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# Time-Periodic Solutions of the Hydrodynamic Equations for a Reacting Mixture in *n*-Dimension

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## Abstract

In this paper, we investigate the existence of time-periodic solutions to the *n*-dimension hydrodynamic model for a reacting mixture with a time-periodic external force when the dimension $n \geq 5$ is under some smallness assumption. The energy method combined with the spectral analysis is used to obtain the optimal decay estimates on the linearized solution operator. We study the existence and uniqueness of the time-periodic solution in some suitable function space by using a fixed point method and the decay estimates. Furthermore, we obtain the time asymptotic stability of the time-periodic solution.

## Keywords

Time-Periodic Solutions, Compressible Navier-Stokes Equation, Radiative and Reactive Gases, Decay Rate

## 1. Introduction

Mathematical models for mixtures of the hydrodynamic equations in the space $\mathbb{R}^n$ have been studied for quite a few years. Notice that if the radiation effect is neglected, the existence, uniqueness and dynamic behavior of solutions were recognized by Chen [1] [2] and Li [3] under the initial value satisfies certain assumptions; if the radiation effect is considered, for (1.1) in one dimensional, the existence and uniqueness of the global solution to the Cauchy problem is obtained by Liao and Zhao [4] under the assumption of constant viscosity coefficient; the global existence and uniqueness of solutions for initial boundary value problems of viscous radiative reactive gases were achieved very well by Liao and Zhao [5], Ducomet [6], Jiang and Zheng [7] [8] and Umehara [9] [10]. Besides, for (1.1) in multidimensional, global existence and exponential stability of spherically symmetric solutions in a bounded annular domain for compressible viscous radiative reactive gases were well obtained by Qin, Zhang, and Su [11] and Liao, Wang and Zhao [12] for spherical solutions in an exterior domain. When the initial data is in the neighborhood of the trivial stable solution, the global existence and uniqueness of the strong solution of Cauchy problem are proved effectively by Wang and Wen [13]. In addition, there are lots of researches on the time-periodic solution of the Navier-Stokes system, what we want to say most is that the existence of a time-periodic solution for the Navier-Stokes equations with time-periodic external force under some assumptions when the space dimension $n > 5$ was well proved by Ma, Ukai and Yang [14]; The existence of time-periodic solutions for compressible Navier-Stokes equations under general external forces when space dimension $n = 4$ was defined by Jin [15]; For the time-periodic parallel flow problem in $n$-dimensional space, there is a time-periodic solution for the Navier-Stokes equation with a special time-periodic external force was controlled by Brezina and Kagei by [16]. In summary, it's still open whether the time-periodic solution to (1.1) exists and is unique in $n$ dimensions.

In this paper, we consider the Cauchy problem of a model for the combustion of the hydrodynamic equations with a time-periodic external force in $\mathbb{R}^n$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u^2) + \nabla P(\rho, \theta) = \operatorname{div}\mathbb{S} + \rho f, \\ (\rho e)_t + \operatorname{div}(\rho e u) + P\operatorname{div}u = -\operatorname{div}\boldsymbol{Q} + \mathbb{S} : \nabla u + Kq\varphi(\theta)\rho Z, \\ (\rho Z)_t + \operatorname{div}(\rho u Z) = -K\varphi(\theta)\rho Z + \operatorname{div}F \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^n, t \in [0, +\infty)$, where $\rho(t,x)$, $u(t,x) = (u_1, u_2, \cdots, u_n)(t,x)$, $\theta(t,x)$, $Z(t,x)$ denote the density, the velocity, the temperature and the mass fraction of the reactant, respectively. The last term on the right side of the energy Equation $(1.1)_3$ is the reactant energy difference $Kq\varphi(\theta)\rho Z$, which means the difference between the rate of gained energy for the product and that of lost energy for the reactant. The constant $K > 0$ is the reaction rate. $q$ is the difference of the stoichiometric coefficients for components appearing as reactant and product. $\rho Z = \rho Z(t,x)$ represents the density of the reactant. $\varphi = \varphi(\theta)$ is the reaction function which is assumed to satisfy the first-order Arrhenius law as follows (see [6]):

$$\varphi(\theta) = \begin{cases} 0, & 0 \le \theta \le \theta_I; \\ \theta^\beta \mathrm{e}^{-\frac{A}{\theta}}, & \theta > \theta_I, \beta \ge 0, \end{cases}$$

where $A$ is a positive constant and stands for the activation energy. $\theta_I \ge 0$ is the ignition temperature. Combustion will occur when the temperature of the given fluid particle rise above $\theta_I$. Then, the reactant is transformed to the product via an irreversible reaction governed by the function $\theta_I$.

The heat flux $\boldsymbol{Q} = \boldsymbol{Q}(\rho, \theta, \nabla\theta)$ satisfies the Fourier law

$$\mathbf{Q} = -\kappa(\rho,\theta)\nabla\theta, \tag{1.2}$$

where $\kappa(\rho,\theta) = \kappa_1 + \kappa_2 \frac{\theta^b}{\rho} > 0$ is the heat conductivity coefficient. $\kappa_1$, $\kappa_2$ and $b$ are positive constants.

The function $F$ that we assumed denotes the diffusion velocity and satisfy Fick's law

$$F = D\nabla Z = d_0\rho\nabla Z, \tag{1.3}$$

where $D = d_0\rho$ stands for the reactant flux diffusion coefficient and the positive constant $d_0$ is the species diffusion in the reaction.

The total pressure $P$ in the gas and the corresponding specific internal energy $e$ have the following form

$$P = P(\rho,\theta) = R\rho\theta + \frac{a}{3}\theta^4, \quad e = e(\rho,\theta) = c_v\theta + a\frac{\theta^4}{\rho}, \tag{1.4}$$

where constants $R > 0$, $a$ and $c_v$ are positive.

The viscous stress tensor is given in system (1.1) by

$$\mathbb{S} = \mu\left(\nabla u + \nabla^{\mathrm{T}} u\right) + \lambda \mathrm{div} u \mathbb{I}_{n\times n}, \tag{1.5}$$

where $\mu$ is the heat viscosity coefficient, $\lambda = \varsigma - \frac{2}{n}\mu$ with the bulk viscosity coefficient $\varsigma \geq 0$. Thus

$$\mathbb{S} : \nabla u = \sum_{i,j} \frac{\mu}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right)^2 + \lambda|\mathrm{div} u|^2. \tag{1.6}$$

In what follows, we first make two assumptions:

A.1. $f(t,x) = (f_1, f_2, \cdots, f_n)(t,x)$ is periodic on time with period $T > 0$.

A.2. $P(\rho,\theta)$ is a smooth function in a neighborhood of the constant state $(\bar{\rho}, 0, \bar{\theta}, 0)$ with $\bar{\rho} > 0$, $\bar{\theta} > 0$. In addition, $P(\rho,\theta)$ satisfies $P_\rho(\bar{\rho},\bar{\theta}) > 0$ and $P_\theta(\bar{\rho},\bar{\theta}) > 0$.

In this paper, our main purpose is to obtain a time-periodic solution of (1.1) around the constant state $(\rho_\infty, 0, \theta_\infty, 0)$ which has the same period as the periodic function $f(t,x)$. Our main idea is to combine the energy method with spectral analysis to get the optimal decay estimates of the linearized solution operator $S_u(t,s)$, which we will introduce in Section 2 and obtain the decay rates of $S_u(t,s)$ in Section 4.

Let $N \geq n+2$, we define the solution space by

$$\begin{aligned} X(0,T;a') = \Big\{&(\rho,u,\theta,Z)(t);\rho(t,x) \in C^0\left(0,T;H^{N-1}(\mathbb{R}^n)\right) \cap C^1\left(0,T;H^{N-2}(\mathbb{R}^n)\right), \\ &(u,\theta,Z)(t,x) \in C^0\left(0,T;H^{N-1}(\mathbb{R}^n)\right) \cap C^1\left(0,T;H^{N-3}(\mathbb{R}^n)\right), \\ &\nabla\rho(t,x) \in L^2\left(0,T;H^{N-1}(\mathbb{R}^n)\right), \nabla(u,\theta)(t,x) \in L^2\left(0,T;H^{N}(\mathbb{R}^n)\right), \\ &Z(t,x) \in L^2\left(0,T;H^{N-1}(\mathbb{R}^n)\right), 0 \leq Z(t,x) \leq 1, \left\|\!\left|(\rho,u,\theta,Z)\right|\!\right\| \leq a'\Big\}, \end{aligned} \tag{1.7}$$

with some constant $a' > 0$. The corresponding norm $|||\cdot|||$ is defined as follows:

$$\begin{aligned}|||(\rho,u,\theta,Z)|||^2 &= \sup_{0\le t\le T}\|(\rho,u,\theta,Z)(t)\|_{N-1}^2 \\ &+\int_0^T\left(\|\nabla\rho(t)\|_{N-1}^2+\|\nabla(u,\theta)(t)\|_N^2+\|Z(t)\|_{N+1}^2\right)\mathrm{d}t.\end{aligned} \tag{1.8}$$

In this paper, we will study the existence and uniqueness of time-periodic solutions and associate optimal time-decay estimates for the time-periodic solution which we obtained in $\mathbb{R}^n$. Now we are in a position to state our main results.

**Theorem 1.1.** *Let* $n\ge 5$, $N\ge n+2$. *Suppose* $f(t,x)\in C^0\left(0,T;H^{N-1}\right)(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$. *Under assumptions A.1-A.2, there exists a fixed constant* $a_0>0$ *given in the proof, such that if* $\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}\le h_0$ *for some sufficiently small constant* $h_0>0$, *then the system* (1.1) *admits a unique time-periodic solution* $\left(\rho^{per},u^{per},\theta^{per},Z^{per}\right)$ *with period* $T>0$, *which satisfies* $\left(\rho^{per}-\bar{\rho},u^{per},\theta^{per}-\bar{\theta},Z^{per}\right)\in X(0,T;a_0)$.

We consider the Cauchy problem of the system (1.1) with the initial data

$$(\rho_0,u_0,\theta_0,Z_0)(x)=(\rho,u,\theta,Z)\big|_{t=t_0} \tag{1.9}$$

for some fixed initial time $t_0\in\mathbb{R}$. When the initial data is a small perturbation of the periodic solution $\left(\rho^{per},u^{per},\theta^{per},Z^{per}\right)$ stated in Theorem 1.1, we obtain the stability of the solution around this time-periodic solution.

**Theorem 1.2.** *Assume that* $\left(\rho^{per},u^{per},\theta^{per},Z^{per}\right)$ *is stated in Theorem 1.1. For each* $t_0\in\mathbb{R}$, *let* $\left\|\left(\rho_0-\rho^{per}(t_0),u_0-u^{per}(t_0),\theta_0-\theta^{per}(t_0),Z_0-Z^{per}(t_0)\right)\right\|_{N-1}$ *is sufficiently small under the same conditions of Theorem 1.1, then the Cauchy problem* (1.1) *and* (1.9) *has a unique global solution* $(\rho,u,\theta,Z)$ *which satisfies*

$$\rho-\rho^{per}\in C^0\left(t_0,\infty;H^{N-1}(\mathbb{R}^n)\right)\cap C^1\left(t_0,\infty;H^{N-2}(\mathbb{R}^n)\right), \tag{1.10}$$

$$u-u^{per},\theta-\theta^{per}\in C^0\left(t_0,\infty;H^{N-1}(\mathbb{R}^n)\right)\cap C^1\left(t_0,\infty;H^{N-3}(\mathbb{R}^n)\right), \tag{1.11}$$

$$Z-Z^{per}\in C^0\left(t_0,\infty;H^{N-1}(\mathbb{R}^n)\right)\cap C^1\left(t_0,\infty;L^2(\mathbb{R}^n)\right), \tag{1.12}$$

and it holds that

$$\begin{aligned}&\|(\rho,u,\theta,Z)(t)\|_{N-1}^2+\int_{t_0}^t\left(\|\nabla\rho(t)\|_{N-2}^2+\|\nabla(u,\theta)(t)\|_{N-1}^2+\|Z(t)\|_N^2\right)\mathrm{d}s \\ &\le C_0\left\|\left(\rho_0-\rho^{per}(t_0),u_0-u^{per}(t_0),\theta_0-\theta^{per}(t_0),Z_0-Z^{per}(t_0)\right)\right\|_{N-1}^2.\end{aligned} \tag{1.13}$$

Moreover, if $\left(\rho_0-\rho^{per}(t_0),u_0-u^{per}(t_0),\theta_0-\theta^{per}(t_0),Z_0-Z^{per}(t_0)\right)\in L^1(\mathbb{R}^n)$, then exists a constant $C_0$ such that

$$\begin{aligned}&\left\|\left(\rho-\rho^{per},u-u^{per},\theta-\theta^{per},Z-Z^{per}\right)\right\|_{N-1} \\ &\le C_0(1+t)^{-\frac{n}{4}}\left\|\left(\rho_0-\rho^{per}(t_0),u_0-u^{per}(t_0),\theta_0-\theta^{per}(t_0),Z_0-Z^{per}(t_0)\right)\right\|_{H^{N-1}\cap L^1}.\end{aligned} \tag{1.14}$$

**Notations** For a multi-index $k=(k_1,k_2,\cdots,k_n)$, we denote $|k|=\sum_{i=1}^n k_i$ and

$\partial_x^k = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\cdots\partial_{x_n}^{k_n}$. $\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-ix\cdot\xi}\mathrm{d}x$, $f(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}(x)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi$, for $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$. Here, $\|\cdot\|_{H^m} = \|\cdot\|_m$ for $m\ge 0$ in the Sobolev Space $H^m(\mathbb{R}^n)$.

The rest of the paper is organized as follows. In the second section, we introduce appropriate variable transformations to linearize the transformed equations. In Section 3, we first show the energy estimates of the solution to (1.1). Then we study the periodicity of the solution of the linearized system with respect to time. Finally, we obtain the time-periodic solution of the nonlinear equation according to the expression of the solution of the equations. In Section 4, the proof of Theorem 1.1 is given. In the last section, we study the stability of the time-periodic solution.

## 2. Reformations

From the system (1.1), we have

$$\begin{cases}\rho_t + \mathrm{div}(\rho u) = 0,\\ \rho\left(u_t + (u\cdot\nabla)u\right) + \nabla P(\rho,\theta) = \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u + \rho f,\\ \rho e_t + \rho u\cdot\nabla e + P\mathrm{div}u = \mathrm{div}\left(\kappa(\rho,\theta)\nabla\theta\right) + \mathbb{S}:\nabla u + Kq\varphi(\theta)\rho Z,\\ \rho Z_t + \rho u\cdot\nabla Z = -K\varphi(\theta)\rho Z + d_0\rho\Delta Z + d_0\nabla\rho\nabla Z.\end{cases} \tag{2.1}$$

In terms of the definition of $e = e(\rho,\theta)$ in (1.3), we can see that

$$e_\rho(\rho,\theta) = -\frac{a\theta^4}{\rho^2} < 0,\quad e_\theta(\rho,\theta) = c_v + \frac{4a\theta^3}{\rho} > 0. \tag{2.2}$$

By using $(2.1)_1$, we have

$$e_t + u\cdot\nabla e = e_\theta(\rho,\theta)\partial_t\theta - e_\rho(\rho,\theta)\rho\mathrm{div}u + e_\theta(\rho,\theta)u\cdot\nabla\theta. \tag{2.3}$$

Taking a change of variables by $\sigma = \rho - \bar{\rho}$, $u = u$, $v = \theta - \bar{\theta}$, $Z = Z$, and using (2.3), then the problem (1.1) can be reformulated as

$$\begin{cases}\sigma_t + \bar{\rho}\nabla u + u\cdot\nabla\sigma = S_1(\sigma,u),\\ u_t + \dfrac{P_\rho(\bar{\rho},\bar{\theta})}{\bar{\rho}}\nabla\sigma + \dfrac{P_\theta(\bar{\rho},\bar{\theta})}{\bar{\rho}}\nabla v - \dfrac{\mu}{\bar{\rho}}\Delta u - \dfrac{(\mu+\lambda)}{\bar{\rho}}\nabla\mathrm{div}u = S_2(\sigma,u,v) + f,\\ v_t + \dfrac{P(\bar{\rho},\bar{\theta}) - e_\rho(\bar{\rho},\bar{\theta})\bar{\rho}^2}{e_\theta(\bar{\rho},\bar{\theta})\bar{\rho}}\mathrm{div}u - \dfrac{\kappa(\bar{\rho},\bar{\theta})}{e_\theta(\bar{\rho},\bar{\theta})\bar{\rho}}\Delta v - \dfrac{Kq\varphi(\bar{\theta})}{e_\theta(\bar{\rho},\bar{\theta})}Z = S_3(\sigma,u,v,Z),\\ Z_t - d_0\Delta Z + K\varphi(\bar{\theta})Z = S_4(\sigma,u,v,Z),\end{cases} \tag{2.4}$$

where

$$S_1(\sigma,u) = -\sigma\nabla\cdot u,$$

$$S_2(\sigma,u,v) = -(u\cdot\nabla)u + g_1(\sigma,v)\nabla\sigma + g_2(\sigma,v)\nabla v - h(\sigma)\frac{\mu}{\bar{\rho}}\Delta u - h(\sigma)\frac{(\mu+\lambda)}{\bar{\rho}}\nabla\mathrm{div}u,$$

$$S_3(\sigma,u,v,Z) = -u\cdot\nabla v + h_1(\sigma,v)\operatorname{div}u - h_2(\sigma,v)\Delta v - h_3(\sigma,v)Z$$
$$+ l_1(\sigma,v)\mathbb{S}:\nabla u + l_2(\sigma,v)\nabla\sigma\nabla v + l_3(\sigma,v)|\nabla v|^2,$$
$$S_4(\sigma,u,v,Z) = -u\cdot\nabla Z + h_4(v)Z + l_4(\sigma)\nabla\sigma\nabla Z,$$

with

$$g_1(\sigma,v) = \frac{P_\rho(\bar\rho,\bar\theta)}{\bar\rho} - \frac{P_\rho(\sigma+\bar\rho, v+\bar\theta)}{\sigma+\bar\rho},$$
$$g_2(\sigma,v) = \frac{P_\theta(\bar\rho,\bar\theta)}{\bar\rho} - \frac{P_\theta(\sigma+\bar\rho, v+\bar\theta)}{\sigma+\bar\rho},$$
$$h_1(\sigma,v) = \frac{P(\bar\rho,\bar\theta) - e_\rho(\bar\rho,\bar\theta)\bar\rho^2}{e_\theta(\bar\rho,\bar\theta)\bar\rho} - \frac{P(\sigma+\bar\rho,v+\bar\theta) - e_\rho(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)^2}{e_\theta(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)},$$
$$h_2(\sigma,v) = \frac{\kappa(\bar\rho,\bar\theta)}{e_\theta(\bar\rho,\bar\theta)\bar\rho} - \frac{\kappa(\sigma+\bar\rho,v+\bar\theta)}{e_\theta(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)},$$
$$h_3(\sigma,v) = \frac{Kq\varphi(\bar\theta)}{e_\theta(\bar\rho,\bar\theta)} - \frac{Kq\varphi(v+\bar\theta)}{e_\theta(\sigma+\bar\rho,v+\bar\theta)}, \quad h_4(v) = K\varphi(\bar\theta) - K\varphi(v+\bar\theta),$$
$$l_1(\sigma,v) = \frac{1}{e_\theta(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)}, \quad l_2(\sigma,v) = \frac{\kappa_\rho(\sigma+\bar\rho,v+\bar\theta)}{e_\theta(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)},$$
$$l_3(\sigma,v) = \frac{\kappa_\theta(\sigma+\bar\rho,v+\bar\theta)}{e_\theta(\sigma+\bar\rho,v+\bar\theta)(\sigma+\bar\rho)}, \quad l_4(\sigma) = \frac{d_0}{\sigma+\bar\rho}, \quad h(\sigma) = \frac{\sigma}{\sigma+\bar\rho}.$$

Taking a change of variables again by

$$\sigma = \sigma, \quad \omega = \alpha u, \quad v = \beta v, \quad r = \frac{q\beta}{e_\theta(\bar\rho,\bar\theta)}Z, \tag{2.5}$$

and

$$\alpha = \frac{\bar\rho}{\sqrt{P_\rho(\bar\rho,\bar\theta)}}, \quad \beta = \sqrt{\frac{P_\theta(\bar\rho,\bar\theta)}{P_\rho(\bar\rho,\bar\theta)}\frac{e_\theta(\bar\rho,\bar\theta)\bar\rho^2}{P(\bar\rho,\bar\theta) - e_\rho(\bar\rho,\bar\theta)\bar\rho^2}},$$
$$\lambda_1 = \sqrt{P_\rho(\bar\rho,\bar\theta)}, \quad \lambda_2 = \sqrt{P_\theta(\bar\rho,\bar\theta)\frac{P(\bar\rho,\bar\theta) - e_\rho(\bar\rho,\bar\theta)\bar\rho^2}{e_\theta(\bar\rho,\bar\theta)\bar\rho^2}}, \quad \lambda_3 = K\varphi(\bar\theta),$$
$$\lambda_4 = \frac{\sqrt{P_\rho(\bar\rho,\bar\theta)}}{\bar\rho}, \quad \gamma_1 = \frac{\kappa(\bar\rho,\bar\theta)}{e_\theta(\bar\rho,\bar\theta)\bar\rho^2}, \quad \bar\mu = \frac{\mu}{\bar\rho}, \quad \bar\lambda = \frac{\lambda}{\bar\rho}.$$

Then, the regularized problem (2.4) can be reformulated as

$$\begin{cases} \sigma_t + \lambda_1\nabla\omega + \lambda_4\omega\cdot\nabla\sigma = G_1(\sigma,\omega), \\ \omega_t + \lambda_1\nabla\sigma + \lambda_2\nabla v - \bar\mu\Delta\omega - (\bar\mu+\bar\lambda)\nabla\operatorname{div}\omega = G_2(\sigma,\omega,v) + \alpha f, \\ v_t + \lambda_2\operatorname{div}\omega - \gamma_1\Delta v - \lambda_3 r = G_3(\sigma,\omega,v,r), \\ r_t - d_0\Delta r + \lambda_3 r = G_4(\sigma,u,v,r), \end{cases} \tag{2.6}$$

where

$$\left(G_1,G_2,G_3,G_4\right)=\left(S_1,\alpha S_2,\beta S_3,\frac{q\beta}{e_\theta\left(\bar{\rho},\bar{\theta}\right)}S_4\right)\left(\sigma,\frac{1}{\alpha}\omega,\frac{1}{\beta}v,\frac{e_\theta\left(\bar{\rho},\bar{\theta}\right)}{q\beta}r\right). \quad (2.7)$$

Notice that $G_1,G_2,G_3,G_4$ have the following properties:

$$G_1(\sigma,\omega)\sim\sigma\nabla\cdot\omega,$$

$$G_2(\sigma,\omega,v)\sim(\omega\cdot\nabla)\omega+\sigma\nabla\sigma+v\nabla\sigma+\sigma\nabla v+v\nabla v+\sigma\Delta\omega+\sigma\nabla\mathrm{div}\omega,$$

$$G_3(\sigma,\omega,v,r)\sim\omega\cdot\nabla v+\sigma\mathrm{div}\omega+v\mathrm{div}\omega+\sigma\Delta v+v\Delta v+\sigma r+vr+\sigma\mathbb{S}:\nabla\omega$$
$$+\mathbb{S}:\nabla\omega+\sigma\nabla\sigma\nabla v+v\nabla\sigma\nabla v+\sigma|\nabla v|^2+v|\nabla v|^2,$$

$$G_4(\sigma,\omega,v,r)\sim\omega\cdot\nabla r+vr+\sigma\nabla\sigma\nabla r.$$

Let $U=(\sigma,\omega,v,r)$, $G=(G_1,G_2,G_3,G_4)$, $F=(0,\alpha f,0,0)$.

Then, we can get

$$\begin{pmatrix}\sigma_t\\ \omega_t\\ v_t\\ r_t\end{pmatrix}=\begin{pmatrix}-\lambda_1\mathrm{div}\omega-\lambda_4\omega\cdot\nabla\sigma+G_1(U)\\ -\lambda_1\nabla\sigma-\lambda_2\nabla v+\left(\bar{\mu}\Delta\omega+\left(\bar{\mu}+\bar{\lambda}\right)\nabla\mathrm{div}\omega\right)+G_2(U)+\alpha f\\ -\lambda_2\mathrm{div}\omega+\gamma_1\Delta v+\lambda_3 r+G_3(U)\\ d_0\Delta r-\lambda_3 r+G_4(U)\end{pmatrix}. \quad (2.8)$$

Referred to the way of [14], we are using A and $\mathrm{B}_\omega$ to denote the $(n+2)\times(n+2)$ matrix differential operators, we can write as

$$\mathrm{A}=\begin{pmatrix}0 & -\lambda_1\mathrm{div} & 0 & 0\\ -\lambda_1\nabla & \bar{\mu}\Delta+\left(\bar{\mu}+\bar{\lambda}\right)\nabla\mathrm{div} & -\lambda_2\nabla & 0\\ 0 & -\lambda_2\mathrm{div} & \gamma_1\Delta & \lambda_3\\ 0 & 0 & 0 & d_0\Delta-\lambda_3\end{pmatrix},\quad \mathrm{B}_\omega=\begin{pmatrix}-\lambda_4\omega\cdot\nabla & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

From (2.6), we can write as

$$U_t=\left(\mathrm{A}+\mathrm{B}_\omega\right)U+G(U)+F. \quad (2.9)$$

To obtain the periodic solution of the above problem, we consider the following linear system

$$\begin{cases}\sigma_t+\lambda_1\mathrm{div}\omega+\lambda_4 u\cdot\nabla\sigma=G_1(W),\\ \omega_t+\lambda_1\nabla\sigma+\lambda_2\nabla v-\left(\bar{\mu}\Delta\omega+\left(\bar{\mu}+\bar{\lambda}\right)\nabla\mathrm{div}\omega\right)=G_2(W)+\alpha f,\\ v_t+\lambda_2\mathrm{div}\omega-\gamma_1\Delta v-\lambda_3 r=G_3(W),\\ r_t-d_0\Delta r+\lambda_3 r=G_4(W),\end{cases} \quad (2.10)$$

for any given $W=(\rho,u,\theta,Z)$ satisfying

$$\rho(t)\in H^N\left(\mathbb{R}^n\right),(u,\theta,Z)(t)\in H^{N+1}\left(\mathbb{R}^n\right),\forall t\geq 0.$$

That's to say (2.10) can be written by

$$U_t=\left(\mathrm{A}+\mathrm{B}_u\right)U+G(W)+F. \quad (2.11)$$

From (2.11), we use the Duhamels principle to determine the solution of the system (2.10)

$$U(t)=S_u(t,s)U(s)+\int_s^t S_u(t,\tau)(G(W)+F)(\tau)\mathrm{d}\tau,\quad t\ge s, \tag{2.12}$$

where $S_u(t,s)$ is the corresponding linearized solution operator.

## 3. Energy Estimates

In this section, we assume that $f(t,x)\in H^{N-1}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ for all $t\ge 0$. We list the following inequalities for later use; cf. [17].

**Lemma 3.1.** *Let* $u\in C_0^\infty(\mathbb{R}^n)$, *we have*

$$\begin{aligned}&(1)\quad \|u\|_{L^\infty}^2\le C\|\nabla^{m+1}u\|\|\nabla^{m-1}u\|,\quad n=2m;\\&(2)\quad \|u\|_{L^\infty}^2\le C\|\nabla^{m+1}u\|\|\nabla^{m}u\|,\quad n=2m+1.\end{aligned} \tag{3.1}$$

**Lemma 3.2.** *Assume that* $f,g\in C_0^\infty(\mathbb{R}^n)$. *If* $\forall\varepsilon>0$, *we have*

$$\left|\int_{\mathbb{R}^n} f\cdot g\cdot h\mathrm{d}x\right|\le\varepsilon\|\nabla^{m-1}f\|_2^2+\frac{C}{\varepsilon}\|g\|^2\|h\|^2, \tag{3.2}$$

$$\left|\int_{\mathbb{R}^n} f\cdot g\cdot h\mathrm{d}x\right|\le\varepsilon\|f\|^2+\frac{C}{\varepsilon}\|\nabla^{m-1}g\|_2^2\|h\|^2, \tag{3.3}$$

where $m$ is defined in Lemma 3.1.

### 3.1. The Energy Estimates on the Higher Order Derivatives

**Lemma 3.3.** *Assume that* $n\ge 5$, $N\ge n+2$. *Let* $(\sigma,\omega,v,r)$ *be the solution of* (2.10), *it holds that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1\|\nabla U(t)\|_{N-1}^2+\sum_{1\le|k|\le N-1}\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\right)\\&+c_2\left(\|\nabla^2\sigma(t)\|_{N-2}^2+\|\nabla^2(\omega,v)(t)\|_{N-1}^2+\|\nabla r(t)\|_N^2\right)\\&\le C\left(\|W(t)\|_{N-1}^2+\|W(t)\|_{N-1}^4\right)\left(\|\nabla\rho(t)\|_{N-1}^2+\|\nabla(u,\theta)(t)\|_N^2+\|Z(t)\|_{N+1}^2\right)\\&\quad+C\|f(t)\|_{N-1}^2+C\left(\|u(t)\|_{N-1}^2+\|u(t)\|_{N-1}\right)\|\nabla^2\sigma(t)\|_{N-2}^2\\&\quad+C\|\nabla^N u(t)\|\|\nabla^2\sigma(t)\|_{N-3}\|\nabla^2\sigma(t)\|_{N-2},\end{aligned} \tag{3.4}$$

where constants $c_1>0$ and $c_2>0$ with $c_1$ is large suitably.

*Proof.* For each multi-index $k$ with $1\le|k|\le N$. Applying $\partial_x^k$ to $(2.10)_1$ - $(2.10)_3$, multiplying them by $\partial_x^k\sigma$, $\partial_x^k\omega$, $\partial_x^k v$ respectively, and using Young's inequality and integration by parts over $\mathbb{R}^n$, one can get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x^k\sigma\|^2+\|\partial_x^k\omega\|^2+\|\partial_x^k v\|^2\right)+\bar{\mu}\|\partial_x^k\nabla\omega\|^2+(\bar{\mu}+\bar{\lambda})\|\partial_x^k\mathrm{div}\omega\|^2+\gamma_1\|\partial_x^k\nabla v\|^2\\&=-\lambda_4\int_{\mathbb{R}^n}\partial_x^k\sigma\cdot\partial_x^k(u\cdot\nabla\sigma)\mathrm{d}x+\int_{\mathbb{R}^n}\partial_x^k\sigma\cdot\partial_x^k G_1(W)\mathrm{d}x+\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\partial_x^k G_2(W)\mathrm{d}x\\&\quad+\alpha\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\partial_x^k f\mathrm{d}x+\int_{\mathbb{R}^n}\partial_x^k v\cdot\partial_x^k G_3(W)\mathrm{d}x+\lambda_3\int_{\mathbb{R}^n}\partial_x^k v\cdot\partial_x^k r\mathrm{d}x\\&=I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned} \tag{3.5}$$

For each multi-index $k$ with $|k|=1$, using Hölder's inequality and Lemma 3.1, we have

$$I_1=\lambda_4\int_{\mathbb{R}^n}\left(\partial_x^k\partial_x^k\sigma\right)\cdot\left(u\cdot\nabla\sigma\right)\mathrm{d}x\le\lambda_4\left\|u\nabla^2\sigma\right\|_{L^1}\left\|\nabla\sigma\right\|_{L^\infty}\le C\|u\|\left\|\nabla^2\sigma\right\|\left\|\nabla^m\sigma\right\|_2,$$

where $m=\frac{n-1}{2}$ if $n$ is odd, $m=\frac{n}{2}$ if $n$ is even. For each multi-index $k$ with $2\le|k|\le N$, using Lemma 3.1, Lemma 3.2 and the fact that for $n\ge5$ and $N\ge n+2$, $m+2+|k|<2N$, we have

$$\begin{aligned}I_1&=-\frac{1}{2}\lambda_4\int_{\mathbb{R}^n}\left|\partial_x^k\sigma\right|^2\cdot\nabla\cdot u\mathrm{d}x+C_k^l\sum_{0\le l<k}\int_{\mathbb{R}^n}\left(\partial_x^k\sigma\right)\cdot\left(\partial_x^{k-l}u\cdot\partial_x^l\nabla\sigma\right)\mathrm{d}x\\&\le C\|\nabla u\|_{L^\infty}\left\|\partial_x^k\sigma\right\|^2+C_k^l\sum_{0\le l<k}\left\|\partial_x^{k-l}u\cdot\partial_x^l\nabla\sigma\right\|\left\|\partial_x^k\sigma\right\|\\&\le C\|\nabla u\|_{N-2}\left\|\nabla^2\sigma\right\|_{N-2}^2+C\left\|\nabla^N u\right\|\left\|\nabla^2\sigma\right\|_{N-3}\left\|\nabla^2\sigma\right\|_{N-2}.\end{aligned}$$

Thus for $n\ge5$ and $N\ge n+2$, $m\ge2$ and $m+2<n+1<N$, we can get

$$I_1\le C\|u\|_{N-1}\left\|\nabla^2\sigma\right\|_{N-2}^2+C\left\|\nabla^N u\right\|\left\|\nabla^2\sigma\right\|_{N-3}\left\|\nabla^2\sigma\right\|_{N-2}.\tag{3.6}$$

For the term $I_3$, let $\forall k'\le k$ with $|k'|=1$, we have

$$I_3=-\int_{\mathbb{R}^n}\partial_x^{k'}\partial_x^k\omega\partial_x^{k-k'}G_2(W)\mathrm{d}x\le\varepsilon\left\|\nabla\partial_x^k\omega\right\|^2+C_\varepsilon\left\|\partial_x^{k-k'}G_2(W)\right\|^2.\tag{3.7}$$

Now, we need to prove that the term

$$\begin{aligned}\left\|\partial_x^{k-k'}G_2(W)\right\|\le&\left\|\partial_x^{k-k'}\left((u\cdot\nabla)u\right)\right\|+\left\|\partial_x^{k-k'}(\rho\nabla\rho)\right\|+\left\|\partial_x^{k-k'}(\theta\nabla\rho)\right\|+\left\|\partial_x^{k-k'}(\rho\nabla\theta)\right\|\\&+\left\|\partial_x^{k-k'}(\theta\nabla\theta)\right\|+\left\|\partial_x^{k-k'}(\rho\Delta u)\right\|+\left\|\partial_x^{k-k'}(\rho\nabla\mathrm{div}u)\right\|,\end{aligned}\tag{3.8}$$

can be estimated by $C\|(\rho,u,\theta)\|_{N-1}\left(\|\nabla\rho\|_{N-1}+\|\nabla(u,\theta)\|_N\right)$, where we show the estimate on the term of $\partial_x^{k-k'}(\rho\nabla\mathrm{div}u)$ which can be written as

$$\begin{aligned}\partial_x^{k-k'}(\rho\nabla\mathrm{div}u)=&\sum_{k''\le k-k',|k''|\le m}\partial_x^{k''}\rho\partial_x^{k-k'-k''}(\nabla\mathrm{div}u)\\&+\sum_{k''\le k-k',|k''|\ge m+1}\partial_x^{k''}\rho\partial_x^{k-k'-k''}(\nabla\mathrm{div}u).\end{aligned}\tag{3.9}$$

Notice that for any $k''\le k-k'$ with $|k''|\le m$, $N-1\ge|k''|+(m+1)$; for any $k''\le k-k'$ with $|k''|\ge m+1$, $N+1\ge|k-k'-k''|+2+(m+1)$. Then, Lemma 3.1 implies that

$$\left\|\partial_x^{k-k'}(\rho\nabla\mathrm{div}u)\right\|\le C\|\rho\|_{N-1}\|\mathrm{div}u\|_N,$$

which is the desired estimate. Note that the other terms can be estimated similarly. Hence, we have

$$I_3\le\varepsilon\left\|\nabla^2\omega\right\|_{N-1}^2+C_\varepsilon\|(\rho,u,\theta)\|_{N-1}^2\left(\|\nabla\rho\|_{N-1}^2+\|\nabla(u,\theta)\|_N^2\right).\tag{3.10}$$

Similar to $I_1$ and $I_3$, it follows from Lemma 3.1 and Lemma 3.2 that

$$I_2\le\varepsilon\left\|\partial_x^k\sigma\right\|^2+C_\varepsilon\left\|\partial_x^k(\rho\cdot\nabla u)\right\|^2\le\varepsilon\left\|\nabla^2\sigma\right\|_{N-2}^2+C_\varepsilon\|\rho\|_{N-1}^2\|\nabla u\|_N^2,\tag{3.11}$$

and

$$
\begin{aligned}
I_5 &\le \varepsilon\left\|\nabla^2 v\right\|_{N-1}^2 + C_\varepsilon\|W\|_{N-1}^2\left(\|\nabla\rho\|_{N-1}^2 + \left\|\nabla(u,\theta)\right\|_N^2 + \|Z\|_{N+1}^2\right) \\
&\quad + C_\varepsilon\left\|\partial_x^{k-k'}\left\{\rho\cdot\mathbb{S}:\nabla u + \mathbb{S}:\nabla u\right\}\right\| \\
&\quad + C_\varepsilon\left\|\partial_x^{k-k'}\left\{\rho\cdot\nabla\rho\nabla\theta + \theta\cdot\nabla\rho\nabla\theta + \rho\cdot|\nabla\theta|^2 + \theta\cdot|\nabla\theta|^2\right\}\right\| \\
&\le \varepsilon\left\|\nabla^2 v\right\|_{N-1}^2 + C_\varepsilon\left(\|W\|_{N-1}^2 + \|W\|_{N-1}^4\right)\left(\|\nabla\rho\|_{N-1}^2 + \left\|\nabla(u,\theta)\right\|_N^2 + \|Z\|_{N+1}^2\right).
\end{aligned}
$$

Now we turn to $I_4$, for any $k' \le k$ with $|k'| = 1$, we have

$$
I_4 = -\alpha\int_{\mathbb{R}^n}\partial_x^{k'}\partial_x^k\omega\cdot\partial_x^{k-k'}f\mathrm{d}x \le \varepsilon\left\|\nabla^2\omega\right\|_{N-1}^2 + C_\varepsilon\|f\|_{N-1}^2. \tag{3.12}
$$

For the term $I_6$, we can get

$$
I_6 \le \frac{\lambda_3}{2}\left\|\partial_x^k v\right\|^2 + \frac{\lambda_3}{2}\left\|\partial_x^k r\right\|^2 \le \frac{\lambda_3}{2}\left\|\nabla^2 v\right\|_{N-1}^2 + \frac{\lambda_3}{2}\|\nabla r\|_{N-1}^2. \tag{3.13}
$$

Now, we estimate the unknown function $Z$ and $\nabla Z$. Multiplying $\partial_x^k$ (2.10)$_4$ by $\partial_x^k r$ and integrating with respect to $x$ over $\mathbb{R}^n$, we can get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^k r\right\|^2 + d_0\left\|\partial_x^k\nabla r\right\|^2 + \lambda_3\left\|\partial_x^k r\right\|^2 = \int_{\mathbb{R}^n}\partial_x^k r\partial_x^k G_4(W)\mathrm{d}x. \tag{3.14}
$$

Similar to the estimation on $I_3$, let $\forall k' \le k$ with $|k'| = 1$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}\partial_x^k r\partial_x^k G_4(W)\mathrm{d}x &\le -\int_{\mathbb{R}^n}\partial_x^{k+k'}r\cdot\partial_x^{k-k'}G_4(W)\mathrm{d}x \\
&\le \varepsilon\left\|\nabla\partial_x^k r\right\|^2 + C_\varepsilon\left\|\partial_x^{k-k'}G_4(W)\right\|^2.
\end{aligned} \tag{3.15}
$$

Then, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla r\|_{N-1}^2 + (d_0 - \varepsilon)\left\|\nabla^2 r\right\|_{N-1}^2 + \lambda_3\|\nabla r\|_{N-1}^2 \\
&\le C_\varepsilon\left(\|W\|_{N-1}^2 + \|W\|_{N-1}^4\right)\left(\|\nabla\rho\|_{N-1}^2 + \left\|\nabla(u,\theta)\right\|_N^2 + \|Z\|_{N+1}^2\right).
\end{aligned} \tag{3.16}
$$

Meanwhile, to estimate $\left\|\nabla\partial_x^k\sigma\right\|^2$ for $|k| = 1, 2, \cdots, N-1$, by applying $\partial_x^k$ to (2.10)$_2$, multiplying them by $\nabla\partial_x^k\sigma$ and integrating them on $\mathbb{R}^n$, we obtain

$$
\begin{aligned}
\lambda_1\int_{\mathbb{R}^n}\left|\nabla\partial_x^k\sigma\right|^2\mathrm{d}x &= -\int_{\mathbb{R}^n}\partial_x^k\omega_t\cdot\nabla\partial_x^k\sigma\mathrm{d}x - \lambda_2\int_{\mathbb{R}^n}\nabla\partial_x^k v\cdot\nabla\partial_x^k\sigma\mathrm{d}x \\
&\quad + \bar{\mu}\int_{\mathbb{R}^n}\Delta\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x + \left(\bar{\mu} + \bar{\lambda}\right)\int_{\mathbb{R}^n}\nabla\partial_x^k\mathrm{div}\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n}\partial_x^k\left(G_2(W) + \alpha f\right)\cdot\nabla\partial_x^k\sigma\mathrm{d}x.
\end{aligned} \tag{3.17}
$$

By applying $\nabla\partial_x^k$ to (2.10)$_1$, multiplying them by $\partial_x^k\omega$, integrating them on $\mathbb{R}^n$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma_t\mathrm{d}x + \lambda_1\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\nabla\omega\mathrm{d}x + \lambda_4\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k(u\cdot\nabla\sigma)\mathrm{d}x \\
&= \int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k G_1(W)\mathrm{d}x.
\end{aligned} \tag{3.18}
$$

Meanwhile, using integration by parts and putting (3.18) into (3.17), we have

$$
\begin{aligned}
&\lambda_1\left\|\nabla\partial_x^k\sigma\right\|_{N-1}^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\\
&=-\lambda_1\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\nabla\omega\mathrm{d}x-\lambda_4\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\left(u\cdot\nabla\sigma\right)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^kG_1\left(W\right)\mathrm{d}x-\lambda_2\int_{\mathbb{R}^n}\nabla\partial_x^kv\cdot\nabla\partial_x^k\sigma\mathrm{d}x+\bar{\mu}\int_{\mathbb{R}^n}\Delta\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\\
&\quad+\left(\bar{\mu}+\bar{\lambda}\right)\int_{\mathbb{R}^n}\nabla\partial_x^k\mathrm{div}\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x+\int_{\mathbb{R}^n}\partial_x^k\left(G_2\left(W\right)+\alpha f\right)\cdot\nabla\partial_x^k\sigma\mathrm{d}x.
\end{aligned}
\tag{3.19}
$$

By using Hölder's inequality, it holds that

$$
\begin{aligned}
&-\lambda_1\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\nabla\omega\mathrm{d}x-\lambda_4\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\left(u\cdot\nabla\sigma\right)\mathrm{d}x+\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^kG_1\left(W\right)\mathrm{d}x\\
&\le C\left\|\nabla\partial_x^k\omega\right\|^2+C\left\|\partial_x^k\left(u\cdot\nabla\sigma\right)\right\|^2+C\left\|\partial_x^kG_1\left(W\right)\right\|^2.
\end{aligned}
\tag{3.20}
$$

And by using Youngs inequality, we obtain

$$
\begin{cases}
\bar{\mu}\int_{\mathbb{R}^n}\Delta\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\le\frac{\lambda_1}{8}\left\|\nabla\partial_x^k\sigma\right\|^2+\frac{\bar{\mu}^2}{\lambda_1}\left\|\nabla^2\partial_x^k\omega\right\|^2;\\
\left(\bar{\mu}+\bar{\lambda}\right)\int_{\mathbb{R}^n}\nabla\partial_x^k\mathrm{div}\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\le\frac{\lambda_1}{8}\left\|\nabla\partial_x^k\sigma\right\|^2+\frac{\left(\bar{\mu}+\bar{\lambda}\right)^2}{\lambda_1}\left\|\nabla^2\partial_x^k\omega\right\|^2;\\
-\lambda_2\int_{\mathbb{R}^n}\nabla\partial_x^kv\cdot\nabla\partial_x^k\sigma\mathrm{d}x\le\frac{\lambda_1}{8}\left\|\nabla\partial_x^k\sigma\right\|^2+\frac{\lambda_2^2}{\lambda_1}\left\|\nabla\partial_x^kv\right\|^2;\\
\int_{\mathbb{R}^n}\partial_x^k\left(G_2\left(W\right)+\alpha f\right)\cdot\nabla\partial_x^k\sigma\mathrm{d}x\le\frac{\lambda_1}{8}\left\|\nabla\partial_x^k\sigma\right\|^2+\frac{1^2}{\lambda_1}\left\|\partial_x^k\left(G_2\left(W\right)+\alpha f\right)\right\|^2.
\end{cases}
\tag{3.21}
$$

Similar to the estimation on $I_3$, we have

$$
\begin{aligned}
&\left\|\partial_x^k\left(u\cdot\nabla\sigma\right)\right\|^2+\left\|\partial_x^kG_1\left(W\right)\right\|^2+\left\|\partial_x^k\left(G_2\left(W\right)+\alpha f\right)\right\|^2\\
&\le C\left(\left\|\left(\rho,u,\theta\right)\right\|_{N-1}^2+\left\|\left(\rho,u,\theta\right)\right\|_{N-1}^4\right)\left(\left\|\nabla\rho\right\|_{N-1}^2+\left\|\nabla\left(u,\theta\right)\right\|_N^2\right)\\
&\quad+C\left\|\partial_x^kf\right\|^2+C\left\|u\right\|_{N-1}^2\left\|\nabla^2\sigma\right\|_{N-2}^2.
\end{aligned}
\tag{3.22}
$$

Hence, from (3.19) to (3.22), we can get

$$
\begin{aligned}
&\frac{\lambda_1}{2}\left\|\nabla^2\sigma\right\|_{N-2}^2+\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|k|\le N-1}\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\\
&\le C\left(\left\|W\right\|_{N-1}^2+\left\|W\right\|_{N-1}^4\right)\left(\left\|\nabla\rho\right\|_{N-1}^2+\left\|\nabla\left(u,\theta\right)\right\|_N^2\right)+C\left\|\nabla^2\left(\omega,v\right)\right\|_{N-2}^2\\
&\quad+C\left\|f\right\|_{N-1}^2+C\left\|u\right\|_{N-1}^2\left\|\nabla^2\sigma\right\|_{N-2}^2.
\end{aligned}
\tag{3.23}
$$

Combining $I_i\left(i=1,2,3,4,5,6\right)$, (3.17) and (3.23) yields the inequality (3.4). This completes the proof of the lemma.

## 3.2. The Energy Estimates on the Lower Order Derivatives

In this subsection, the usual energy method does not work here for the system (2.10) because the zero order derivative term $\int_{\mathbb{R}^n}|v|^2\mathrm{d}x$ cannot be controlled. In order to overcome this difficulty, we would like to rewrite the system (2.10) and then obtain a new system about some modes

$$\sigma, \quad \omega, \quad V = v + r, \quad r. \tag{3.24}$$

Then the system (2.10) could be rewritten as

$$\begin{cases} \sigma_t + \lambda_1 \operatorname{div}\omega + \lambda_4 u \cdot \nabla \sigma = G_1(W), \\ \omega_t + \lambda_1 \nabla \sigma + \lambda_2 \nabla V - \left(\bar{\mu}\Delta\omega + \left(\bar{\mu} + \bar{\lambda}\right)\nabla \operatorname{div}\omega\right) - \lambda_2 \nabla r = G_2(W) + \alpha f, \\ V_t + \lambda_2 \operatorname{div}\omega - \gamma_1 \Delta V + (\gamma_1 - d_0)\Delta r = G_3(W) + G_4(W), \\ r_t - d_0 \Delta r + \lambda_3 r = G_4(W), \end{cases} \tag{3.25}$$

**Lemma 3.4.** *Assume that* $n \geq 5$, $N \geq n+2$. *Let* $(\sigma, \omega, V, r)$ *be the solution of* (2.10), *it holds that*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(c_3 \left\|(\sigma,\omega,V,r)(t)\right\|^2 + \int_{\mathbb{R}^n} \omega \cdot \nabla \sigma \mathrm{d}x\right) \\ &+ c_4\left(\left\|\nabla\sigma(t)\right\|^2 + \left\|\nabla(\omega,V)(t)\right\|^2 + \left\|r(t)\right\|^2\right) \\ &\leq C\left(\left\|u(t)\right\| + \left\|u(t)\right\|^2 + \varepsilon\right)\left\|\nabla\sigma(t)\right\|_m^2 + C\left\|\nabla^m(\omega,V)(t)\right\|_1^2 + C\left\|f(t)\right\|_{L^1}^2 \\ &\quad + C\left(\left\|(\rho,u,\theta,Z)(t)\right\|_{m+1}^2 + \left\|(\rho,u,\theta,Z)(t)\right\|_{m+1}^4\right)\left(\left\|\nabla(\rho,u,\theta)(t)\right\|_1^2 + \left\|Z\right\|_1^2\right), \end{aligned} \tag{3.26}$$

where constants $c_3 > 0$, $c_4 > 0$ and $\varepsilon > 0$ with $c_3$ is suitably large and $\varepsilon$ is small enough.

*Proof.* Multiplying $(3.25)_1$, $(3.25)_2$ and $(3.25)_3$ by $\sigma$, $\omega$ and $V$, integration by part over $\mathbb{R}^n$ respectively, one obtains

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sigma\|^2 + \|\omega\|^2 + \|V\|^2\right) + \bar{\mu}\|\nabla\omega\|^2 + \left(\bar{\mu} + \bar{\lambda}\right)\|\operatorname{div}\omega\|^2 + \gamma_1\|\nabla V\|^2 \\ &= -\lambda_4 \int_{\mathbb{R}^n} \sigma \cdot (u \cdot \nabla\sigma)\mathrm{d}x + \int_{\mathbb{R}^n} \sigma \cdot G_1(W)\mathrm{d}x + \int_{\mathbb{R}^n} \omega \cdot G_2(W)\mathrm{d}x + \lambda_2 \int_{\mathbb{R}^n} \omega \cdot \nabla r \mathrm{d}x \\ &\quad + \alpha \int_{\mathbb{R}^n} \omega \cdot f \mathrm{d}x + \int_{\mathbb{R}^n} V \cdot \left(G_3(W) + G_4(W)\right)\mathrm{d}x - (\gamma_1 - d_0)\int_{\mathbb{R}^n} V \cdot \Delta r \mathrm{d}x \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned} \tag{3.27}$$

For the term $J_1$, we obtain

$$J_1 \leq C\|\nabla\sigma\|\|u\sigma\| \leq C\|u\|\|\nabla\sigma\|\left\|\nabla^{m-1}\sigma\right\|_2.$$

For $J_2$, by using Lemma 3.2, we have

$$J_2 = -\int_{\mathbb{R}^n} \sigma \cdot (\rho \operatorname{div} u)\mathrm{d}x \leq \varepsilon\left\|\nabla^{m-1}\sigma\right\|_2^2 + C_\varepsilon\|\rho\|^2\|\nabla u\|^2.$$

For $J_3$ and $J_6$, Lemma 3.1 and Lemma 3.2 give

$$J_3 \leq \varepsilon\left\|\nabla^{m-1}\omega\right\|_2^2 + C_\varepsilon\left\|(\rho,u,\theta)\right\|^2\left\|\nabla(\rho,u,\theta)\right\|_1^2.$$

$$J_6 \leq \varepsilon\left\|\nabla^{m-1}V\right\|_2^2 + C_\varepsilon\left(\left\|(\rho,u,\theta,Z)\right\|_{m+1}^2 + \left\|(\rho,u,\theta,Z)\right\|_{m+1}^4\right)\left(\left\|\nabla(\rho,u,\theta)\right\|_1^2 + \|Z\|_1^2\right).$$

For $J_4$, one can get

$$J_4 = -\lambda_2 \int_{\mathbb{R}^n} \operatorname{div}\omega \cdot r \mathrm{d}x \leq \frac{\bar{\mu} + \bar{\lambda}}{2}\|\operatorname{div}\omega\|^2 + \frac{\lambda_2^2}{2\left(\bar{\mu} + \bar{\lambda}\right)}\|r\|^2.$$

For $J_5$, we obtain

$$J_5 = \alpha \int_{\mathbb{R}^n} \omega \cdot f \mathrm{d}x \le \varepsilon \left\| \nabla^{m-1} \omega \right\|_2^2 + C_\varepsilon \| f \|_{L^1}^2 .$$

For $J_7$, we have

$$J_7 = (\gamma_1 - d_0) \int_{\mathbb{R}^n} \nabla V \cdot \nabla r \mathrm{d}x \le \frac{\gamma_1}{2} \|\nabla V\|^2 + \frac{(\gamma_1 - d_0)^2}{2\gamma_1} \|\nabla r\|^2 .$$

Due to $n \ge 5$, $N \ge n+2$, we get that $m-1 \ge 1$ and $m+1 \ge N-1$. Now we estimate the unknown function $Z$ and $\nabla Z$. Multiplying (3.25)$_4$ by $r$ and integrating with respect to $x$ over $\mathbb{R}^n$, we can get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r\|^2 + d_0 \|\nabla r\|^2 + (\lambda_3 - \varepsilon)\|r\|^2 \\ &\le C_\varepsilon \left( \left\|\nabla^{m-1} u\right\|_2^2 \|\nabla Z\|^2 + \left\|\nabla^{m-1}\theta\right\|_2^2 \|Z\|^2 + \left\|\nabla^{m-1}(\rho\nabla\rho)\right\|^2 \|\nabla Z\|^2 \right) \\ &\le C\left( \|(\rho,u,\theta,Z)\|_{m+1}^2 + \|(\rho,u,\theta,Z)\|_{m+1}^4 \right)\left( \|\nabla(\rho,u,\theta)\|_1^2 + \|Z\|_1^2 \right). \end{aligned} \tag{3.28}$$

Next, we need to estimate $\|\nabla\sigma\|^2$, similar to the estimation on $\left\|\nabla\partial_x^k\sigma\right\|^2$, we obtain

$$\begin{aligned} &\frac{\lambda_1}{2}\|\nabla\sigma\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \omega\cdot\nabla\sigma \mathrm{d}x \\ &\le C\|\nabla(\omega,V)\|_1^2 + C\|(\rho,u,\theta)\|_{m+1}^2 \|\nabla(\rho,u,\theta)\|_1^2 + C\|f\|_{L^1}^2 + C\|u\|^2\|\nabla\sigma\|^2 . \end{aligned} \tag{3.29}$$

Combining $J_j\ (j = 1,2,3,4,5,6,7)$, (3.28) and (3.29) yields the inequality (3.26). This completes the proof of the lemma.

## 4. Conclusion

In this section, we will present the proof of two Theorems in Section 1.

### 4.1. The Proof of Theorem 1.1

Let $S_0(t,s)$ be the solution operator for the case when $u = 0$, by spectral analysis, we have the following time decay properties for $S_0(t,s)$, cf. [13].

**Proposition 4.1.** *Assume* $1 \le p \le 2$, $n \ge 5$, *for any integer* $\alpha \ge 0$, *if* $U(s) \in H^\alpha(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, *then for* $\forall t \ge s$, *we have*

$$\left\|\partial_x^\alpha\left(S_0(t,s)U(t)\right)\right\| \le C(1+t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{\alpha}{2}} \|U(s)\|_{H^\alpha \cap L^p} . \tag{4.1}$$

By Proposition 4.1 and the estimates obtained in Section 3, the solution operator $S_u(t,s)$ has the decay estimates as follows.

**Proposition 4.2.** *Let* $(\sigma,\omega,v,r)$ *be a smooth solution to the system* (2.10), *integers* $n \ge 5$ *and* $\alpha \ge \left[\frac{n}{2}\right] + 2$, *there exist three positive constant* $c_5$ *and* $c_6$ *with* $c_5$ *being suitably large and* $\varepsilon(s,\alpha) > 0$, *such that for* $\forall t \ge s$, $s \in \mathbb{R}$ *and* $U(s) \in H^\alpha(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *if* $\sup_{t \ge s}\|u(t)\|_\alpha \le \varepsilon(s,\alpha)$.

Then it holds that

$$1)\ \left\|S_u(t,s)U(t)\right\| \le C(1+t-s)^{-\frac{n}{4}} \|U(s)\|_{H^\alpha \cap L^1} ; \tag{4.2}$$

$$2)\ \left\|\nabla\left(S_u(t,s)U(t)\right)\right\|_{\alpha-1} \le C(1+t-s)^{-\frac{n}{4}-\frac{1}{2}}\left\|U(s)\right\|_{H^\alpha \cap L^1}. \tag{4.3}$$

From the assumptions of time-periodic solution and global solution given above, we will prove Theorem 1.1 as follows and the proof is divided into two steps.

*Proof.* (Theorem 1.1). 1) Choosing the time $s=-kT$ for $k \in N$, we can suppose that there exists a time-periodic solution $U=U(t)$, $t \in \mathbb{R}$ with period $T>0$ for the system (2.10) with the initial data $U_0 = U(s)$ at any given time $s \in \mathbb{R}$. Then (2.12) can be written

$$U(t) = S_u(t,-kT)U_0 + \int_{-kT}^{t} S_u(t,s)\left(G(W)(s)+F(s)\right)\mathrm{d}s, \tag{4.4}$$

where $W=(\rho,u,\theta,Z)$, $U=(\sigma,\omega,v,r)$ and *f* are periodic on time with period $T>0$ in the system (2.10). Due to $N \ge n+2 > \left[\frac{n}{2}\right]+2$, from proposition 4.2, if $g \in H^N \cap L^1$, we obtain

$$\left\|S_u(t,s)g\right\|_N \le C(1+t-s)^{-\frac{n}{4}}\left\|g\right\|_{H^N \cap L^1} \to 0 \quad \text{as} \quad s \to -\infty. \tag{4.5}$$

In addition, since $L^2 \cap L^1$ is dense in $L^2$, from (4.5), we have

$$\left\|S_u(t,s)g\right\|_N \to 0 \quad \text{as} \quad s \to -\infty, k \to +\infty, \forall g \in H^N.$$

Therefore, since $\frac{n}{4}>1$ when $n \ge 5$, the fact that $\left\|S_u(t,s)g\right\|_N \le C(1+t-s)^{-\frac{n}{4}}\left\|g\right\|_{H^N \cap L^1}$ for $g \in H^N \cap L^1$, which means that the convergence of the integral can be guaranteed in (4.4). Then, it holds that

$$U(t) = \int_{-\infty}^{t} S_u(t,s)\left(G(W)(s)+F(s)\right)\mathrm{d}s, t \ge 0. \tag{4.6}$$

For any given perturbed solution $U=(\sigma,\omega,v,r)$, we can define a map $H[U](t)$:

$$H[U](t) = \int_{-\infty}^{t} S_w(t,s)\left(G(U)(s)+F(s)\right)\mathrm{d}s, t \ge 0. \tag{4.7}$$

And by (4.6), there exists a fixed point which is also the mild solution of (2.10) in map $H[U](t)$; on the contrary, suppose that there exist a unique fixed point in map *H*, denoted by $U_1(t)=(\sigma_1,\omega_1,v_1,r_1)(t)$. Because *F* and *f* have the same time period *T*, we set $U_2(t)=(\sigma_1,\omega_1,v_1,r_1)(T+t)$, then $S_{\omega_1}(t+T,s+T)=S_{\omega_2}(t,s)$ since $w_2(t)=w_1(T+t)$. We have

$$\begin{aligned} U_2(t) &= H[U_1](T+t) \\ &= \int_{-\infty}^{T+t} S_{w_1}(T+t,s)\left(G(U_1)(s)+F(s)\right)\mathrm{d}s \\ &= \int_{-\infty}^{t} S_{w_1}(T+t,T+s)\left(G(U_1)(T+s)+F(T+s)\right)\mathrm{d}s \\ &= \int_{-\infty}^{t} S_{w_2}(t,s)\left(G(U_2)(s)+F(s)\right)\mathrm{d}s \\ &= H[U_2](t). \end{aligned} \tag{4.8}$$

Thus from the uniqueness, $U_1 = U_2$ which is the desired periodic solution to the system (2.10).

2) For the periodic function *f*, we assume that

$$\int_0^T \left( \left| f(\cdot,t) \right|_{H^{N-1}}^2 + \left| f \right|_{L^1}^2 \right) \mathrm{d}t \le h_0,$$

and $h_0 > 0$ is small enough, then for some appropriate constant $\varepsilon_0$, $H$ has a unique fixed point in the space $X(0,T;\varepsilon_0)$. Firstly, Since the period of the time-periodic solution $U = (\sigma,\omega,v,r) = H(W)$ is also *T*, for $n \ge 5$ and $N \ge n+2$, from lemma 3.3 and lemma 3.4, we can choose a small constant $\varepsilon_1$, a large enough $c_7 > 0$ and a constant $c_8 > 0$ such that if $\sup_{0\le t\le T} \left\| u(t) \right\|_{N-1} \le \varepsilon_1$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left( c_7 \left\|(\sigma,\omega,v,r)\right\|_N^2 + \sum_{|k|\le N-1} \int_{\mathbb{R}^n} \partial_x^k \omega \cdot \nabla \partial_x^k \sigma \mathrm{d}x \right) \\
&+ c_8 \left( \left\|\nabla(\omega,v)\right\|_N^2 + \left\|r\right\|_{N+1}^2 + \left\|\nabla\sigma\right\|_{N-1}^2 \right) \\
&\le C\left( \left\|(\rho,u,\theta,Z)\right\|_{N-1}^2 + \left\|(\rho,u,\theta,Z)\right\|_{N-1}^4 \right)\left( \left\|\nabla\rho\right\|_{N-1}^2 + \left\|\nabla(u,\theta)\right\|_N^2 + \left\|Z\right\|_{N+1}^2 \right) \\
&\quad + C\left\| f \right\|_{H^{N-1}\cap L^1}^2 + C\left\|\nabla^N u\right\| \left\|\nabla^2\sigma\right\|_{N-3} \left\|\nabla^2\sigma\right\|_{N-2}.
\end{aligned} \tag{4.9}$$

Since the period of $U(t) = (\sigma,\omega,v,r)(\cdot,t)$ is *T*, and the inequality (4.9) is integrated with respect to $t$ in $[0,t]$, it can be got

$$\begin{aligned}
&\int_0^T \left( \left\|\nabla(\omega,v)\right\|_{N-1}^2 + \left\|r\right\|_{N-1}^2 + \left\|\nabla\sigma\right\|_{N-2}^2 \right)\mathrm{d}t \\
&\le C\int_0^T \left( \left\|W\right\|_{N-1}^2 + \left\|W\right\|_{N-1}^4 \right)\left( \left\|\nabla\rho\right\|_{N-1}^2 + \left\|\nabla(u,\theta)\right\|_N^2 + \left\|Z\right\|_{N+1}^2 \right)\mathrm{d}t \\
&\quad + C\int_0^T \left\| f \right\|_{H^{N-1}\cap L^1}^2 \mathrm{d}t + C\int_0^T \left\|\nabla^N u\right\| \left\|\nabla^2\sigma\right\|_{N-3} \left\|\nabla^2\sigma\right\|_{N-2} \mathrm{d}t.
\end{aligned} \tag{4.10}$$

Then, from the definition of $\left\|\left|(\rho,u,\theta,Z)\right|\right\|^2$, one can get

$$\begin{aligned}
&\int_0^T \left( \left\|\nabla(\omega,v)\right\|_{N-1}^2 + \left\|r\right\|_{N-1}^2 + \left\|\nabla\sigma\right\|_{N-2}^2 \right)\mathrm{d}t \\
&\le C\left( \left\|\left|W\right|\right\|^4 + \left\|\left|W\right|\right\|^6 \right) + C\int_0^T \left\| f \right\|_{H^{N-1}\cap L^1}^2 \mathrm{d}t + \frac{1}{4}\sup_{0\le t\le T}\left\|\nabla^2\sigma\right\|_{N-3}^2 \\
&\quad + C_0 \sup_{0\le t\le T}\left\|\nabla^2\sigma\right\|_{N-3} \int_0^T \left\|\nabla^2\sigma\right\|_{N-2}^2 \mathrm{d}t.
\end{aligned} \tag{4.11}$$

And choosing a small constant $\varepsilon_2 > 0$ such that $C_0\varepsilon_2 \le \frac{1}{2}$. Then if $\sup_{0\le t\le T}\left\|\nabla^2\sigma(t)\right\|_{N-3} \le \varepsilon_2$, from (4.11), it can be written as

$$\begin{aligned}
&\int_0^T \left( \left\|\nabla(\omega,v)\right\|_{N-1}^2 + \left\|r\right\|_{N-1}^2 + \left\|\nabla\sigma\right\|_{N-2}^2 \right)\mathrm{d}t \\
&\le C\left( \left\|\left|W\right|\right\|^4 + \left\|\left|W\right|\right\|^6 \right) + CT\sup_{0\le t\le T}\left\| f \right\|_{H^{N-1}\cap L^1}^2 + \frac{1}{2}\sup_{0\le t\le T}\left\|\nabla^2\sigma\right\|_{N-3}^2.
\end{aligned} \tag{4.12}$$

Secondly, from (4.6), one gets

$$\begin{aligned}\|U(t)\|_{N-1} &\le C\int_{-\infty}^{t}(1+t-s)^{-\frac{n}{4}}\left(\|G(W)(s)\|_{H^{N-1}\cap L^1}+\|f(s)\|_{H^{N-1}\cap L^1}\right)ds \\ &\le C\sum_{j=0}^{\infty}\int_{t-(j+1)T}^{t-jT}(1+t-s)^{-\frac{n}{4}}\|G(W)(s)\|_{H^{N-1}\cap L^1}ds \\ &\quad +C\int_{-\infty}^{t}(1+t-s)^{-\frac{n}{4}}\|f(s)\|_{H^{N-1}\cap L^1}ds.\end{aligned} \tag{4.13}$$

From (2.10), we have

$$\|G(W)(s)\|_{H^{N-1}\cap L^1}=\sum_{i=1}^{4}\|G_i(W)(s)\|_{H^{N-1}\cap L^1}. \tag{4.14}$$

Therefore, due to $\frac{n}{4}>1$ for $n\ge 5$, by (4.13), we can obtain

$$\begin{aligned}\|U(t)\|_{N-1} &\le C\sum_{j=0}^{\infty}\left\{\int_0^T(1+(j+1)T-s)^{-\frac{n}{2}}ds\right\}^{\frac{1}{2}}\left\{\int_0^T\left(\|W(s)\|_{N-1}^2+\|W(s)\|_{N-1}^4\right)\right. \\ &\quad \left.\times\left(\|\nabla\rho(s)\|_{N-1}^2+\|\nabla(u,\theta)(s)\|_N^2+\|Z\|_{N+1}^2\right)ds\right\}^{\frac{1}{2}}+C\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1} \\ &\le C\left(|||W|||^2+|||W|||^3\right)+C\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}.\end{aligned} \tag{4.15}$$

From (4.12) and (4.15), there exist two positive constants $C_1$, $C_2$ independent of $W$, such that

$$|||H(W)|||\le C_1\left(|||W|||^2+|||W|||^3\right)+C_2\sup_{0\le t\le T}\|f(t)\|_{H^{N-1}\cap L^1}. \tag{4.16}$$

Finally, set $W_1=(\rho_1,u_1,\theta_1,Z_1)$, $W_2=(\rho_2,u_2,\theta_2,Z_2)$, let $\bar{U}_1=H(W_1)$, $\bar{U}_2=H(W_2)$, if we set $\tilde{U}=\bar{U}_1-\bar{U}_2$. By (2.8), we have

$$|||H(W_1)-H(W_2)|||\le C_3\left(|||W_1|||+|||W_1|||^2+|||W_2|||+|||W_2|||^2\right)|||W_1-W_2|||, \tag{4.17}$$

where there is no relationship between $C_3$ and $W$. Assume that $a=|||W|||$ and $h=\int_0^T\left(|f(\cdot,t)|_{H^{N-1}}^2+|f|_{L^1}^2\right)dt$, we choose $0<a'\le b=\min\{\varepsilon(0,N-1),\varepsilon_1,\varepsilon_2\}$ and sufficiently small $h>0$, such that

$$C_1\left(a'^2+a'^3\right)+C_2h\le a' \quad\text{and}\quad 2C_3\left(a'+a'^2\right)<1.$$

Then, we can get through simple calculation

$$a'<\min\left\{\sqrt{\frac{1}{(4C_1)^2}-\frac{C_2h}{2C_1}}+\frac{1}{4C_1},1,\sqrt{\frac{1}{2C_3}+\frac{1}{4}}-\frac{1}{2},b\right\} \tag{4.18}$$

and

$$a'\ge\lim_{h\to 0}\left(-\sqrt{\frac{1}{(4C_1)^2}-\frac{C_2h}{2C_1}}+\frac{1}{4C_1}\right)=0. \tag{4.19}$$

Hence if $h\le h_1$ for a positive constant $h_1$, the inequalities (4.18) and (4.19) are both allowed to make sense. When $a_0$ satisfies (4.18) and (4.19), for $h_0 \le h_1$, $H$ is a contraction in the complete space $X(0,T;a_0)$. Then $H$ has a unique fixed point in $X(0,T;a_0)$. We have proved the Theorem 1.1.

### 4.2. The Proof of Theorem 1.2

In this section, the proving of Theorem 1.2 can be arranged as follows. Firstly, under the same conditions of Theorem 1.1, in order to prove the global existence of the solution to the Cauchy problem (1.1) and (1.9), we can suppose that $t_0 = 0$ without loss of generality. Let $(\rho^{per}, u^{per}, \theta^{per}, Z^{per})$ be the time-periodic solution constructed in Theorem 1.1 and $(\rho, u, \theta, Z)$ be a solution to (1.1) and (1.9). As in Section 2, denote

$$(\sigma,\omega,v,r) = \left(\rho-\bar{\rho}, \alpha u, \beta(\theta-\bar{\theta}), \frac{q\beta}{e_\theta(\bar{\rho},\bar{\theta})} Z\right),$$

$$(\sigma^{per},\omega^{per},v^{per},r^{per}) = \left(\rho^{per}-\bar{\rho}, \alpha u^{per}, \beta(\theta^{per}-\bar{\theta}), \frac{q\beta}{e_\theta(\bar{\rho},\bar{\theta})} Z^{per}\right).$$

Then the difference

$(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r}) = \left(\rho-\rho^{per}, \alpha(u-u^{per}), \beta(\theta-\theta^{per}), \frac{q\beta}{e_\theta(\bar{\rho},\bar{\theta})}(Z-Z^{per})\right)$ is a solution to the Cauchy problem

$$\begin{cases} \tilde{\sigma}_t + \lambda_1 \nabla \tilde{\omega} \\ = \tilde{G}_1(\tilde{\sigma}+\sigma^{per}, \tilde{\omega}+\omega^{per}, \tilde{v}+v^{per}, \tilde{r}+r^{per}) - \tilde{G}_1(\sigma^{per},\omega^{per},v^{per},r^{per}), \\ \tilde{\omega}_t + \lambda_1 \nabla\tilde{\sigma} + \lambda_2 \nabla \tilde{v} - \left(\bar{\mu}\Delta\tilde{\omega} + (\bar{\mu}+\bar{\lambda})\nabla \mathrm{div}\tilde{\omega}\right) \\ = \tilde{G}_2(\tilde{\sigma}+\sigma^{per}, \tilde{\omega}+\omega^{per}, \tilde{v}+v^{per}, \tilde{r}+r^{per}) - \tilde{G}_2(\sigma^{per},\omega^{per},v^{per},r^{per}), \\ \tilde{v}_t + \lambda_2 \mathrm{div}\tilde{\omega} - \gamma_1 \Delta \tilde{v} - \lambda_3 \tilde{r} \\ = \tilde{G}_3(\tilde{\sigma}+\sigma^{per}, \tilde{\omega}+\omega^{per}, \tilde{v}+v^{per}, \tilde{r}+r^{per}) - \tilde{G}_3(\sigma^{per},\omega^{per},v^{per},r^{per}), \\ \tilde{r}_t - d_0 \Delta \tilde{r} + \lambda_3 \tilde{r} \\ = \tilde{G}_4(\tilde{\sigma}+\sigma^{per}, \tilde{\omega}+\omega^{per}, \tilde{v}+v^{per}, \tilde{r}+r^{per}) - \tilde{G}_4(\sigma^{per},\omega^{per},v^{per},r^{per}), \end{cases} \tag{4.20}$$

$$\begin{aligned} &(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})\big|_{t=0} = (\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0) \\ &= \left(\rho_0-\rho^{per}(0), \alpha(u_0-u^{per}(0)), \beta(\theta_0-\theta^{per}(0)), \frac{q\beta}{e_\theta(\bar{\rho},\bar{\theta})}(Z_0-Z^{per}(0))\right), \end{aligned} \tag{4.21}$$

where $\tilde{G}_1(\sigma,\omega,v,r) = G_1(\sigma,\omega,v,r) - \lambda_4 \nabla\sigma\cdot\omega$, $\tilde{G}_2(\sigma,\omega,v,r) = G_2(\sigma,\omega,v,r)$, $\tilde{G}_3(\sigma,\omega,v,r) = G_3(\sigma,\omega,v,r)$. Then our problem is turned to prove the global existence and decay estimates on the solution to the Cauchy problem (4.20) and (4.21). Define the function space of Cauchy problem (4.20) and (4.21) by $\tilde{X}(0,\infty)$, where for $0 \le t_1 < t_2 \le \infty$. Thus we have

$$\tilde{X}(t_1,t_2)=\Big\{(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})(t);\tilde{\sigma}(t,x)\in C^0\left(t_1,t_2;H^{N-1}(\mathbb{R}^n)\right)\cap C^1\left(t_1,t_2;H^{N-2}(\mathbb{R}^n)\right),$$
$$(\tilde{\omega},\tilde{v})(t,x)\in C^0\left(t_1,t_2;H^{N-1}(\mathbb{R}^n)\right)\cap C^1\left(t_1,t_2;H^{N-3}(\mathbb{R}^n)\right),$$
$$\nabla\tilde{\sigma}\in L^2\left(t_1,t_2;H^{N-2}(\mathbb{R}^n)\right),\nabla\tilde{\omega},\nabla\tilde{v}\in L^2\left(t_1,t_2;H^{N-1}(\mathbb{R}^n)\right),$$
$$\tilde{r}(t,x)\in L^2\left(t_1,t_2;H^N(\mathbb{R}^n)\right)\Big\}, \tag{4.22}$$

with the norm $\tilde{N}(t_1,t_2)$ is given by

$$\tilde{N}(t_1,t_2)^2=\sup_{t_1\le t\le t_2}\left\|(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})(t)\right\|_{N-1}^2 + \int_{t_1}^{t_2}\left(\left\|\nabla\tilde{\sigma}(t)\right\|_{N-2}^2+\left\|\nabla(\tilde{\omega},\tilde{v})(t)\right\|_{N-1}^2+\left\|\tilde{r}(t)\right\|_N^2\right)\mathrm{d}t. \tag{4.23}$$

Notice that $(\sigma^{per},\omega^{per},v^{per},r^{per})\in\tilde{X}(0,T)$. As usual, the local existence of the Cauchy problem (4.20) and (4.21) can be given by the standard argument of the contracting map theorem. Hence we omit the details of this proof.

**Proposition 4.3.** (*The local existence*). *Assume that* $(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\in H^{N-1}(\mathbb{R}^n)$ *such that* $\inf_{x\in\mathbb{R}^n}\{\tilde{\sigma}_0+\bar{\rho},\tilde{v}_0+\bar{\theta}\}>0$ *and* $0\le\tilde{r}_0(x)\le 1$. *Then there exists a constant* $T_0>0$ *depending on* $\tilde{N}(0,0)$ *such that the Cauchy problem* (4.20) *and* (4.21) *has a unique solution* $(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})\in X(0,T_0)$, *which satisfies*

$$\tilde{N}(0,T_0)\le C_4\tilde{N}(0,0), \tag{4.24}$$

where $C_4$ is independent of $\tilde{N}(0,0)$.

**Proposition 4.4.** (*A priori estimates*). *Suppose that* $(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\in H^{N-1}(\mathbb{R}^n)$, *let* $(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})\in X(0,T_1)$ *for some positive constant* $T_1$ *be a solution of the Cauchy problem* (4.20) *and* (4.21). *Thus if the solution for sufficiently small positive constants* $\delta$ *and* $C_5$ *satisfies*

$$\sup_{0\le t\le T_1}\left\|(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})(t)\right\|\le\delta. \tag{4.25}$$

It holds the following estimate that

$$\left\|(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})(t)\right\|_{N-1}^2+\int_0^t\left(\left\|\nabla\tilde{\sigma}(s)\right\|_{N-2}^2+\left\|\nabla(\tilde{\omega},\tilde{v})(s)\right\|_{N-1}^2+\left\|\tilde{r}(s)\right\|_N^2\right)\mathrm{d}t \le C_5\left\|(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\right\|_{N-1}^2, \tag{4.26}$$

where $\forall t\in[0,T_1]$.

**Remark 4.1.** *Here* $C_4$ *is independent of* $\varepsilon$ *and* $\delta$, *set* $\delta=\max\left\{2\varepsilon,\frac{3\sqrt{C_4}\varepsilon}{2}\right\}$, *such that*

$$\left\|(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})(t)\right\|_{H^{N-1}}^2\le C_4\left\|(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\right\|_{N-1}^2\le\left(\frac{2\delta}{3}\right)^2. \tag{4.27}$$

Then, the global solution to the Cauchy problem (4.20) and (4.21) will be obtained by combining Propositions 4.3 and 4.4 by the standard continuity argument. Finally, we have the following decay property of the solution $(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})$.

**Proposition 4.5.** [14] *Assume that* $(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\in H^{N-1}(\mathbb{R}^n)$ *be such that* $\|(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\|_{H^{N-1}}$ *is small enough and* $\|(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\|_{L^1}$ *is bounded. Then the solution* $(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})$ *to* (4.20) *and* (4.21) *has the decay property*

$$\|(\tilde{\sigma},\tilde{\omega},\tilde{v},\tilde{r})\|_{N-1}\le C(1+t)^{-\frac{n}{4}}\|(\tilde{\sigma}_0,\tilde{\omega}_0,\tilde{v}_0,\tilde{r}_0)\|_{H^{N-1}\cap L^1}. \tag{4.28}$$

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix

In this section, we will give the proof of propositions used by Theorem 1.1 and 1.2.

### A.1. The Proof of Proposition 4.2

*Proof.* Let us rewrite the problem (2.11) when $G(W)=0$, $f=0$ as follow:

$$U(t)=S_0(t,s)U(s)+\int_s^t S_0(t,\tau)(-\mathrm{B}_u U)(\tau)\mathrm{d}\tau. \tag{A.1}$$

In addition, by the proof of Lemma 3.3 in Section 3, when $G(W)=0$, $f=0$, and if the condition $\sup_{t\geq s}\|u(t)\|_\alpha \leq \varepsilon(s,\alpha)$ is tenable, we can easily get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(c_5\|\nabla(\sigma,\omega,v,r)(\cdot,t)\|_{\alpha-1}^2+\sum_{1\leq|k|\leq\alpha-1}\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x\right)\\&+c_6\left(\|\nabla^2(\omega,v)\|_{\alpha-1}^2+\|\nabla r\|_{\alpha-1}^2+\|\nabla^2\sigma\|_{\alpha-2}^2\right)\leq 0.\end{aligned} \tag{A.2}$$

We can construct an energy function $\phi_\alpha(U(t))$ on the higher order derivatives by

$$\phi_\alpha(U(t))=c_5\|\nabla(\sigma,\omega,v,r)(\cdot,t)\|_{\alpha-1}^2+\sum_{1\leq|k|\leq\alpha-1}\int_{\mathbb{R}^n}\partial_x^k\omega\cdot\nabla\partial_x^k\sigma\mathrm{d}x. \tag{A.3}$$

Because of the sufficient large constant $c_5>0$, we obtain

$$\phi_\alpha(U(t))\sim\|\nabla(\sigma,\omega,v,r)(\cdot,t)\|_{\alpha-1}^2. \tag{A.4}$$

Notice that if there exist a constant $\eta>0$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\alpha(U(t))+\eta\phi_\alpha(U(t))\leq C\|\nabla U(t)\|^2. \tag{A.5}$$

We solve above inequality by solving general solutions of ordinary differential equations, then

$$\phi_\alpha(U(t))\leq\phi_\alpha(U(s))\mathrm{e}^{-\eta(t-s)}+C\int_s^t\mathrm{e}^{-\eta(t-\tau)}\|\nabla U(\tau)\|^2\mathrm{d}\tau. \tag{A.6}$$

Thus, we calculate the time decay estimate (i) by using (6.1) and Proposition 4.1, then when taking $p=1$, $\alpha=0$:

$$\|U(t)\|\leq C(1+t-s)^{-\frac{n}{4}}\|U(s)\|_{L^2\cap L^1}+C\int_s^t(1+t-\tau)^{-\frac{n}{4}}\|\mathrm{B}_uU(\tau)\|_{H^2\cap L^1}\mathrm{d}\tau; \tag{A.7}$$

when taking $p=1$, $\alpha=1$:

$$\|\nabla U(t)\|\leq C(1+t-s)^{-\frac{n}{4}-\frac{1}{2}}\|U(s)\|_{H^1\cap L^1}+C\int_s^t(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|\mathrm{B}_uU(\tau)\|_{H^1\cap L^1}\mathrm{d}\tau. \tag{A.8}$$

Notice that

$$\begin{aligned}&\|\mathrm{B}_uU(\tau)\|_{H^1}=\|-\lambda_4u\cdot\nabla\sigma\|_{H^1}\leq C\left(\|u\|_{L^\infty}+\|\nabla u\|_{L^\infty}\right)\|\nabla\sigma\|_{H^1},\\&\|\mathrm{B}_uU(\tau)\|_{L^1}=\|-\lambda_4u\cdot\nabla\sigma\|_{L^1}\leq C\|u\|\|\nabla\sigma\|.\end{aligned}$$

Since $\alpha \ge \left[\frac{n}{2}\right] + 2$, from Lemma 3.1, we obtain $\|u\|_{L^\infty} \le \|u\|_\alpha$, $\|\nabla u\|_{L^\infty} \le \|u\|_\alpha$.

Thus, we can easily get

$$\begin{aligned}\|U(t)\| &\le C(1+t-s)^{-\frac{n}{4}}\|U(s)\|_{L^2\cap L^1} \\ &\quad + C\left(\sup_{t\ge s}\|u(t)\|_\alpha\right)\int_s^t (1+t-\tau)^{-\frac{n}{4}}\|\nabla U(\tau)\|\mathrm{d}\tau,\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}\|\nabla U(t)\| &\le C(1+t-s)^{-\frac{n}{4}-\frac{1}{2}}\|U(s)\|_{H^1\cap L^1} \\ &\quad + C\left(\sup_{t\ge s}\|u(t)\|_\alpha\right)\int_s^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|\nabla U(\tau)\|\mathrm{d}\tau.\end{aligned} \tag{A.10}$$

According to the definition of the energy equation $\phi_\alpha(U(t))$, the time decay estimation of $\phi_\alpha(U(t))$ can be further obtained, assume that for the time decay estimation (i),

$$\tilde{\phi}_\infty(t) = \sup_{s\le\tau\le t}(1+\tau-s)^{\frac{n}{2}}\phi_\alpha(U(\tau)), \tag{A.11}$$

for the time decay estimation (ii),

$$\bar{\phi}_\infty(t) = \sup_{s\le\tau\le t}(1+\tau-s)^{2\left(\frac{n}{4}+\frac{1}{2}\right)}\phi_\alpha(U(\tau)). \tag{A.12}$$

We use the definition of $\tilde{\phi}_\infty(t)$, (A.4) and (A.10), taking (A.11) and (A.12) into (A.6) to get

$$\begin{aligned}&\phi_\alpha(U(t)) \\ &\le \phi_\alpha(U(s))\mathrm{e}^{-\eta(t-s)} + C\left(\|U(s)\|^2_{H^1\cap L^1} + \left(\sup_{t\ge s}\|u(t)\|_\alpha\right)^2\tilde{\phi}_\infty(t)\right)\int_s^t \mathrm{e}^{-\eta(t-\tau)}(1+t-\tau)^{-\frac{n}{2}}\mathrm{d}\tau \\ &\le C(1+t-s)^{-\frac{n}{2}}\left(\phi_\alpha(U(s)) + \|U(s)\|^2_{H^1\cap L^1} + \left(\sup_{t\ge s}\|u(t)\|_\alpha\right)^2\tilde{\phi}_\infty(t)\right),\end{aligned} \tag{A.13}$$

$$\begin{aligned}\tilde{\phi}_\infty(t) &\le C\left(\phi_\alpha(U(s)) + \|U(s)\|^2_{H^1\cap L^1} + \left(\sup_{t\ge s}\|u(t)\|_\alpha\right)^2\tilde{\phi}_\infty(t)\right) \\ &\le C\left(\phi_\alpha(U(s)) + \|U(s)\|^2_{H^1\cap L^1}\right) \\ &\le \|U(s)\|^2_{H^1\cap L^1}.\end{aligned} \tag{A.14}$$

Similar to (A.13) and (A.14), by (A.12), we have

$$\phi_\alpha(U(t)) \le C(1+t-s)^{2\left(-\frac{n}{4}-\frac{1}{2}\right)}\left(\phi_\alpha(U(s)) + \|U(s)\|^2_{H^1\cap L^1} + \left(\sup_{t\ge s}\|u(t)\|_\alpha\right)^2\bar{\phi}_\infty(t)\right), \tag{A.15}$$

$$\bar{\phi}_\infty(t) \le C\left(\phi_\alpha(U(s)) + \|U(s)\|^2_{H^1\cap L^1}\right) \le \|U(s)\|^2_{H^1\cap L^1}. \tag{A.16}$$

We can obtain that

$$\left\|U(t)\right\|^2 \le C(1+t-s)^{-\frac{n}{2}}\left\|U(s)\right\|^2_{H^\alpha\cap L^1},$$
$$\left\|\nabla U(t)\right\|^2_{\alpha-1} \le C(1+t-s)^{2\left(-\frac{n}{4}-\frac{1}{2}\right)}\left\|U(s)\right\|^2_{H^\alpha\cap L^1}.$$

Then, we complete the proof of the proposition 4.2.

## A.2. The Proof of Proposition 4.4

*Proof.* Before proving this proposition, we need to recall the third section of energy estimation. By (4.22) and the smallness condition on $(\sigma^{per},\omega^{per},v^{per},r^{per})$, then there exist some positive constants $d_1,d_2,d_3,d_4$ and $\varepsilon>0$ with $d_1,d_3$ being suitably large and $d_2,d_4$ being sufficiently small, we can get by direct calculation of the energy estimates for $(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(d_1\sum_{1\le|k|\le N-1}\left\|\partial_x^k(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)(t)\right\|^2+\sum_{1\le|k|\le N-2}\int_{\mathbb{R}^n}\partial_x^k\tilde\omega(t)\cdot\nabla\partial_x^k\tilde\sigma(t)\mathrm{d}x\right)\\&+d_2\left(\left\|\nabla^2(\tilde\omega,\tilde v(t))\right\|^2_{N-2}+\left\|\nabla\tilde r(t)\right\|^2_{N-1}+\left\|\nabla^2\tilde\sigma(t)\right\|^2_{N-3}\right)\le C\left\|\nabla(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)(t)\right\|^2,\end{aligned}\tag{A.17}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(d_3\left\|(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)(t)\right\|^2+\int_{\mathbb{R}^n}\tilde\omega(t)\cdot\nabla\tilde\sigma(t)\mathrm{d}x\right)\\&+d_4\left(\left\|\nabla(\tilde\omega,\tilde v)(t)\right\|^2+\left\|\tilde r(t)\right\|^2+\left\|\nabla\tilde\sigma(t)\right\|^2\right)\le C\left\|\nabla^2(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)(t)\right\|^2_{m-1}.\end{aligned}\tag{A.18}$$

From (A.17) and (A.18), by noticing that $m-1<N-3$, we can choose a constant $d_5$ suitably large such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(d_5\left\|(\tilde\sigma,\tilde\omega,\tilde v,\tilde r)(t)\right\|^2_{N-1}+\sum_{1\le|k|\le N-2}\int_{\mathbb{R}^n}\partial_x^k\tilde\omega(t)\cdot\nabla\partial_x^k\tilde\sigma(t)\mathrm{d}x\right)\\&+C\left(\left\|\nabla(\tilde\omega,\tilde v)(t)\right\|^2_{N-1}+\left\|\tilde r(t)\right\|^2_N+\left\|\nabla\tilde\sigma(t)\right\|^2_{N-2}\right)\le 0,\end{aligned}\tag{A.19}$$

which implies (4.26) is hold. This completes the proof of proposition 4.4.

## A.3. Some Useful Formulas

Here, we list some known formulas.

**Lemma A.1.** (*Duhamels principle*). *Assume that the function* $u(x_1,x_2,\cdots,x_n,t,\tau)$ *is the solution of the Cauchy problem*

$$\begin{cases}u_{tt}-m^2\Delta u=0,t>\tau,\\u|_{t=\tau}=0,\\u_t|_{t=\tau}=h(x_1,x_2,\cdots,x_n,\tau)\end{cases}\tag{A.20}$$

then the function $v(x_1,x_2,\cdots,x_n,t)=\int_0^t u(x_1,x_2,\cdots,x_n,t,\tau)\mathrm{d}\tau$ is the solution of the Cauchy problem

$$\begin{cases}v=\dfrac{\partial^2 v}{\partial t^2}-m^2\Delta v=h(x_1,x_2,\cdots,x_n,t),\\v(x_1,x_2,\cdots,x_n,0)=0,\\v_t(x_1,x_2,\cdots,x_n,0)=0\end{cases}\tag{A.21}$$

**Lemma A.2.** (*Hölder inequality*). *Assume that* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, *we have*

$$\int_{\mathbb{R}^n} |uv| \, \mathrm{d}x \le \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}. \tag{A.22}$$