A Family of the Inertial Manifolds for a Class of Generalized Kirchhoff-Type Coupled Equations

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Abstract
The paper considers the long-time behavior for a class of generalized high-order Kirchhoff-type coupled equations, under the corresponding hypothetical conditions, according to the Hadamard graph transformation method, obtain the equivalent norm in space $E_k (k = 1, 2, \cdots, 2m)$, and we obtain the existence of a family of the inertial manifolds while such equations satisfy the spectral interval condition.

Keywords
Kirchhoff-Type Coupled Equations, Spectral Interval Condition, A Family of the Inertial Manifolds

1. Introduction
This paper investigates the following primal value problems of a system of generalized Kirchhoff-type coupled equations:

$$
\begin{align*}
&u_t + M \left( \|D^m u\|_p^p + \|D^m v\|_p^p \right) (-\Delta)^{2m} u + \beta (-\Delta)^{2m} u_j + g_1 (u,v) = f_1 (x), \\
v_t + M \left( \|D^m u\|_p^p + \|D^m v\|_p^p \right) (-\Delta)^{2m} v + \beta (-\Delta)^{2m} v_j + g_2 (u,v) = f_2 (x), \\
u (x,0) = u_0 (x), u_t (x,0) = u_1 (x), x \in \Omega, \\
v (x,0) = v_0 (x), v_t (x,0) = v_1 (x), x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, \frac{\partial v}{\partial n} = 0 (i = 0, 1, 2, \cdots, 2m).
\end{align*}
$$

where $\Omega$ is a bounded region with a smooth boundary in $R^n$, $\partial \Omega$ represents the boundary of $\Omega$, $u_0 (x), u_1 (x)$ and $v_0 (x), v_1 (x)$ are known functions, where $g_j (u,v), f_j (u,v) (j = 1, 2)$ are nonlinear terms and external interference terms, respectively, and are known functions on $\Omega \times (0,T)$, $\beta$ is the normal
number, \( M \left( \| D^n u \|_p + \| D^n v \|_p \right) \) is a non-negative first-order continuous derivative function, and \( m > 1 \) is the normal number, \( \int_{\Omega} \| D^n u \|_p^m \, dx \).

In order to overcome the research difficulties, G. Foias, G. R. Sell and R. Temam [1] proposed the concept of inertial manifolds, which greatly promoted the study of infinite-dimensional dynamical systems. Where the inertial manifold is a positive, finite-dimensional Lipschitz manifold, and the existence of an inertial manifold depends on the establishment of a spectral interval condition. Therefore, the research on a family of inertial manifolds is of great significance from both theoretical and practical aspects, and the relevant theoretical achievements can be referred to [2]-[9].

Guoguang Lin, Lingjuan Hu [10] studied a system of coupled wave equations of higher-order Kirchhoff type with strong damping terms

\[
\begin{align*}
\frac{\partial^i u}{\partial t^i} + M \left( \left\| \nabla u \right\|_p^m + \left\| \nabla v \right\|_p^m \right) \left( -\Delta \right)^m u + \beta \left( -\Delta \right)^n u_i + g_i(u, v) &= f_1(x), \\
\frac{\partial^i v}{\partial t^i} + M \left( \left\| \nabla u \right\|_p^m + \left\| \nabla v \right\|_p^m \right) \left( -\Delta \right)^m v + \beta \left( -\Delta \right)^n v_i + g_2(u, v) &= f_2(x), \\
u(x, 0) &= u_0(x), u_i(x, 0) = u_i(x), x \in \Omega, \\
v(x, 0) &= v_0(x), v_i(x, 0) = v_i(x), x \in \Omega, \\
\beta \frac{\partial^i u}{\partial \nu^i} &= 0, \beta \frac{\partial^i v}{\partial \nu^i} = 0 \quad (i = 0, 1, 2, \ldots, 2m - 1) \quad x \in \partial \Omega.
\end{align*}
\]

where \( \Omega \) is a bounded region with a smooth boundary in \( R^n \), \( \partial \Omega \) represents the boundary of \( \Omega \), \( g_j(u, v)(j = 1, 2) \) is a nonlinear source term, \( f_1(x), f_2(x) \) is an external force interference term, and \( \beta \left( -\Delta \right)^n u, \beta \left( -\Delta \right)^n v \) (\( \beta \geq 0 \)) is a strong dissipation terms. Using the Hadamard graph transformation method, the Lipschitz constant \( I_F \) of \( F \) is further estimated, and the inertial manifolds that satisfies the spectral interval condition is obtained.

Lin Guoguang, Liu Xiaomei [11] studied a family of inertial manifolds for a class of generalized higher-order Kirchhoff equations with strong dissipation terms

\[
\begin{align*}
u_i(x, t) = 0, \frac{\partial^i u}{\partial \nu^i} &= 0, i = 1, 2, \ldots, 2m - 1, x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0(x), u_i(x, 0) = u_i(x), x \in \Omega \subset R^n.
\end{align*}
\]

where \( m \in N^+ \), \( \Omega \subset R^n \), \( n \geq 1 \) is a bounded domain with a smooth boundary in \( \partial \Omega \), \( f(x) \) is an external force term, \( M \left( \left\| \nabla u \right\|_p^m \right) \) is the stress term of Kirchhoff equation, \( \beta \left( -\Delta \right)^m u \) is a strong dissipative term, \( \left\| \nabla \right\|_p^m (u_i + u) \) is a nonlinear source term. Based on appropriate assumptions and the Hadamard graph transformation method, the spectral interval condition is verified, and the existence of a family of the inertial manifolds of the equation is obtained.

On the basis of previous research, rigid term strengthening becomes
and this paper seeks a family of inertial manifolds. When defining the equivalence norm in space $E_k$, by making reasonable assumptions, it is obtained that the equation satisfies the spectral interval condition so that there is a family of inertial manifolds.

2. Preliminaries

For narrative convenience, we introduce the following symbols and assumptions:

Set $\mathcal{D} = \mathcal{D}_k \cap \mathcal{E}_k$. Consider Hilbert space family $V_\alpha = H^2(\Omega) \cap H^1_0(\Omega)$, whose inner product and norm are $(\cdot, \cdot)_{\alpha} = (-\Delta)^{\alpha/2}(-\Delta)^{\alpha/2}$ and $\|\cdot\|_{\alpha} = ||(-\Delta)^{\alpha/2}||$, respectively. Apparently there are $(\cdot, \cdot)_{\alpha} = (-\Delta)^{\alpha/2}(-\Delta)^{\alpha/2}$ and $\|\cdot\|_{\alpha} = ||(-\Delta)^{\alpha/2}||$, respectively. Apparently there are

\[
V_0 = L^2(\Omega), V_{2m} = H^2(\Omega) \cap H^1_0(\Omega), V_{2m+k} = H^2(\Omega) \cap H^1_0(\Omega),
\]

\[
V_1 = H^1(\Omega) \cap H^0_0(\Omega), E_0 = V_{2m+k} \times V_{2m+k} \times V_0,
\]

\[
E_k = V_{2m+k} \times V_k \times V_{2m+k} \times V_k, (k = 1, 2, \cdots, 2m).
\]

The assumption is as follows:

Let $M(s)$ be a continuous function on interval $D_1 (D_1 \in \Omega)$, and $M(s) \in C^1(\mathbb{R}^+)$:

\[
1 \leq \mu_0 \leq M(s) \leq \mu_k, \text{ set } M(s) = \left\|D^w u\right\|_{E_k} + \left\|D^v v\right\|_{E_k}.
\]

3. A Family of Inertial Manifolds

**Definition 1** [12] lets $S = \{S(t)\}_{t \geq 0}$ be the solution semigroup on Banach space $E_k = H^2(\Omega) \times H^1_0(\Omega)(k = 1, 2, \cdots, 2m)$, and a subset $\mu_k \subset E_k$ satisfies:

1) $\mu_k$ is finite-dimensional Lipschitz popular;
2) $\mu_k$ is positively unchanging, $\{S(t)\}_{t \geq 0} : \forall u_0 \in \mu_k, S(t)u_0 \subset \mu_k, t \geq 0$;
3) $\mu_k$ attracts the solution orbit exponentially, i.e. for any $u \in E_k$, the existence constant $\eta > 0, c > 0$ makes $\text{dist}(S(t)u, \mu) \leq ce^{-t}$, $t \geq 0$.

Then $\mu_k$ is called $E_k$ is a family of inertial manifolds.

In order to describe the spectral interval condition, first consider that the nonlinear term $F : E_k \rightarrow E_k$ is integrally bounded and continuous, and has a positive Lipschitz constant $l_F$, and its operator $A$ has several eigenvalues and eigenfunctions of the positive real part.

**Definition 2** [12] Set operator $A : X \rightarrow X$ has several eigenvalues of positive real numbers, and $F \in C_0(X, X)$ satisfies the Lipschitz condition:

\[
\left\|F(u) - F(v)\right\|_{X} \leq l_F \left\|u - v\right\|_{X}, \quad u, v \in X,
\]

The point spectrum of the operator $A$ can be divided into two parts $\sigma_1$ and $\sigma_2$, and $\sigma_1$ is finite,
and conditions

\[
\Lambda_2 - \Lambda_1 > 4l_f
\]

are satisfied.

Where the continuous projection \( P_i : X \to X_i, P_2 : X \to X_2 \), there is orthogonal decomposition \( X = X_1 \oplus X_2 \), then the operator \( A \) satisfies the spectral interval condition.

**Lemma 1** \((g_i : V_{2m+k} \times V_{2m+k} \to V_{2m+k} \times V_{2m+k} \quad (i = 1, 2))\) is a uniform bounded and integral Lipschitz continuous function.

Proof: \( \forall (\tilde{u}, \tilde{v}), (u, v) \in V_{2m+k} \times V_{2m+k} \) \((k = 1, 2, \ldots, 2m)\),

\[
\|g_i(\tilde{u}, \tilde{v}) - g_i(u, v)\|_{2m+k,2m+k} \\
= \|g_{1u}(u + \theta(\bar{u} - u), v + \theta(\bar{v} - v))(\bar{u} - u) \\
+ g_{1v}(u + \theta(\bar{u} - u), v + \theta(\bar{v} - v))(\bar{v} - v)\|_{2m+k,2m+k} \\
\leq \|g_{1u}(u + \theta(\bar{u} - u), v + \theta(\bar{v} - v))(\bar{u} - u)\|_{2m+k,2m+k} \\
+ \|g_{1v}(u + \theta(\bar{u} - u), v + \theta(\bar{v} - v))(\bar{v} - v)\|_{2m+k,2m+k} \\
\leq l(\|\tilde{u} - u\|_{2m+k} + \|\tilde{v} - v\|_{2m+k}).
\]

Similarly, there are

\[
\|g_2(\tilde{u}, \tilde{v}) - g_2(u, v)\|_{2m+k,2m+k} \leq l(\|\tilde{u} - u\|_{2m+k} + \|\tilde{v} - v\|_{2m+k}),
\]

where \( l \) is the Lipschitz constant of \( g_i, \theta \in (0,1) \).

**Lemma 2** \([12]\) lets the sequence of eigenvalues \( \{\mu_j\}_{j=1} \) is a non-subtractive sequence, then \( \exists \forall N_0 \in N \), for \( \forall N \geq N_0 \), \( \mu_N \) and \( \mu_{N+1} \) are consecutive adjacent values.

In order to verify that the operator satisfies the spectral interval condition, so as to draw the conclusion that there is a family of inertial manifolds in questions (1)-(5), the following definitions and assumptions can be made first.

Based on the above relevant conditions, consider the first-order development equation equivalent to Equations (1)-(5), as follows:

\[
U_j + A'U = F(U)
\]

Of which \( U = (u, z, v, q) \),

\[
A' = \begin{pmatrix}
0 & -I & 0 & 0 \\
M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m} & 0 & 0 \\
0 & 0 & 0 & -I \\
0 & 0 & M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m}
\end{pmatrix},
\]
In order to determine the eigenvalue of matrix operator \( A' \), first consider graph module
\[
(U,V)_{E_k} = \left( M(s)D^{2m+1}u, M(s)D^{2m+1}v \right) + \left( D^4z, D^4z \right)
\]
in inner product in \( E_k \).

Where \( U = (u,z,v,q), V = (u',z',v',q'), \) and \( u',z',v',q' \) represent the conjugation of \( u,z,v,q \) respectively. In addition, operator \( A' \) is monotonic, and for \( U \in D(A') \), there is
\[
(A'U,U)_{E_k} = -\left( M(s)D^{2m+1}z, M(s)(-\Delta)^2u + \beta(-\Delta)^2q \right)
\]
\[
+ \left( D^4\bar{z}, M(s)D^{2m+1}u \right)
\]
\[
+ \left( D^4\bar{z}, M(s)D^{2m+1}v \right) - \left( M(s)D^{2m+1}q, D^{2m+1}\bar{v} \right)
\]
\[
+ \left( D^{2m+1}\bar{q}, M(s)D^{2m+1}v \right) + \left( D^{2m+1}\bar{q}, \beta D^{2m+1}q \right)
\]
\[
= \beta \left\| D^{2m+1}z \right\|^2 + \left\| D^{2m+1}q \right\|^2 \geq 0.
\]

Therefore, \( (A'U,U)_{E_k} \) is a nonnegative real number.

In order to further determine the eigenvalue of the matrix operator \( A' \), the following characteristic equation can be considered,
\[
A'U = \lambda U, U = (u,z,v,q) \in E_k, \tag{8}
\]

That is
\[
\begin{align*}
-z &= \lambda u, \\
M(s)(-\Delta)^2u + \beta(-\Delta)^2z &= \lambda z, \\
-q &= \lambda v, \\
M(s)(-\Delta)^2v + \beta(-\Delta)^2q &= \lambda q.
\end{align*}
\]

Thus \( u,v \) meet the eigenvalue problem
\[
\begin{align*}
\lambda^2u - \lambda\beta(-\Delta)^2u + M(s)(-\Delta)^2u &= 0, \\
\lambda^2v - \lambda\beta(-\Delta)^2v + M(s)(-\Delta)^2v &= 0, \\
\frac{\partial^i u}{\partial n^i} \bigg|_{\partial\Omega} &= \frac{\partial^i v}{\partial n^i} \bigg|_{\partial\Omega} = 0, i = 0,1,2,\ldots,2m-1,
\end{align*}
\]
Take the inner product of \((-\Delta)^k u, (-\Delta)^k v\) and Equations (1) and (2) above respectively, with
\[
\begin{align*}
\lambda^2 \|D^4 u\|^2 - \lambda \beta \|D^{2m+4} u\|^2 + M(s) \|D^{2m+4} v\|^2 &= 0, \\
\lambda^2 \|D^4 v\|^2 - \lambda \beta \|D^{2m+4} v\|^2 + M(s) \|D^{2m+4} u\|^2 &= 0,
\end{align*}
\]
That is
\[
\lambda^2 \left( \|D^4 u\|^2 + \|D^4 v\|^2 \right) - \lambda \beta \left( \|D^{2m+4} u\|^2 + \|D^{2m+4} v\|^2 \right) + M(s) \left( \|D^{2m+4} u\|^2 + \|D^{2m+4} v\|^2 \right) = 0. \tag{9}
\]
Equation (9) is a univariate quadratic equation about \(\lambda\). Replace \(u, v\) with \(u_j, v_j\). For each positive integer \(j\), Equation (8) has paired eigenvalues
\[
\lambda_j^\pm = \frac{\beta \mu_j \pm \sqrt{(\beta \mu_j)^2 - 4M(s)\mu_j}}{2},
\]
where \(\mu_j\) is the characteristic root of \((-\Delta)^{2m}\) in \(V_{2m} \times V_{2m}\), then \(\mu_j = \lambda_j^{2m}\).

If \((\beta \mu_j)^2 \geq 4M(s)\mu_j\), then \(\mu_j \geq \frac{4M(s)}{\beta^2}\), the eigenvalues of operator \(A'\) are all real numbers, and the corresponding eigenfunction form is
\[
U_j^\pm = (u_j, -\lambda_j^\pm u_j, v_j, -\lambda_j^\pm v_j).
\]

For convenience, mark for any \(j \geq 1\), there are
\[
\|D^{2m+4} u_j\|^2 + \|D^{2m+4} v_j\|^2 = \mu_j, \quad \|D^4 u_j\|^2 + \|D^4 v_j\|^2 = 1, \\
\|D^{2m+4} u_j\|^2 + \|D^{2m+4} v_j\|^2 = \frac{1}{\mu_j}.
\]

**Theorem 1:** Assumes that \(I\) is the Lipschitz constant of \(g_i(u, v)(i = 1, 2)\). When \(N_0 \in N_+\) is sufficiently large, for \(\forall N \geq N_0\), the following inequality holds
\[
(\mu_{N+1} - \mu_N) \left( \beta - \sqrt{\beta^2 \mu_i - 4M(s)} \right) \geq \frac{32I}{\sqrt{\beta^2 \mu_i - 4M(s)}} + 1. \tag{10}
\]
Then all operators \(A'\) satisfy the spectral interval condition (6).

**Proof.** Because \(\mu_j \geq \frac{4M(s)}{\beta^2}\) and the eigenvalues of \(A'\) are positive real numbers, \(\lambda_j^{\pm}\) and \(\lambda_j^{\pm}\) are single increment sequences.

The following four steps are taken to prove theorem 1:

**Step 1:** Because \(\lambda_j^{\pm}\) and \(\lambda_j^{\pm}\) are non subtractive columns, according to lemma 2, there are \(\exists N_0 \in N_+\), for \(\forall N \geq N_0\), \(\lambda_N^{\pm}\) and \(\lambda_{N+1}^{\pm}\) are continuous adjacent values.

Therefore, there is \(\lambda_N^{\pm}\) so that \(\lambda_N^{\pm}\) and \(\lambda_{N+1}^{\pm}\) are continuous adjacent values, and the eigenvalue of \(A'\) can be decomposed into
\[ \sigma_1 = \{ \lambda_r^- | 1 \leq r \leq N \}, \sigma_2 = \{ \lambda_r^+ | 1 \leq r \leq j \} \]

Step 2: Consider the corresponding decomposition of \( E_k \) into

\[ E_{k_1} = \text{span} \left\{ U_r^- | \lambda_r^- \in \sigma_1 \right\}, \]
\[ E_{k_2} = \text{span} \left\{ U_r^+, U_r^+ | \lambda_r^+ \in \sigma_2 \right\} \]

The equivalent inner product \( \langle (U,V) \rangle_{E_k} \) given below makes \( E_{k_1}, E_{k_2} \) orthogonal.

Further decompose \( E_{k_5} = E_H \oplus E_R \), of which

\[ E_H = \text{span} \left\{ U_r^+ | 1 \leq r \leq N \right\}, E_R = \text{span} \left\{ U_r^+ | j \geq N \right\} \]

Because \( E_{k_5} \) and \( E_H \) are finite dimensional subspaces, \( U_N \in E_{k_5}, U_{N+1} \in E_R \), and \( E_{k_1} \) and \( E_R \) are orthogonal, while \( E_{k_5} \) and \( E_H \) are not orthogonal, \( E_{k_1} \) and \( E_{k_2} \) are not orthogonal. So we need to redefine the equivalent norm on \( E_k \), so that \( E_{k_1} \) and \( E_H \) are orthogonal. Order \( E_N = E_{k_1} \oplus E_H \).

Construct two functions \( \Phi : E_N \rightarrow R, \Psi : E_R \rightarrow R \) of which,

\[
\Phi(U,V) = 2\beta(\beta - 1) \left( D^{2m+1} u, D^{(2m+1)} u' \right) + 2\beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} \bar{u} \right) \\
+ 2\beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} u' \right) + 4 \left( D^{(2m+1)} \bar{z}, D^{(2m+1)} \bar{z} \right) \\
- 4M(s) \left( D^k u, D^k \bar{u} \right) + 2\beta \left( D^{2m+1} \bar{u}, D^{2m+1} u' \right) \\
+ 2\beta(\beta - 1) \left( D^{2m+1} v, D^{2m+1} \bar{v} \right) + 2\beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} \bar{v} \right) \\
+ 2\beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} v' \right) + 4 \left( D^{(2m+1)} \bar{q}, D^{(2m+1)} q \right) \\
- 4M(s) \left( D^k v, D^k \bar{v} \right) + 2\beta \left( D^{2m+1} \bar{v}, D^{2m+1} v' \right),
\]

\[
\Psi(U,V) = 2\beta \left( D^{2m+1} \bar{u}, D^{2m+1} u' \right) + \beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} u \right) \\
+ \beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} u' \right) + 4 \left( D^{(2m+1)} \bar{z}, D^{(2m+1)} \bar{z} \right) \\
- 2M(s) \left( D^k u, D^k \bar{u} \right) + 2\beta(\beta - 1) \left( D^{2m+1} u, D^{(2m+1)} u' \right) \\
+ 2\beta \left( D^{2m+1} \bar{v}, D^{2m+1} v' \right) + \beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} v \right) \\
+ \beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} v' \right) + 4 \left( D^{(2m+1)} \bar{q}, D^{(2m+1)} q \right) \\
- 2M(s) \left( D^k v, D^k \bar{v} \right) + 2\beta(\beta - 1) \left( D^{2m+1} v, D^{2m+1} \bar{v} \right).
\]

Among them \( U = (u, z, v, q), V = (u', z', v', q') \in E_N \) or \( E_R \).

For \( U = (u, z, v, q) \in E_N \), then

\[
\Phi(U, U) = 2\beta(\beta - 1) \left( D^{2m+1} u, D^{(2m+1)} \bar{u} \right) + 2\beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} u \right) \\
+ 2\beta \left( D^{(2m+1)} \bar{z}, D^{2m+1} u' \right) + 4 \left( D^{(2m+1)} \bar{z}, D^{(2m+1)} \bar{z} \right) \\
- 4M(s) \left( D^k u, D^k \bar{u} \right) + 2\beta \left( D^{2m+1} \bar{u}, D^{2m+1} u \right) \\
+ 2\beta(\beta - 1) \left( D^{2m+1} v, D^{2m+1} \bar{v} \right) + 2\beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} v \right) \\
+ 2\beta \left( D^{(2m+1)} \bar{q}, D^{2m+1} v' \right) + 4 \left( D^{(2m+1)} \bar{q}, D^{(2m+1)} q \right).
\]
\[-4M(s)(D^4v, D^4\overline{v}) + 2\beta(D^{2m+4}v, D^{2m+4}\overline{v})\]

\[\geq 2\beta(\beta-1)(\|D^{2m+k-1}v\|^2 + \|D^{2m+k-1}\overline{v}\|^2) - 4\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}u\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{-(2m+k)}\overline{q}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-4M(s)(D^4u, D^4\overline{u}) + 2\beta(\|D^{2m+k}u\| + \|D^{2m+k}\overline{u}\|)\]

\[\geq 2\beta(\beta-1)(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{u}\|^2) - 4\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}v\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{-(2m+k)}\overline{q}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-2M(s)(D^4v, D^4\overline{v}) + 2\beta(\beta-1)(\|D^{2m+k-1}v\| + \|D^{2m+k-1}\overline{v}\|)\]

\[+ 2\beta(D^{2m+k}v, D^{2m+k}\overline{v}) + \beta\left(\|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\right)\]

\[-2M(s)(D^4u, D^4\overline{u}) + 2\beta(\beta-1)(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{u}\|^2)\]

\[\geq 2\beta(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{u}\|^2) - 2\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}u\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-2M(s)(D^4v, D^4\overline{v}) + 2\beta(\beta-1)(\|D^{2m+k}v\|^2 + \|D^{2m+k}\overline{v}\|^2)\]

\[\geq 2\beta(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{v}\|^2) - 4\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}v\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-2M(s)(D^4u, D^4\overline{u}) + 2\beta(\beta-1)(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{u}\|^2)\]

\[= \beta^2(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{v}\|^2) - 2M(s)(\|D^4u\|^2 + \|D^4\overline{u}\|^2)\]

\[\geq \left(\beta^2\mu_1 - 2M(s)\right)(\|D^4u\|^2 + \|D^4\overline{u}\|^2).\]

For any \(k\), there is \(\beta^2\mu_1 \geq 4M(\mu_k)\). According to hypothesis

\[1 \leq \mu_0 \leq M(s) \leq \mu_1 \leq \frac{\beta^2\mu_1}{4},\] then \(\Phi(U, \overline{U}) \geq 0\), that is, \(\Phi\) is positive definite.

Similarly, for any \(U = (u, z, v, q) \in E_B\), there is

\[\Psi(U, \overline{U}) = 2\beta(D^{2m+k}v, D^{2m+k}\overline{v}) + \beta(D^{-(2m+k)}z, D^{2m+k}u)\]

\[+ \beta\left(\|D^{-(2m+k)}z\|\|D^{2m+k}u\| + \|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\right)\]

\[-2M(s)(\|D^4v\|^2 + \|D^4\overline{v}\|^2) - 2\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}u\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-2M(s)(\|D^4u\|^2 + \|D^4\overline{u}\|^2) - 2\beta(\beta-1)(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{u}\|^2)\]

\[\geq 2\beta(\|D^{2m+k}u\|^2 + \|D^{2m+k}\overline{v}\|^2) - 2\beta(\|D^{-(2m+k)}z\|\|D^{2m+k}u\|)\]

\[+ \|D^{-(2m+k)}q\|\|D^{2m+k}\overline{v}\|^2 + 4\left(\|D^{-(2m+k)}z\|^2 + \|D^{-(2m+k)}q\|^2\right)\]

\[-2M(s)(\|D^4v\|^2 + \|D^4\overline{v}\|^2) - 2\beta(\beta-1)(\|D^{2m+k}v\|^2 + \|D^{2m+k}\overline{v}\|^2)\]

\[\geq \left(\beta^2\mu_1 - 2M(s)\right)(\|D^4u\|^2 + \|D^4\overline{u}\|^2).\]
So there are $\forall U = (u, z, v, q) \in E_R$, $\Psi(U, U) \geq 0$, then $\Psi$ is also positive definite.

Redefine the inner product of $E_k$:

$$
\left( (U, V) \right)_{E_k} = \Phi(P_N U, P_N V) + \Psi(P_N U, P_N V)
$$

(11)

where $P_N$ and $P_R$ are projections of $E_k \rightarrow E_N$ and $E_k \rightarrow E_R$, respectively.

Here, Equation (11) is written as

$$
\left( (U, V) \right)_{E_k} = \Phi(U, V) + \Psi(U, V)
$$

Under the redefined inner product of $E_k$, to prove that $E_{k_1}$ and $E_{k_2}$ are orthogonal, we only need to prove that $E_{k_1}$ and $E_{k_2}$ are orthogonal, that is,

$$
\left( (U_j^-, U_j^+) \right)_{E_k} = \Phi(U_j^-, U_j^+) = 0.
$$

Because there are $U_j^- \in E_{k_1}, U_j^+ \in E_{k_2}$, that is

$$
\Phi(U_j^-, U_j^+) = 2\beta(\beta - 1)\left( (D^{2m+k} u_j, D^{2m+k} u_j) - 2\beta\lambda_j^+ \lambda_j^- \left( (D^{2m+k} u_j, D^{2m+k} u_j) \right) \right)
$$

$$
-2\beta\lambda_j^+ \left( (D^{2m+k} u_j, D^{2m+k} u_j) \right) + 4\lambda_j^+ \lambda_j^- \left( (D^{2m+k} u_j, D^{2m+k} u_j) \right) \right)
$$

$$
-4M(s)\left( (D^{m+k} u_j, D^{m+k} u_j) + 2\beta\lambda_j^- \left( (D^{m+k} u_j, D^{m+k} u_j) \right) \right) \right)
$$

$$
+ 2\beta(\beta - 1)\left( (D^{m+k} v_j, D^{m+k} v_j) - 2\beta\lambda_j^+ \lambda_j^- \left( (D^{m+k} v_j, D^{m+k} v_j) \right) \right)
$$

$$
-2\beta\lambda_j^- \left( (D^{m+k} v_j, D^{m+k} v_j) \right) + 4\lambda_j^+ \lambda_j^- \left( (D^{m+k} v_j, D^{m+k} v_j) \right) \right)
$$

$$
-4M(s)\left( (D^{m+k} v_j, D^{m+k} v_j) + 2\beta\lambda_j^+ \lambda_j^- \left( (D^{m+k} v_j, D^{m+k} v_j) \right) \right) \right)
$$

$$
= 2\beta(\beta - 1)\left( (D^{2m+k} u_j, D^{2m+k} u_j) + (D^{2m+k} v_j, D^{2m+k} v_j) \right) - 2\beta(\lambda_j^+ + \lambda_j^-) \left( (D^{2m+k} u_j, D^{2m+k} u_j) \right)
$$

$$
+ 4\lambda_j^+ \lambda_j^- \left( (D^{2m+k} u_j, D^{2m+k} u_j) \right) \right)
$$

$$
= -4M(\mu_j) + 2\beta^2\mu_j - 2\beta(\lambda_j^+ + \lambda_j^-) + 4\lambda_j^+ \lambda_j^- \frac{1}{\mu_j}.
$$

Because of Equation (9), there are

$$
\lambda_j^+ + \lambda_j^- = \beta \mu_j, \lambda_j^+ \lambda_j^- = M(\mu_j) \mu_j.
$$

So

$$
\Phi(U_j^-, U_j^+) = -4M(\mu_j) + 2\beta^2\mu_j - 2\beta(\lambda_j^+ + \lambda_j^-) + 4\lambda_j^+ \lambda_j^- \frac{1}{\mu_j} = 0.
$$

Step 3: according to the orthogonal decomposition established above, let's prove that $A'$ satisfies the spectral interval condition. First estimate the Lipschitz constant $l_F$ of $F$, where

$$
F(U) = \left( f_1(x) - g_1(u, v), 0, f_2(x) - g_2(u, v) \right)^T
$$

According to lemma 1, $g_1(u, v): V_{2m+k} \times V_{2m+k} \rightarrow V_{2m+k} \times V_{2m+k}$ are uniformly bounded and Lipschitz continuous, if $U = (u, z, v, q) \in E_k$, $U_j = (u_j, z_j, v_j, q_j) \in P_i U (i = 1, 2)$.
Then
\[ P_1u = u_1, P_1v = v_1, P_2u = u_2, P_2v = v_2. \]
\[
\|U\|_{E_k}^2 = \Phi(PU, PU) + \Psi(PU, PU) \\
\geq (\beta^2 \mu_k - 4M(s))(\|D^k Pu\|^2 + \|D^k P_v\|^2) \\
+ (\beta^2 \mu_k - 2M(s))(\|D^k P_2u\|^2 + \|D^k P_2v\|^2) \\
\geq (\beta^2 \mu_k - 4M(s))(\|D^k u\|^2 + \|D^k v\|^2).
\]

Given \( U = (u, z, v, q) \), \( V = (\bar{u}, \bar{z}, \bar{v}, \bar{q}) \) \( \in E_k \), we can get
\[
\|F(U) - F(V)\|_{E_k} \\
= \|g_1(u, v) - g_1(\bar{u}, \bar{v})\|_{2^{n\times n} \times 2^{n\times n}} + \|g_2(u, v) - g_2(\bar{u}, \bar{v})\|_{2^{n\times n} \times 2^{n\times n}} \\
\leq 2I \left( \|u - \bar{u}\|_{2^{n\times n}} + \|v - \bar{v}\|_{2^{n\times n}} \right) \\
\leq \frac{4I}{\sqrt{\beta^2 \mu_k - 4M(s)}} \|U - V\|_{E_k}.
\]

So
\[
l_F \leq \frac{4I}{\sqrt{\beta^2 \mu_k - 4M(s)}} \quad (13)
\]

From (13), if
\[
\Lambda_2 - \Lambda_1 = \lambda_{N+1} - \lambda_N > \frac{16I}{\sqrt{\beta^2 \mu_k - 4M(s)}} \quad (14)
\]

Then the spectral interval condition (6) holds.

Step 4: according to the above paired eigenvalues, there are
\[
\Lambda_2 - \Lambda_1 = \lambda_{N+1} - \lambda_N = \frac{\beta}{2}(\mu_{N+1} - \mu_N) + \sqrt{R(N) - R(N+1)} \\
\left( \sqrt{\beta^2 \mu_k - 4M(s)} \right) \quad (15)
\]

Of which, \( R(N) = \beta^2 \mu_k - 4M(s) \mu_N \).

There are \( N_0 \in \mathbb{N} \), for \( \forall N \geq N_0 \), let \( R_0(N) = \frac{R(N)}{\sqrt{\beta^2 \mu_k - 4M(s)}} \), there are
\[
\sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \mu_k - 4M(s)}(\mu_{N+1} - \mu_N) \\
= \sqrt{\beta^2 \mu_k - 4M(s)} \left( \mu_{N+1} - \frac{\sqrt{R(N+1)}}{\sqrt{\beta^2 \mu_k - 4M(s)}} \right) - \mu_N + \frac{\sqrt{R(N)}}{\sqrt{\beta^2 \mu_k - 4M(s)}} \\
= \sqrt{\beta^2 \mu_k - 4M(s)} \left( (\mu_{N+1} - R_0 (N+1)) - (\mu_N - R_0(N)) \right) \quad (16)
\]

And because of \( \lim_{N \to +\infty} (\mu_N - R_0(N)) = \lim_{N \to +\infty} \left( \mu_N - \frac{R(N)}{\sqrt{\beta^2 \mu_k - 4M(s)}} \right) = 0 \), there are
\[
\lim_{N \to +\infty} \sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \mu_k - 4M(s)}(\mu_{N+1} - \mu_N) = 0. \quad (17)
\]
According to the hypothesis (10) of Theorem 1 and Equations (13)-(17), there are
\[ \Lambda_2 - \Lambda_1 \geq \frac{1}{2} \left( \mu_{N+1} - \mu_N \right) \left( \beta - \sqrt{\beta^2 \mu_1 - 4M(s)} - 1 \right) \]
\[ \geq \frac{16l}{\sqrt{\beta^2 \mu_1 - 4M(s)}} \geq 4l_F. \]

Theorem 1 is proved.

**Theorem 2** [12] Through theorem1, operator \( A' \) satisfies the spectral interval condition, and problems (1)-(5) have a family of inertial manifolds \( \mu_k \), and \( \mu_k \in E_k \). The form is as follows,
\[ \mu_k = \text{graph}(\Gamma) \in E_k := \{ \xi + \Gamma(\xi) : \xi \in E_k \} \]
where \( \Gamma : E_k \to E_k \) is Lipschitz continuous and has Lipschitz constant \( l_F \), and \( \text{graph}(\Gamma) \) represents the graph of \( \Gamma \).

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

**References**


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