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A Family of the Inertial Manifolds for a Class of Generalized Kirchhoff-Type Coupled Equations

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Abstract

The paper considers the long-time behavior for a class of generalized high-order Kirchhoff-type coupled equations, under the corresponding hypothetical conditions, according to the Hadamard graph transformation method, obtain the equivalent norm in space $E_k\left(k=1,2,\cdots,2m\right)$, and we obtain the existence of a family of the inertial manifolds while such equations satisfy the spectral interval condition.

Keywords

Kirchhoff-Type Coupled Equations, Spectral Interval Condition, A Family of the Inertial Manifolds

1. Introduction

This paper investigates the following primal value problems of a system of generalized Kirchhoff-type coupled equations:

$$\begin{cases} u_{tt}+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g_1(u,v)=f_1(x), & (1)\\ v_{tt}+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}v_t+g_2(u,v)=f_2(x), & (2)\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, & (3)\\ v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, & (4)\\ \dfrac{\partial^i u}{\partial n^i}=0,\dfrac{\partial^i v}{\partial n^i}=0\left(i=0,1,2,\cdots,2m\right). & (5)\end{cases}$$

where Ω is a bounded region with a smooth boundary in R^n, $\partial\Omega$ represents the boundary of Ω, $u_0(x),u_1(x)$ and $v_0(x),v_1(x)$ are known functions, where $g_j(u,v)$, $f_j(u,v)\left(j=1,2\right)$ are nonlinear terms and external interference terms, respectively, and are known functions on $\Omega\times(0,T)$, β is the normal number, $M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)$ is a non-negative first-order continuous derivative function, and $m>1$ is the normal number, $\left\|D^m u\right\|_p^p=\int_\Omega\left|D^m u\right|^p \mathrm{d}x$.

In order to overcome the research difficulties, G. Foias, G. R. Sell and R. Temam [1] proposed the concept of inertial manifolds, which greatly promoted the study of infinite-dimensional dynamical systems. Where the inertial manifold is a positive, finite-dimensional Lipschitz manifold, and the existence of an inertial manifold depends on the establishment of a spectral interval condition. Therefore, the research on a family of inertial manifolds is of great significance from both theoretical and practical aspects, and the relevant theoretical achievements can be referred to [2]-[9].

Guoguang Lin, Lingjuan Hu [10] studied a system of coupled wave equations of higher-order Kirchhoff type with strong damping terms

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \\ v_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \\ v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, \\ \dfrac{\partial^i u}{\partial n^i}=0, \dfrac{\partial^i v}{\partial n^i}=0 \quad (i=0,1,2,\cdots,2m-1) \ \ x\in\partial\Omega. \end{cases}$$

where Ω is a bounded region with a smooth boundary in R^n, $\partial\Omega$ represents the boundary of Ω, $g_j(u,v)(j=1,2)$ is a nonlinear source term, $f_1(x), f_2(x)$ is an external force interference term, and $\beta(-\Delta)^m u$, $\beta(-\Delta)^m v$ $(\beta\geq 0)$ is a strong dissipation terms. Using the Hadamard graph transformation method, the Lipschitz constant l_F of F is further estimated, and the inertial manifolds that satisfies the spectral interval condition is obtained.

Lin Guoguang, Liu Xiaomei [11] studied a family of inertial manifolds for a class of generalized higher-order Kirchhoff equations with strong dissipation terms

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u+\beta(-\Delta)^{2m} u_t+|u|^\rho(u_t+u)=f(x), \\ u(x,t)=0, \dfrac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,2m-1, \ x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega\subset R^n. \end{cases}$$

where $m\in N^+$, $\Omega\subset R^n(n\geq 1)$ is a bounded domain with a smooth boundary in $\partial\Omega$, $f(x)$ is an external force term, $M\left(\left\|\nabla^m u\right\|_p^p\right)$ is the stress term of Kirchhoff equation, $\beta(-\Delta)^{2m} u_t$ is a strong dissipative term, $|u|^\rho(u_t+u)$ is a nonlinear source term. Based on appropriate assumptions and the Hadamard graph transformation method, the spectral interval condition is verified, and the existence of a family of the inertial manifolds of the equation is obtained.

On the basis of previous research, rigid term strengthening becomes

$M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)(-\Delta)^{2m} u$ and $M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)(-\Delta)^{2m} v$, and this paper seeks a family of inertial manifolds. When defining the equivalence norm in space E_k, by making reasonable assumptions, it is obtained that the equation satisfies the spectral interval condition so that there is a family of inertial manifolds.

2. Preliminaries

For narrative convenience, we introduce the following symbols and assumptions:

Set $\nabla = D$. Consider Hilbert space family $V_\alpha = D\left((-\Delta)^{\alpha/2}\right), \alpha \in R$, whose inner product and norm are $(\bullet,\bullet)_{V_\alpha} = \left((-\Delta)^{\alpha/2}, (-\Delta)^{\alpha/2}\right)$ and $\|\bullet\|_{V_\alpha} = \left\|(-\Delta)^{\alpha/2}\right\|$, respectively. Apparently there are

$$V_0 = L^2(\Omega), V_{2m} = H^{2m}(\Omega) \cap H_0^1(\Omega), V_{2m+k} = H^{2m+k}(\Omega) \cap H_0^1(\Omega),$$
$$V_k = H^k(\Omega) \cap H_0^1(\Omega), E_0 = V_{2m} \times V_0 \times V_{2m} \times V_0,$$
$$E_k = V_{2m+k} \times V_k \times V_{2m+k} \times V_k, (k = 1, 2, \cdots, 2m).$$

The assumption is as follows:

Let $M(s)$ be a continuous function on interval $D_1 (D_1 \in \Omega)$, and $M(s) \in C^1(R^+)$:

$$1 \le \mu_0 \le M(s) \le \mu_1, \text{ set } M(s) = M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right).$$

3. A Family of Inertial Manifolds

Definition 1 [12] lets $S = \{S(t)\}_{t \ge 0}$ be the solution semigroup on Banach space $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega) (k = 1, 2, \cdots, 2m)$, and a subset $\mu_k \subset E_k$ satisfies:

1) μ_k is finite-dimensional Lipschitz popular;

2) μ_k is positively unchanging, $\{S(t)\}_{t \ge 0} : \forall u_0 \in \mu_k, S(t) u_0 \subset \mu_k, t \ge 0$;

3) μ_k attracts the solution orbit exponentially, *i.e.* for any $u \in E_k$, the existence constant $\eta > 0, c > 0$ makes $dist(S(t)u, \mu) \le ce^{-\eta t}$, $t \ge 0$.

Then μ_k is called E_k is a family of inertial manifolds.

In order to describe the spectral interval condition, first consider that the nonlinear term $F : E_k \to E_k$ is integrally bounded and continuous, and has a positive Lipschitz constant l_F, and its operator A has several eigenvalues and eigenfunctions of the positive real part.

Definition 2 [12] Set operator $A : X \to X$ has several eigenvalues of positive real numbers, and $F \in C_b(X, X)$ satisfies the Lipschitz condition:

$$\|F(u) - F(v)\|_X \le l_F \|u - v\|_X, \quad u, v \in X,$$

The point spectrum of the operator A can be divided into two parts σ_1 and σ_2, and σ_1 is finite,

$$\Lambda_1 = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma_1\},$$
$$\Lambda_2 = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma_2\},$$
$$\mathrm{X}_i = span\{\omega_j \mid \lambda_j \in \sigma_i\}, i = 1,2.$$

and conditions

$$\Lambda_2 - \Lambda_1 > 4l_F \tag{6}$$

are satisfied.

Where the continuous projection $P_1 : \mathrm{X} \to \mathrm{X}_1, P_2 : \mathrm{X} \to \mathrm{X}_2$, there is orthogonal decomposition $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$, then the operator A satisfies the spectral interval condition.

Lemma 1 $g_i : V_{2m+k} \times V_{2m+k} \to V_{2m+k} \times V_{2m+k}\ (i = 1,2)$ is a uniform bounded and integral Lipschitz continuous function.

Proof: $\forall(\tilde{u},\tilde{v}),(u,v) \in V_{2m+k} \times V_{2m+k}\ (k = 1,2,\cdots,2m)$,

$$\begin{aligned}
&\left\| g_1(\tilde{u},\tilde{v}) - g_1(u,v) \right\|_{V_{2m+k}\times V_{2m+k}} \\
&= \left\| g_{1u}\left(u+\theta(\tilde{u}-u), v+\theta(\tilde{v}-v)\right)(\tilde{u}-u) \right. \\
&\quad \left. + g_{1v}\left(u+\theta(\tilde{u}-u), v+\theta(\tilde{v}-v)\right)(\tilde{v}-v) \right\|_{V_{2m+k}\times V_{2m+k}} \\
&\le \left\| g_{1u}\left(u+\theta(\tilde{u}-u), v+\theta(\tilde{v}-v)\right)(\tilde{u}-u) \right\|_{V_{2m+k}\times V_{2m+k}} \\
&\quad + \left\| g_{1v}\left(u+\theta(\tilde{u}-u), v+\theta(\tilde{v}-v)\right)(\tilde{v}-v) \right\|_{V_{2m+k}\times V_{2m+k}} \\
&\le l\left(\|\tilde{u}-u\|_{V_{2m+k}} + \|\tilde{v}-v\|_{V_{2m+k}} \right).
\end{aligned}$$

Similarly, there are

$$\left\| g_2(\tilde{u},\tilde{v}) - g_2(u,v) \right\|_{V_{2m+k}\times V_{2m+k}} \le l\left(\|\tilde{u}-u\|_{V_{2m+k}} + \|\tilde{v}-v\|_{V_{2m+k}} \right)$$

where l is the Lipschitz constant of g_i, $\theta \in (0,1)$.

Lemma 2 [12] lets the sequence of eigenvalues $\{\mu_j^-\}_{j\ge1}$ is a non-subtractive sequence, then $\exists N_0 \in N_+$, for $\forall N \ge N_0$, μ_N^- and μ_{N+1}^- are consecutive adjacent values.

In order to verify that the operator satisfies the spectral interval condition, so as to draw the conclusion that there is a family of inertial manifolds in questions (1)-(5), the following definitions and assumptions can be made first.

Based on the above relevant conditions, consider the first-order development equation equivalent to Equations (1)-(5), as follows:

$$U_t + A'U = F(U) \tag{7}$$

Of which $U = (u,z,v,q)$,

$$A' = \begin{pmatrix} 0 & -I & 0 & 0 \\ M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m} & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m} \end{pmatrix},$$

$$F(U)=\begin{pmatrix}0\\ f_1(x)-g_1(u,v)\\ 0\\ f_2(x)-g_2(u,v)\end{pmatrix},$$

$$D(A')=\left\{(u,v)\in V_{2m+k}\times V_{2m+k}\mid (u,v)\in V_0\times V_0,\left(D^{2m+k}u,D^{2m+k}v\right)\in V_0\times V_0\right\}$$
$$\times V_k\times V_k,$$

$$s=\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p.$$

In order to determine the eigenvalue of matrix operator A', first consider graph module
$$(U,V)_{E_k}=\left(M(s)D^{2m+k}u,D^{2m+k}\overline{u'}\right)+\left(D^k\overline{z'},D^k z)\right)+\left(M(s)D^{2m+k}v,D^{2m+k}\overline{v'}\right)+\left(D^k\overline{q'},D^k q\right)$$
generated by inner product in E_k.

Where $U=(u,z,v,q),V=(u',z',v',q')$, and $\overline{u'},\overline{z'},\overline{v'},\overline{q'}$ represent the conjugation of u',z',v',q' respectively. In addition, operator A' is monotonic, and for $U\in D(A')$, there is

$$\begin{aligned}(A'U,U)_{E_k}&=-\left(M(s)D^{2m+k}z,D^{2m+k}\overline{u}\right)+\left(D^k\overline{z},M(s)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}z\right)\\&\quad-\left(M(s)D^{2m+k}q,D^{2m+k}\overline{v}\right)+\left(\overline{q},M(s)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}q\right)\\&=-\left(M(s)D^{2m+k}z,D^{2m+k}\overline{u}\right)+\left(D^{2m+k}\overline{z},M(s)D^{2m+k}u\right)\\&\quad+\left(D^{2m+k}\overline{z},\beta D^{2m+k}z\right)-\left(M(s)D^{2m+k}q,D^{2m+k}\overline{v}\right)\\&\quad+\left(D^{2m+k}\overline{q},M(s)D^{2m+k}v\right)+\left(D^{2m+k}\overline{q},\beta D^{2m+k}q\right)\\&=\beta\left(\left\|D^{2m+k}z\right\|^2+\left\|D^{2m+k}q\right\|^2\right)\geq 0,\end{aligned}$$

Therefore, $(A'U,U)_{E_k}$ is a nonnegative real number.

In order to further determine the eigenvalue of the matrix operator A', the following characteristic equation can be considered,

$$A'U=\lambda U,U=(u,z,v,q)\in E_k, \tag{8}$$

That is

$$\begin{cases}-z=\lambda u,\\ M(s)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}z=\lambda z,\\ -q=\lambda v,\\ M(s)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}q=\lambda q.\end{cases}$$

Thus u,v meet the eigenvalue problem

$$\begin{cases}\lambda^2u-\lambda\beta(-\Delta)^{2m}u+M(s)(-\Delta)^{2m}u=0,\\ \lambda^2v-\lambda\beta(-\Delta)^{2m}v+M(s)(-\Delta)^{2m}v=0,\\ \left.\dfrac{\partial^i u}{\partial n^i}\right|_{\partial\Omega}=\left.\dfrac{\partial^i v}{\partial n^i}\right|_{\partial\Omega}=0,i=0,1,2,\cdots,2m-1,\end{cases}$$

Take the inner product of $(-\Delta)^k u,(-\Delta)^k v$ and Equations (1) and (2) above respectively, with

$$\begin{cases}\lambda^2\left\|D^k u\right\|^2-\lambda\beta\left\|D^{2m+k}u\right\|^2+M(s)\left\|D^{2m+k}u\right\|^2=0,\\ \lambda^2\left\|D^k v\right\|^2-\lambda\beta\left\|D^{2m+k}v\right\|^2+M(s)\left\|D^{2m+k}v\right\|^2=0,\end{cases}$$

That is

$$\begin{aligned}&\lambda^2\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)-\lambda\beta\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)\\&+M(s)\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)=0.\end{aligned} \tag{9}$$

Equation (9) is a univariate quadratic equation about λ. Replace u,v with u_j,v_j. For each positive integer *j*, Equation (8) has paired eigenvalues

$$\lambda_j^{\pm}=\frac{\beta\mu_j\pm\sqrt{(\beta\mu_j)^2-4M(s)\mu_j}}{2},$$

where μ_j is the characteristic root of $(-\Delta)^{2m}$ in $V_{2m}\times V_{2m}$, then $\mu_j=\lambda_1 j^{\frac{2m}{n}}$.

If $(\beta\mu_j)^2\ge 4M(s)\mu_j$, then $\mu_j\ge\frac{4M(s)}{\beta^2}$, the eigenvalues of operator A' are all real numbers, and the corresponding eigenfunction form is

$$U_j^{\pm}=\left(u_j,-\lambda_j^{\pm}u_j,v_j,-\lambda_j^{\pm}v_j\right)$$

For convenience, mark for any $j\ge 1$, there are

$$\begin{aligned}&\left\|D^{2m+k}u_j\right\|^2+\left\|D^{2m+k}v_j\right\|^2=\mu_j,\ \left\|D^k u_j\right\|^2+\left\|D^k v_j\right\|^2=1,\\&\left\|D^{-2m-k}u_j\right\|^2+\left\|D^{-2m-k}v_j\right\|^2=\frac{1}{\mu_j}.\end{aligned}$$

Theorem 1: Assumes that *l* is the Lipschitz constant of $g_i(u,v)(i=1,2)$. When $N_0\in N_+$ is sufficiently large, for $\forall N\ge N_0$, the following inequality holds

$$(\mu_{N+1}-\mu_N)\left(\beta-\sqrt{\beta^2\mu_1-4M(s)}\right)\ge\frac{32l}{\sqrt{\beta^2\mu_1-4M(s)}}+1. \tag{10}$$

Then all operators A' satisfy the spectral interval condition (6).

Proof. Because $\mu_j\ge\frac{4M(s)}{\beta^2}$ and the eigenvalues of A' are positive real numbers, $\{\lambda_j^-\}_{j\ge1}$ and $\{\lambda_j^+\}_{j\ge1}$ are single increment sequences.

The following four steps are taken to prove theorem 1:

Step 1: Because $\{\lambda_j^-\}_{j\ge1}$ and $\{\lambda_j^+\}_{j\ge1}$ are non subtractive columns, according to lemma 2, there are $\exists N_0\in N_+$, for $\forall N\ge N_0$, λ_N^- and λ_{N+1}^- are continuous adjacent values.

Therefore, there is *N*, so that λ_N^- and λ_{N+1}^- are continuous adjacent values, and the eigenvalue of A' can be decomposed into

$$\sigma_1=\left\{\lambda_r^- \mid 1\le r\le N\right\},\sigma_2=\left\{\lambda_r^+,\lambda_j^\pm \mid 1\le r\le N\le j\right\}$$

Step 2: Consider the corresponding decomposition of E_k into

$$E_{k_1}=span\left\{U_r^-\middle|\lambda_r^-\in\sigma_1\right\},$$
$$E_{k_2}=span\left\{U_r^+,U_j^\pm\middle|\lambda_r^+,\lambda_j^\pm\in\sigma_2\right\}$$

The equivalent inner product $\left(\left(U,V\right)\right)_{E_k}$ given below makes E_{k_1},E_{k_2} orthogonal.

Further decompose $E_{k_2}=E_H\oplus E_R$, of which

$$E_H=span\left\{U_r^+ \mid 1\le r\le N\right\},E_R=span\left\{U_j^\pm\middle| j\ge N\right\}$$

Because E_{k_1} and E_H are finite dimensional subspaces, $U_N^-\in E_{k_1}$, $U_{N+1}^-\in E_R$, and E_{k_1} and E_R are orthogonal, while E_{k_1} and E_H are not orthogonal, E_{k_1} and E_{k_2} are not orthogonal. So we need to redefine the equivalent norm on E_k, so that E_{k_1} and E_H are orthogonal. Order $E_N=E_{k_1}\oplus E_H$.

Construct two functions $\Phi:E_N\to R,\Psi:E_R\to R$ of which,

$$\begin{aligned}\Phi\left(U,V\right)=&2\beta\left(\beta-1\right)\left(D^{2m+k}u,D^{-(2m+k)}\overline{u'}\right)+2\beta\left(D^{-(2m+k)}\overline{z'},D^{2m+k}u\right)\\&+2\beta\left(D^{-(2m+k)}\overline{z},D^{2m+k}u'\right)+4\left(D^{-(2m+k)}\overline{z'},D^{-(2m+k)}z\right)\\&-4M\left(s\right)\left(D^ku,D^k\overline{u'}\right)+2\beta\left(D^{2m+k}\overline{u},D^{2m+k}u'\right)\\&+2\beta\left(\beta-1\right)\left(D^{2m+k}v,D^{2m+k}\overline{v'}\right)+2\beta\left(D^{-(2m+k)}\overline{q'},D^{2m+k}v\right)\\&+2\beta\left(D^{-(2m+k)}\overline{q},D^{2m+k}v'\right)+4\left(D^{-(2m+k)}\overline{q'},D^{-(2m+k)}q\right)\\&-4M\left(s\right)\left(D^kv,D^k\overline{v'}\right)+2\beta\left(D^{2m+k}\overline{v},D^{2m+k}v'\right),\end{aligned}$$

$$\begin{aligned}\Psi\left(U,V\right)=&2\beta\left(D^{2m+k}\overline{u},D^{2m+k}u'\right)+\beta\left(D^{-(2m+k)}\overline{z'},D^{2m+k}u\right)\\&+\beta\left(D^{-(2m+k)}\overline{z},D^{2m+k}u'\right)+4\left(D^{-(2m+k)}\overline{z'},D^{-(2m+k)}z\right)\\&-2M\left(s\right)\left(D^ku,D^k\overline{u'}\right)+2\beta\left(\beta-1\right)\left(D^{2m+k}u,D^{-(2m+k)}\overline{u'}\right)\\&+2\beta\left(D^{2m+k}\overline{v},D^{2m+k}v'\right)+\beta\left(D^{-(2m+k)}\overline{q'},D^{2m+k}v\right)\\&+\beta\left(D^{-(2m+k)}\overline{q},D^{2m+k}v'\right)+4\left(D^{-(2m+k)}\overline{q'},D^{-(2m+k)}q\right)\\&-2M\left(s\right)\left(D^kv,D^k\overline{v'}\right)+2\beta\left(\beta-1\right)\left(D^{2m+k}v,D^{2m+k}\overline{v'}\right).\end{aligned}$$

Among them $U=\left(u,z,v,q\right),V=\left(u',z',v',q'\right)\in E_N$ or E_R.

For $U=\left(u,z,v,q\right)\in E_N$, then

$$\begin{aligned}\Phi\left(U,U\right)=&2\beta\left(\beta-1\right)\left(D^{2m+k}u,D^{-(2m+k)}\overline{u}\right)+2\beta\left(D^{-(2m+k)}\overline{z},D^{2m+k}u\right)\\&+2\beta\left(D^{-(2m+k)}\overline{z},D^{2m+k}u\right)+4\left(D^{-(2m+k)}\overline{z},D^{-(2m+k)}z\right)\\&-4M\left(s\right)\left(D^ku,D^k\overline{u}\right)+2\beta\left(D^{2m+k}\overline{u},D^{2m+k}u\right)\\&+2\beta\left(\beta-1\right)\left(D^{2m+k}v,D^{2m+k}\overline{v}\right)+2\beta\left(D^{-(2m+k)}\overline{q},D^{2m+k}v\right)\\&+2\beta\left(D^{-(2m+k)}\overline{q},D^{2m+k}v\right)+4\left(D^{-(2m+k)}\overline{q},D^{-(2m+k)}q\right)\end{aligned}$$

$$
\begin{aligned}
&-4M(s)\left(D^k v, D^k \overline{v}\right)+2\beta\left(D^{2m+k}\overline{v}, D^{2m+k}v\right)\\
&\geq 2\beta(\beta-1)\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-4\beta\left(\left\|D^{-(2m+k)}z\right\|\left\|D^{2m+k}u\right\|\right.\\
&\quad\left.+\left\|D^{-(2m+k)}q\right\|\left\|D^{-(2m+k)}v\right\|\right)+4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-4M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)+2\beta\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)\\
&\geq 2\beta(\beta-1)\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-\beta^2\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)+4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-4M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)+2\beta\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)\\
&=\beta^2\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-4M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)\\
&\geq\left(\beta^2\mu_1-4M(s)\right)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right).
\end{aligned}
$$

For any *k*, there is $\beta^2\mu_k \geq 4M(\mu_k)$. According to hypothesis $1\leq\mu_0\leq M(s)\leq\mu_1\leq\frac{\beta^2\mu_k}{4}$, then $\Phi(U,U)\geq 0$, that is, Φ is positive definite.

Similarly, for any $U=(u,z,v,q)\in E_R$, there is

$$
\begin{aligned}
\Psi(U,U)&=2\beta\left(D^{2m+k}\overline{u}, D^{2m+k}u\right)+\beta\left(D^{-(2m+k)}\overline{z}, D^{2m+k}u\right)\\
&\quad+\beta\left(D^{-(2m+k)}\overline{z}, D^{2m+k}u\right)+4\left(D^{-(2m+k)}\overline{z}, D^{-(2m+k)}z\right)\\
&\quad-2M(s)\left(D^k u, D^k\overline{u}\right)+2\beta(\beta-1)\left(D^{2m+k}u, D^{-(2m+k)}\overline{u}\right)\\
&\quad+2\beta\left(D^{2m+k}\overline{v}, D^{2m+k}v\right)+\beta\left(D^{-(2m+k)}\overline{q}, D^{2m+k}v\right)\\
&\quad+\beta\left(D^{-(2m+k)}\overline{q}, D^{2m+k}v\right)+4\left(D^{-(2m+k)}\overline{q}, D^{-(2m+k)}q\right)\\
&\quad-2M(s)\left(D^k v, D^k\overline{v}\right)+2\beta(\beta-1)\left(D^{2m+k}v, D^{2m+k}\overline{v}\right).\\
&\geq 2\beta\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-2\beta\left(\left\|D^{-(2m+k)}z\right\|\left\|D^{2m+k}u\right\|\right.\\
&\quad\left.+\left\|D^{-(2m+k)}q\right\|\left\|D^{2m+k}v\right\|\right)+4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-2M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)+2\beta(\beta-1)\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)\\
&\geq 2\beta\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-\beta^2\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)+4\left(\left\|D^{-(2m+k)}z\right\|^2+\left\|D^{-(2m+k)}q\right\|^2\right)\\
&\quad-2M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)+2\beta(\beta-1)\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)\\
&=\beta^2\left(\left\|D^{2m+k}u\right\|^2+\left\|D^{2m+k}v\right\|^2\right)-2M(s)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right)\\
&\geq\left(\beta^2\mu_1-2M(s)\right)\left(\left\|D^k u\right\|^2+\left\|D^k v\right\|^2\right).
\end{aligned}
$$

So there are $\forall U=(u,z,v,q)\in E_R$, $\Psi(U,U)\ge 0$, then Ψ is also positive definite.

Redefine the inner product of E_k:

$$\left((U,V)\right)_{E_k}=\Phi(P_N U,P_N V)+\Psi(P_R U,P_R V) \tag{11}$$

where P_N and P_R are projections of $E_k\to E_N$ and $E_k\to E_R$, respectively Here, Equation (11) is written as

$$\left((U,V)\right)_{E_k}=\Phi(U,V)+\Psi(U,V)$$

Under the redefined inner product of E_k, to prove that E_{k_1} and E_{k_2} are orthogonal, we only need to prove that E_{k_1} and E_H are orthogonal, that is,

$$\left((U_j^-,U_j^+)\right)_{E_k}=\Phi(U_j^-,U_j^+)=0.$$

Because there are $U_j^-\in E_{k_1},U_j^+\in E_H$, that is

$$\begin{aligned}\Phi(U_j^-,U_j^+)&=2\beta(\beta-1)\left(D^{2m+k}u_j,D^{2m+k}\overline{u}_j\right)-2\beta\lambda_j^+\left(D^{-(2m+k)}\overline{u}_j,D^{2m+k}u_j\right)\\&\quad-2\beta\lambda_j^-\left(D^{-(2m+k)}\overline{u}_j,D^{2m+k}u_j\right)+4\lambda_j^-\lambda_j^+\left(D^{-(2m+k)}\overline{u}_j,D^{-(2m+k)}u_j\right)\\&\quad-4M(s)\left(D^k u_j,D^k\overline{u}_j\right)+2\beta\left(D^{2m+k}\overline{u}_j,D^{2m+k}u_j\right)\\&\quad+2\beta(\beta-1)\left(D^{2m+k}v_j,D^{2m+k}\overline{v}_j\right)-2\beta\lambda_j^+\left(D^{-(2m+k)}\overline{v}_j,D^{2m+k}v_j\right)\\&\quad-2\beta\lambda_j^-\left(D^{-(2m+k)}\overline{v}_j,D^{2m+k}v_j\right)+4\lambda_j^-\lambda_j^+\left(D^{-(2m+k)}\overline{v}_j,D^{2m+k}v_j\right)\\&\quad-4M(s)\left(D^k v_j,D^k\overline{v}_j\right)+2\beta\left(D^{2m+k}\overline{v}_j,D^{2m+k}v_j\right)\\&=2\beta(\beta-1)\left(\left\|D^{2m+k}u_j\right\|^2+\left\|D^{2m+k}v_j\right\|^2\right)-2\beta\left(\lambda_j^-+\lambda_j^+\right)\left(\left\|u_j\right\|^2\right.\\&\quad\left.+\left\|v_j\right\|^2\right)+4\lambda_j^-\lambda_j^+\left(\left\|D^{-(2m+k)}u_j\right\|^2+\left\|D^{-(2m+k)}v_j\right\|^2\right)\\&\quad-4M(s)\left(\left\|D^k u_j\right\|^2+\left\|D^k v_j\right\|^2\right)+2\beta\left(\left\|D^{2m+k}u_j\right\|^2+\left\|D^{2m+k}v_j\right\|^2\right)\\&=-4M(\mu_j)+2\beta^2\mu_j-2\beta\left(\lambda_j^-+\lambda_j^+\right)+4\lambda_j^-\lambda_j^+\cdot\frac{1}{\mu_j}.\end{aligned} \tag{12}$$

Because of Equation (9), there are

$$\lambda_j^++\lambda_j^-=\beta\mu_j,\lambda_j^+\cdot\lambda_j^-=M(\mu_j)\mu_j.$$

So $\Phi(U_j^-,U_j^+)=-4M(\mu_j)+2\beta^2\mu_j-2\beta\left(\lambda_j^++\lambda_j^-\right)+4\lambda_j^+\lambda_j^-\cdot\dfrac{1}{\mu_j}=0$.

Step 3: according to the orthogonal decomposition established above, let's prove that A' satisfies the spectral interval condition. First estimate the Lipschitz constant l_F of *F*, where

$$F(U)=\left(0,f_1(x)-g_1(u,v),0,f_2(x)-g_2(u,v)\right)^{\mathrm{T}}$$

According to lemma 1, $g_i(u,v):V_{2m+k}\times V_{2m+k}\to V_{2m+k}\times V_{2m+k}$ are uniformly bounded and Lipschitz continuous, if $U=(u,z,v,q)\in E_k$,

$U_i=(u_i,z_i,v_i,q_i)\in P_iU\ (i=1,2)$, Then

$$P_1u=u_1, P_1v=v_1, P_2u=u_2, P_2v=v_2.$$

$$\begin{aligned}\|U\|_{E_k}^2&=\Phi(P_1U,P_2U)+\Psi(P_1U,P_2U)\\&\ge\left(\beta^2\mu_1-4M(s)\right)\left(\|D^kP_1u\|^2+\|D^kP_1v\|^2\right)\\&\quad+\left(\beta^2\mu_1-2M(s)\right)\left(\|D^kP_2u\|^2+\|D^kP_2v\|^2\right)\\&\ge\left(\beta^2\mu_1-4M(s)\right)\left(\|D^ku\|^2+\|D^kv\|^2\right).\end{aligned}$$

Given $U=(u,z,v,q), V=(\tilde{u},\tilde{z},\tilde{v},\tilde{q})\in E_k$, we can get

$$\begin{aligned}&\|F(U)-F(V)\|_{E_k}\\&=\|g_1(u,v)-g_1(\tilde{u},\tilde{v})\|_{V_{2m+k}\times V_{2m+k}}+\|g_2(u,v)-g_2(\tilde{u},\tilde{v})\|_{V_{2m+k}\times V_{2m+k}}\\&\le 2l\left(\|u-\tilde{u}\|_{V_{2m+k}}+\|v-\tilde{v}\|_{V_{2m+k}}\right)\\&\le\frac{4l}{\sqrt{\beta^2\mu_1-4M(s)}}\|U-V\|_{E_k}.\end{aligned}$$

So

$$l_F\le\frac{4l}{\sqrt{\beta^2\mu_1-4M(s)}} \tag{13}$$

From (13), if

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^- -\lambda_N^- >\frac{16l}{\sqrt{\beta^2\mu_1-4M(s)}} \tag{14}$$

Then the spectral interval condition (6) holds.

Step 4: according to the above paired eigenvalues, there are

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^- -\lambda_N^- =\frac{\beta}{2}\left(\mu_{N+1}-\mu_N\right)+\frac{\sqrt{R(N)}-\sqrt{R(N+1)}}{2}. \tag{15}$$

Of which, $R(N)=\beta^2\mu_N^2-4M(s)\mu_N$.

There are $N_0\in N_+$, for $\forall N\ge N_0$, let $R_0(N)=\sqrt{\frac{R(N)}{\beta^2\mu_1-4M(s)}}$, there are

$$\begin{aligned}&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}\left(\mu_{N+1}-\mu_N\right)\\&=\sqrt{\beta^2\mu_1-4M(s)}\left(\left(\mu_{N+1}-\frac{\sqrt{R(N+1)}}{\sqrt{\beta^2\mu_1-4M(s)}}\right)-\left(\mu_N-\frac{\sqrt{R(N)}}{\sqrt{\beta^2\mu_1-4M(s)}}\right)\right)\\&=\sqrt{\beta^2\mu_1-4M(s)}\left(\left(\mu_{N+1}-R_0(N+1)\right)-\left(\mu_N-R_0(N)\right)\right)\end{aligned} \tag{16}$$

And because of $\lim_{N\to+\infty}\left(\mu_N-R_0(N)\right)=\lim_{N\to+\infty}\left(\mu_N-\sqrt{\frac{R(N)}{\beta^2\mu_1-4M(s)}}\right)=0$, there are

$$\lim_{N\to+\infty}\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}\left(\mu_{N+1}-\mu_N\right)=0. \tag{17}$$

According to the hypothesis (10) of Theorem 1 and Equations (13)-(17), there are

$$\begin{aligned}\Lambda_2-\Lambda_1 &\geq \frac{1}{2}\left(\left(\mu_{N+1}-\mu_N\right)\left(\beta-\sqrt{\beta^2\mu_1-4M(s)}\right)-1\right)\\ &\geq \frac{16l}{\sqrt{\beta^2\mu_1-4M(s)}} \geq 4l_F.\end{aligned} \tag{18}$$

Theorem 1 is proved.

Theorem 2 [12] Through theorem1, operator A' satisfies the spectral interval condition, and problems (1)-(5) have a family of inertial manifolds μ_k, and $\mu_k \in E_k$. The form is as follows,

$$\mu_k = graph(\Gamma) \in E_k := \left\{\varsigma + \Gamma(\varsigma) : \varsigma \in E_{k_1}\right\}$$

where $\Gamma : E_{k_1} \to E_{k_2}$ is Lipschitz continuous and has Lipschitz constant l_F, and $graph(\Gamma)$ represents the graph of Γ.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Foias, C., Sell, G.R. and Temam, R. (1985) Varietes inertielles des equations differentielles disspatives. *Comptes Rendus de I Academie des Science-Series I- Mathematics*, **301**, 139-142.

[2] Lin, G.G. and Xia, X.S. (2018) The Inertial Manifold for Class Kirchhoff-Type Equations with Strongly Damped Terms and Source Terms. *Applied Mathematics*, **9**, 730-737. https://doi.org/10.4236/am.2018.96050

[3] Lou, R.J., Lv, P.H. and Lin, G.G. (2016) Exponential Attractors and Inertial Manifolds for a Class of Generalized Nonlinear Kirchhoff-Sine-Gordon Equation. *Journal of Advances in Mathematics*, **12**, 6361-6375.

[4] Lin, G.G. and Yang, S.M. (2018) The Inertial Manifolds for a Class of Higher-Order Coupled Kirchhoff-Type Equations. *Journal of Applied Mathematics and Physics*, **6**, 1055-1064. https://doi.org/10.4236/jamp.2018.65091

[5] Lin, G.G., Lv, P.H. and Lou, R.J. (2017) Exponential Attractors and Inertial Manifolds for a Class of Nonlinear Generalized Kirchhoff-Boussinesq Model. *Far East Journal of Mathematical Sciences*, **101**, 1913-1945. https://doi.org/10.17654/MS101091913

[6] Lin, G.G. (2019) Dynamic Properties of Several Kinds of the Kirchhoff Equations. Chongqing University Press, Chongqing.

[7] Lin, G.G. and Wang, Y.G. (2022) A Family of Exponential Attractors and Inertial Manifolds for a Class of Higher Order Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **10**, 900-914. https://doi.org/10.4236/jamp.2022.103062

[8] Lin, G.G. and Yang, L.J. (2021) A Family of the Exponential Attractors and the Inertial Manifolds for a Class of Generalized Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **9**, 2399-2413. https://doi.org/10.4236/jamp.2021.910152

[9] Lin, G.G. and Li, S.Y. (2020) Inertial Manifolds for Generalized Higher-Order Kir-

chhoff Type Equations. *Journal of Mathematics Research*, **12**, 67-72. https://doi.org/10.5539/jmr.v12n5p67

[10] Lin, G.G. and Hu, L.J. (2018) The Inertial Manifold for a Class of Nonlinear Higher-Order Coupled Kirchhoff Equations with Strong Linear Damping. *International Journal of Modern Nonlinear Theory and Application*, **7**, 35-47. https://doi.org/10.4236/ijmnta.2018.72003

[11] Lin, G.G. and Liu, X.M. (2022) The Family of Exponential Attractors and Inertial Manifolds for a Generalized Nonlinear Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **10**, 172-189. https://doi.org/10.4236/jamp.2022.101013

[12] Lin, G.G. (2011) Nonlinear Evolution Equation. Yunnan University Press, Yunnan.