Lin, G.G. and Li, S.Y. (2020) Inertial Manifolds for Generalized Higher-Order Kirchhoff Type Equations. *Journal of Mathematics Research*, **12**, 67-72. https://doi.org/10.5539/jmr.v12n5p67

[9] Lin, G.G. and Xia, X.S. (2018) The Inertial Manifold for Class Kirchhoff-Type Equations with Strongly Damped Terms and Source Terms. *Applied Mathematics*, **9**, 730-737. https://doi.org/10.4236/am.2018.96050

[10] Lin, G.G. and Yang, S.M. (2018) The Inertial Manifolds for a Class of Higher-Order Coupled Kirchhoff-Type Equations. *Journal of Applied Mathematics and Physics*, **6**,

1055-1064. https://doi.org/10.4236/jamp.2018.65091

[11] Ai, C.F, Zhu, H.X. and Lin, G.G. (2016) Approximate Inertial Manifold for a Class of the Kirchhoff Wave Equations with Nonlinear Strongly Damped Terms. *International Journal of Modern Nonlinear Theory and Application*, **5**, 218-234. https://doi.org/10.4236/ijmnta.2016.54020

[12] Yuan, Z.Q., Guo, L. and Lin, G.G. (2015) Inertial Manifolds for 2D Generalized MHD System. *International Journal of Modern Nonlinear Theory and Application*, 4, 190-203. https://doi.org/10.4236/ijmnta.2015.43014

[13] Lee, J. and Nguyen, N. (2021) Gromov-Hausdorff Stability of Inertial Manifolds under Perturbations of the Domain and Equation. *Journal of Mathematical Analysis and Applications*, **494**, Article ID: 124623. https://doi.org/10.1016/j.jmaa.2020.124623

[14] Roussel, M.R. (2020) Perturbative-Iterative Computation of Inertial Manifolds of Systems of Delay-Differential Equations with Small Delays. *Algorithms*, **13**, Article 209. https://doi.org/10.3390/a13090209

[15] Vu, T.N.H., Nguyen, T.H. and Le, A.M. (2021) Admissible Inertial Manifolds for Neutral Equations and Applications. *Dynamical Systems*, **36**, 608-630. https://doi.org/10.1080/14689367.2021.1971623

[16] Le, A.M. (2020) Inertial Manifolds for Neutral Functional Differential Equations with Infinite Delay and Applications. *Annales Polonici Mathematici*, **125**, 255-271. https://doi.org/10.4064/ap191219-29-5

[17] Nguyen, T.H. and Le, A.M. (2018) Admissible Inertial Manifolds for Delay Equations and Applications to Fisher-Kolmogorov Model. *Acta Applicandae Mathematicae*, **156**, 15-31. https://doi.org/10.1007/s10440-017-0153-y