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Semilinear Fractional Elliptic Equation of Normal Solution

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Abstract

In this paper, a semilinear elliptic equation of fractional order is constructed by combining the semilinear elliptic equation with the fractional order equation. On this basis, the "standardized" solution, which is often sought by physicists, is studied. In order to overcome the problem of lack of boundness, we set up appropriate conditions to prove the existence of the solution by means of variational theorem and the mountain road theorem.

Subject Areas

Functional Analysis

Keywords

Fractional Laplacian, Variational Method, Semilinear

1. Introduction

In recent years, many practical problems have been solved successfully through in-depth research on nonlinear problems of fractional Laplace equations. For example, there have been studies in finance [1], fluid dynamics [2], quantum mechanics [3], physics [4], materials science [5], crystal dislocation problems [6], semi-permeable thin film problems [7], soft film problems [8], and very small surface problems [9], soft film problems [8] and very small surface problems [9] have been studied. With the continuous expansion of the fields involved and the deepening of the problem research, people continue to put forward new problems and explore solutions. In this paper, we study the following semilinear fractional elliptic equation on $\mathbb{R}^N$.

$$\begin{cases} (-\Delta)^s u - \lambda u = \sum_{i=1}^{m} a_i |u(x)|^{\sigma_i} u(x), \\ \int_{\mathbb{R}^N} |u|^2 = 1, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is a fixed constant, $\lambda \in \mathbb{R}$, $u \in H^s(\mathbb{R}^N)$ and $(-\Delta)^s$ is the fractional Laplacian operator, defined as

$$(-\Delta)^s u(x) = C_s P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \quad x, y \in \mathbb{R}^N, \tag{1.2}$$

where C_s is a constant, dependent on s can be expressed as

$$C_s = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} \mathrm{d}\xi \right)^{-1}, \tag{1.3}$$

and P.V. stands the principal value. And $u \in H^s(\mathbb{R}^N)$, $\phi \in D^{s,2}(\mathbb{R}^N)$, where $H^s(\mathbb{R}^N)$ and $D^{s,2}(\mathbb{R}^N)$ are defined in (1.9) and (1.12),

$$2_s^* = \frac{2N}{N-2s} \tag{1.4}$$

is the fractional Sobolev critical exponent. Next, let us mention some illuminating work (1.1) related to this problem. In paper [10], Bartsch Thomas and Sébastien de Valeriola analyzed the case when the relevant function has no lower bound on the L_2 unit sphere, and finally proved the existence of the solution. The nonlinear eigenvalue problems studied are as follows

$$\begin{cases} -\Delta u - g(u) = \lambda u, \\ \int_{\mathbb{R}^N} u^2 = 1, \end{cases} \tag{1.5}$$

where $u \in H^1(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$, and the function g is superlinear and subcritical, $N \geq 2$. In this paper, we study the following fractional nonlinear eigenvalue problem of the form:

$$(-\Delta)^s u(x) = \lambda u(x) + \sum_{i=1}^m a_i |u(x)|^{\sigma_i} u(x), \lambda \in \mathbb{R}, x \in \mathbb{R}^N, \tag{1.6}$$

where $N \geq 2$, $s \in (0,1)$, for all $1 \leq i \leq m$, with $a_i > 0$, $0 < \sigma_i < \frac{4s}{N-2s}$ if $N \geq 3$ and $\sigma_i > 0$ if $N = 2$. In this article, we set $S(a) = \left\{ u \in H^s(\mathbb{R}^N), |u|_{L^2(\mathbb{R}^N)} = 1 \right\}$. For all $1 \leq i \leq m$, if $N \geq 3$, set that $\frac{4}{N} < \sigma_i < \frac{4}{N-2}$, and if $N = 1, 2$, we set that $\sigma_i > \frac{4}{N}$. And set $I : H^s \to \mathbb{R}$ is a C^1-functional

$$I(u) = \frac{1}{2} \left| (-\Delta)^{\frac{s}{2}} u(x) \right|^2 - \int_{\mathbb{R}^N} \sum_{i=1}^m \frac{a_i}{\sigma_i + 2} |u(x)|^{\sigma_i + 2} \mathrm{d}x, \tag{1.7}$$

$\tilde{I} : H^s \times \mathbb{R} \to \mathbb{R}$ is a C^1-functional, setting

$$\begin{aligned} \tilde{I}(u,t) &:= I(H(u,t)) \\ &= \frac{\mathrm{e}^{2st}}{2} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 - \mathrm{e}^{-tN} \int_{\mathbb{R}^N} \sum_{i=1}^m \frac{a_i}{\sigma_i + 2} \left| \mathrm{e}^{\frac{tN}{2}} u(x) \right|^{\sigma_i + 2} \mathrm{d}x, \end{aligned} \tag{1.8}$$

Using Ekeland's ε-variational principle, we prove the existence of the Palais-Smale sequence.

In this paper, the norm of fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined

$$H^s(\mathbb{R}^N):=\left\{u\in L^2(\mathbb{R}^N):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\right\}, \tag{1.9}$$

and define $X=\left\{u\in H^s(\mathbb{R}^N):\int_{\mathbb{R}^3}V(x)u^2<\infty\right\}$, for $t\in\mathbb{R}$, and $x\in\mathbb{R}^N$, set $H:H^s\times\mathbb{R}\to H^s$ is a continuous map,

$$H(u,t)(x)=\mathrm{e}^{\frac{tN}{2}}u(\mathrm{e}^t x) \tag{1.10}$$

endowed the norm on X by

$$\|u\|_H^2=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\frac{C_s}{2}\int_{\mathbb{R}^N}V(x)|u(x)|^2\,\mathrm{d}x, \tag{1.11}$$

and the corresponding inner product is

$$(u,v)_H=\iint_{\mathbb{R}^3\times\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y+\frac{C_s}{2}\int_{\mathbb{R}^N}V(x)u(x)v(x)\mathrm{d}x.$$

Consider the following fractional critical Sobolev space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N):=\left\{u\in L^2(\mathbb{R}^N):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\right\}, \tag{1.12}$$

with the norm

$$\|u\|_{D^{s,2}}^2:=\frac{C_s}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \tag{1.13}$$

where $D^{s,2}(\mathbb{R}^N)$ is the completeness of $C_0^\infty(\mathbb{R}^N)$. For $1\le p<\infty$, we let

$$|u|_p=\left(\int_{\mathbb{R}^N}|u(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}},\quad u\in L^p(\mathbb{R}^N), \tag{1.14}$$

for any $s\in(0,1)$, the embedded $D^{s,2}(\mathbb{R}^N)\hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous, exist for the best fractional critical Sobolev constant

$$S(u):=\frac{\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y}{\left(\int_{\mathbb{R}^N}|u(x)|^{2_s^*}\,\mathrm{d}x\right)^{2/2_s^*}}. \tag{1.15}$$

In this article, we list all the conditions below on $\sum_{i=1}^m a_i|t|^{\sigma_i}t$:

(H1) $\sum_{i=1}^m a_i|t|^{\sigma_i}t$ is continuous and odd.

(H2) $\alpha,\beta\in\mathbb{R}$, satisfying.

$$\begin{cases} 2+\frac{4s}{N}<\alpha\le\beta<2-\frac{1}{2s}, & \text{if } N\ge 3 \\ 2+\frac{4s}{N}<\alpha\le\beta, & \text{if } N=1,2 \end{cases}$$

such that

$$0<\alpha\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|t|^{\sigma_i+2}\le\sum_{i=1}^{m}a_i\,|t|^{\sigma_i}\,t^2\le\beta\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|t|^{\sigma_i+2}.$$

Our main result is shown in the following.

Theorem 1.1. *For* $N\ge 2$, *under the hypotheses* $(H1)$ *and* $(H2)$ *Equation* (1.1) *admits a couple* $(u_c,\lambda_c)\in H^s(\mathbb{R}^N)\times\mathbb{R}$ *of weak solutions such that*

$$\|u_c\|_{L^2(\mathbb{R}^N)}=1 \quad and \quad \lambda_c<0.$$

2. Preliminary Lemmas

Lemma 2.1. ([11]) *For future reference note that from* (H1) *and* $(H2)$ *it immediately follows that, for all* $\tau\in\mathbb{R}$

$$\begin{cases} \varpi^{\beta}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau|^{\sigma_i+2}\le\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau\varpi|^{\sigma_i+2}\le\varpi^{\alpha}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau|^{\sigma_i+2}, & \text{if } 0\le\varpi\le 1 \\ \varpi^{\alpha}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau|^{\sigma_i+2}\le\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau\varpi|^{\sigma_i+2}\le\varpi^{\beta}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|\tau|^{\sigma_i+2}, & \text{if } \varpi\ge 1. \end{cases}$$

Lemma 2.2. *Fractional-Gagliardo-Nirenberg-Sobolev inequality* (*see* [12]): *for every* $N>2s$ *and* $2<p<2_s^*$, *there exists a constant* C_N, p, s *depending on N, p and s such that*

$$\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\le C_{N,p,s}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^{\frac{N(p-2)}{4s}}\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{p}{2}-\frac{N(p-2)}{4s}},\quad u\in H^s(\mathbb{R}^N).$$

It is equivalent to

$$|u|_p\le C_{N,p,s}\left|(-\Delta)^s u\right|_2^{\gamma_{p,s}}|u|_2^{1-\gamma_{p,s}},\quad u\in H^s(\mathbb{R}^N).$$

with $\gamma_{p,s}=\frac{N}{s}\left(\frac{1}{2}-\frac{1}{p}\right)$.

Lemma 2.3. ([11]) *If* (H1) *and* (H2), *and let* $u\in S(a)$ *be arbitrary but fixed. Then we get:*

1) $\|H(u,t)\|_{D^s}\to+\infty$ and $I(H(u,t))\to-\infty$ as $t\to+\infty$

2) $\|H(u,t)\|_{D^s}\to 0$ and $I(H(u,t))\to 0$ as $t\to-\infty$.

Proof. Since $u\in S(a)$, we get

$$|H(u,t)|_2=1,$$

and through the derivation, we get

$$\|H(u,t)\|_{D^s}=\mathrm{e}^t\|u\|_{D^s}.$$

Because of $N\left(\frac{\alpha-2}{2}\right)>2s$ and $\alpha\le\beta$, we get for $t\ge1$

$$I\left(H\left(u,t\right)\right)=\frac{\mathrm{e}^{2st}}{2}\left|\left(-\Delta\right)^{\frac{s}{2}}u\left(x\right)\right|^{2}-\mathrm{e}^{-tN}\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\mathrm{e}^{\frac{N(\sigma_i+2)|t|}{2}}\left|u\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x$$
$$\le\frac{\mathrm{e}^{2st}}{2}\left\|u\left(x\right)\right\|_{D^s}^{2}-\left(1+\mathrm{e}^{tN\left(\frac{\alpha-2}{2}\right)}\right)\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|u\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x.$$

Thus, we get that $I\left(H\left(u,t\right)\right)\to-\infty$ as $t\to+\infty$. Similarly, when $t<0$, it can be obtained by calculation $I\left(H\left(u,t\right)\right)\to0$ as $s\to-\infty$.□

Lemma 2.4. *If* (H1) *and* (H2), *there exists* $\rho_c>0$ *such that*

$$0<\sup_{u\in\Gamma_1}F\left(u\right)<\inf_{u\in\Gamma_2}F\left(u\right)$$

with

$$\begin{cases}\Gamma_1=\left\{u\in S\left(a\right),\left\|u\right\|_{D^s}^{2}\le\rho_c\right\}\\ \Gamma_2=\left\{u\in S\left(a\right),\left\|u\right\|_{D^s}^{2}=2\rho_c\right\}.\end{cases}$$

Proof. Now we're going to prove that $\sup_{u\in\Gamma_1}F\left(u\right)<\inf_{u\in\Gamma_2}$, there exists $\rho>0$, $v_1,v_2\in S\left(a\right)$, such that $\left\|v_1\right\|_{D^s}^{2}=\rho$ and $\left\|v_2\right\|_{D^s}^{2}=2\rho$. Then, for $\rho>0$ small enough

$$I\left(v_1\right)-I\left(v_2\right)=\frac{1}{2}\left\|v\right\|_{D^s}^{2}-\frac{1}{2}\left\|u\right\|_{D^s}^{2}-\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_1\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_2\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x$$
$$\ge\frac{\rho}{2}-\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_2\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x$$

By lemma 1.1, lemma 1.3 and $\alpha\le\beta$, with any $u\in S\left(a\right)$, we get

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|u\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x\le G\left(1\right)\left\{\left|u\right|_{\alpha}^{\alpha}+\left|u\right|_{\beta}^{\beta}\right\}$$
$$\le C\left\{\left\|u\right\|_{D^s}^{N\left(\frac{\alpha-2}{2}\right)}+\left\|u\right\|_{D^s}^{N\left(\frac{\beta-2}{2}\right)}\right\},$$

for $\left\|u\right\|_{D^s}$ small enough,

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|u\left(x\right)\right|^{\sigma_i+2}\mathrm{d}x\le C\left\|u\right\|_{D^s}^{N\left(\frac{\alpha-2}{2}\right)},$$

thus

$$I\left(v_1\right)-I\left(v_2\right)\ge\frac{\rho}{2}-C\rho^{N\left(\frac{\alpha-2}{4}\right)}\ge\frac{\rho}{4}>0.$$

Next we are going to prove that $0<\sup_{u\in\Gamma_1}I\left(u\right)$, for $u\in\Gamma_1$, as $\rho>0$, we get

$$I\left(u\right)\ge\frac{1}{2}\left\|u\right\|_{D^s}^{2}-C\rho^{N\left(\frac{\alpha-2}{4}\right)}>0$$

□

Lemma 2.5. *If* (H_1) *and* (H_2), *there exist* $u_1, u_2 \in S(a)$ *such that*

1) $\|u_1\|_{D^s}^2 \le \rho_c$
2) $\|u_2\|_{D^s}^2 > 2\rho_c$
3) $I(u_2) \le 0 < I(u_1)$
4) $\tilde{\gamma}(c) = \gamma(c)$

Setting

$$\tilde{\gamma}(c) = \inf_{h\in\tilde{\Gamma}(c)} \max_{t\in[0,1]} \tilde{F}(h(t))$$

with

$$\tilde{\Gamma}(c) = \{h \in C([0,1], S(c)\times\mathbb{R}), h(0) = (u_1,0), h(1) = (u_2,0)\}.$$

We have

$$\tilde{\gamma}(c) > \max\{\tilde{I}(u_1,0), \tilde{I}(u_2,0)\} \equiv \tilde{\gamma}_0(c) > 0.$$

Proof. In order to facilitate readers to read better, we have written down the proof process.

First note that the existence of $u_1, u_2 \in S(a)$ is insured by Lemmas 2.1 and 2.2. Now define

$$\gamma(c) \equiv \inf_{h\in\Gamma(c)} \max_{t\in[0,1]} I(h(t))$$

with

$$\Gamma(c) = \{h \in C([0,1], S(a)), h(0) = u_1, h(1) = u_2\}.$$

By we have

$$\gamma(c) > \max\{I(u_1), I(u_2)\}$$

Moreover

$$\begin{cases} I(u_1) = I(H(u_1,0)) = \tilde{I}(u_1,0) \\ I(u_2) = I(H(u_2,0)) = \tilde{I}(u_2,0). \end{cases}$$

Therefore, if $\tilde{\gamma}(c) \ge \gamma(c)$ holds, our result proves successful. This follows directly from the observation that: for $\tilde{h} \in \tilde{\Gamma}(c)$, there exists $h \in \Gamma(c)$ such that

$$\max_{t\in[0,1]} \tilde{I}(\tilde{h}(t)) = \max_{t\in[0,1]} I(h(t)).$$

Indeed, the setting $\tilde{h}(t) = (\tilde{h}_1(t), \tilde{h}_2(t)) \in S(a)\times\mathbb{R}$ we have, for all $t \in [0,1]$

$$\tilde{F}(\tilde{h}(t)) = \tilde{F}(\tilde{h}_1(t), \tilde{h}_2(t)) = F(H(\tilde{h}_1(t), \tilde{h}_2(t))).$$

and it suffices to set $h(t) = H(\tilde{h}_1(t), \tilde{h}_2(t)) \in \Gamma(c)$.□

Of course, if $\Gamma(c) \subset \tilde{\Gamma}(c)$, we get $\tilde{\gamma}(c) \le \gamma(c)$, it is proved that ended.

Lemma 2.6. ([10]) *If* ($H1$) *and* ($H2$), *for a sequence* $\{\varepsilon_n\} \subset \tilde{\Gamma}(c)$, *such that*

$$\max_{t\in[0,1]}\tilde{I}\left(\varepsilon_n(t)\right)\le\tilde{\gamma}(c)+\frac{1}{n}$$

Thus, there exists a sequence $\left\{(u_n,s_n)\right\}\subset S(a)\times\mathbb{R}$ such that:

1) $\tilde{\gamma}(c)-\frac{1}{n}\le\tilde{I}(u_n,s_n)\le\tilde{\gamma}(c)+\frac{1}{n}$

2) $\min_{t\in[0,1]}\left\|(u_n,s_n)-g_n(t)\right\|_{\mathbf{E}}\le\frac{1}{\sqrt{n}}$

3) $\left\|\tilde{I}'_{|s(c)\times\mathbb{R}}(u_n,s_n)\right\|\le\frac{2}{\sqrt{n}}$, *i.e.*

$$\left|\left\langle\tilde{I}'(u_n,s_n),z_n\right\rangle_{\mathbf{E}^*\times\mathbf{E}}\right|\le\frac{2}{\sqrt{n}}\|z\|_{\mathbf{E}}$$

for all

$$z\in\tilde{T}_{(u_n,s_n)}\equiv\left\{(z_1,z_2)\in\mathbf{E},\langle u_n,z_1\rangle_{L^2}=0\right\}$$

Lemma 2.7. *If we fix n, there exists a Palais-Smale sequence* $(u_k)_k$ *for* $\mathcal{G}_S$ *at the level* c_n *satisfying*

$$\|u_k\|^2+N\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|u_k|^{\sigma_i+2}-\frac{N}{2}\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i|u_k|^{\sigma_i+2}\to 0.$$

For the proof we recall the stretched function from [10]

$$\tilde{\mathcal{G}}:\mathbb{R}\times E\to\mathbb{R},\quad (s,u)\mapsto\mathcal{G}(s*u)$$

Now we define

$$\tilde{\Gamma}_n=\{\tilde{\gamma}:[0,1]\times(S\cap V_n)\to\mathbb{R}\times S\mid\tilde{\gamma}\text{ is continuous, odd in }u$$
$$\text{and such that } m\circ\tilde{\gamma}\in\Gamma_n\},$$

where $m(s,u)=s*u$ and

$$\tilde{c}_n=\inf_{\tilde{\gamma}\in\tilde{\Gamma}_n}\max_{\substack{t\in[0,1]\\ u\in S\cap V_n}}\tilde{\mathcal{G}}\left(\tilde{\gamma}(t,u)\right).$$

By lemma 2.7, there exists a Palais-Smale sequence $\{v_n\}$ for $S(a)$, that is, satisfying

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x$$

and $\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x$ are bounded. There exists $\{v_n\}\subset S(a)$ such that $I(v_n)\to\gamma(c)$ and $\left\|I'_{|s(c)}(v_n)\right\|\to 0$ as $n\to+\infty$. By lemma 2.6, $\tilde{\gamma}(c)=\gamma(c)$ there exists $\{g_n\}\subset\tilde{\Gamma}(c)$ of the form $g_n(t)=\left((g_n)_1(t),0\right)\in H^s$, $\forall t\in[0,1]$, such that $\Phi(g_n)\in\left[\gamma(c)-\frac{1}{n},\gamma(c)+\frac{1}{n}\right]$. Let

$$\partial_s\tilde{I}(u_n,s_n)\equiv\left\langle\tilde{I}'(u_n,s_n),(0,1)\right\rangle_{\mathbf{E}^*\times\mathbf{E}}.$$

From lemma 2.6, we get that as $n\to+\infty$, $\partial_s\tilde{I}(u_n,s_n)\to 0$ with

$$\partial_s \tilde{I}(u_n,s_n)=\|v_n\|_{D^s}^2+N\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}\sum_{i=1}^m a_i|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x \quad (16)$$

where $v_n \equiv H(u_n,s_n)$. Thus using the fact that

$$\tilde{I}(u_n,s_n)=\frac{1}{2}\|v_n\|_{D^s}^2-\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x$$

is also bounded, we see that there exists a constant $C>0$ independent of n such that

$$\left|N\tilde{I}(u_n,s_n)+\partial_s\tilde{I}(u_n,s_n)\right|\le C$$

From (H2) we have

$$\begin{aligned}N\tilde{I}(u_n,s_n)+\partial_s\tilde{I}(u_n,s_n)&=\frac{N+2}{2}\|v_n\|_{D^s}^2-\frac{N}{2}\int_{\mathbb{R}^N}\sum_{i=1}^m a_i|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\\&\le\frac{N+2}{2}\|v_n\|_{D^s}^2-\frac{N\alpha}{2}\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x.\end{aligned}$$

As a result of $\frac{N+2}{2}\|v_n\|_{D^s}^2-\frac{N\alpha}{2}\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\ge -C$. And then since $\tilde{F}(u_n,s_n)$ is bounded, we can get

$$\|v_n\|_{D^{s2}}^2\le 2C+2\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x.$$

After sorting it out, we can get

$$(N+2)\left\{C+\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\right\}-\frac{N\alpha}{2}\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\ge -C.$$

And $\left(N+2-\frac{N\alpha}{2}\right)\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\ge -C$.

Now the lower bound α (see (H2)), proves that $\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n(x)|^{\sigma_i+2}\,\mathrm{d}x$ is bounded and consequently $\left\{\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x\right\}$ also.

Lemma 2.8. *If* (H1) *and* (H2), *there exists a sequence* $\{v_n\}\subset S(a)$ *such that:*

1) $I(v_n)\to\gamma(c)$

2) $\|v_n\|_{H^s}$ and $\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|v_n|^{\sigma_i+2}\,\mathrm{d}x$ are bounded in $\mathbb{R}$

3) $\left|\langle I'(v_n),z\rangle_{E^*\times E}\right|\le\frac{4}{\sqrt{n}}\|z\|_{H^s}$, for all $z\in T_{v_n}\equiv\left\{z\in H^s,\langle v_n,z\rangle_H=0\right\}$.

Proof. We claim $\{v_n\}$ is a Palais-Smale sequence of the type we are looking for. Clearly $\{v_n\}\subset S(c)$ and is bounded in *E*. Point (1) is trivial since $I(v_n)=I(H(u_n,s_n))=\tilde{I}(u_n,s_n)$. Now let $h_n\in T_{v_n}$. We have

$$\begin{aligned}\langle I'(v_n),h_n\rangle_{E^*\times E}&=\int_{\mathbb{R}^N}\nabla v_n(x)\nabla h_n(x)\,\mathrm{d}x-\int_{\mathbb{R}^N}g(v_n(x))h_n(x)\,\mathrm{d}x\\&=\mathrm{e}^{s_n\left(\frac{N+2}{2}\right)}\int_{\mathbb{R}^N}\nabla u_n(\mathrm{e}^{s_n}x)\nabla h_n(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} g\left(\mathrm{e}^{\frac{s_N N}{2}} u_n\left(\mathrm{e}^{s_n} x\right)\right) h_n(x) \mathrm{d}x \\
&= \mathrm{e}^{2s_n} \int_{\mathbb{R}^N} \nabla u_n(x) \mathrm{e}^{-s_n\left(\frac{N+2}{2}\right)} \nabla h_n\left(\mathrm{e}^{-s_n} x\right) \mathrm{d}x \\
&\quad -\mathrm{e}^{-\frac{s_n N}{2}} \int_{\mathbb{R}^N} g\left(\mathrm{e}^{\frac{s_N N}{2}} u_n(x)\right) \mathrm{e}^{-\frac{s_n N}{2}} h_n\left(\mathrm{e}^{-s_n} x\right) \mathrm{d}x.
\end{aligned}
$$

Thus, setting

$$
\tilde{h}_n(x) := H\left(h_n, -s_n\right)(x) = \mathrm{e}^{-\frac{s_n N}{2}} h_n\left(\mathrm{e}^{-s_n} x\right)
$$

where $h_n \in T_{v_n}$.

We see that

$$
\left\langle I'\left(v_n\right), h_n\right\rangle_{E^* \times E} = \left\langle \tilde{I}'\left(u_n, s_n\right), \left(\tilde{h}_n, 0\right)\right\rangle_{\mathbf{E}^* \times \mathbf{E}}
$$

If $h_n \in T_{v_n}$, from the definition of T_{v_n}, we can get $\left\langle h_n, v_n\right\rangle_{L^2} = 0$, and then by taking the derivative, we set $y = \mathrm{e}^{-s_n} x$, so $\int_{\mathbb{R}^N} \mathrm{e}^{-\frac{s_n N}{2}} u_n(x) h_n\left(\mathrm{e}^{-s_n} x\right) \mathrm{d}x$ is equal to $\int_{\mathbb{R}^N} \mathrm{e}^{\frac{s_n N}{2}} u_n\left(\mathrm{e}^{s_n} y\right) h_n(y) \mathrm{d}y$, that is, $\left\langle h_n, v_n\right\rangle_{L^2} = 0 = \left\langle \tilde{h}_n, u_n\right\rangle_{L^2}$, then we have $\left(\tilde{h}_n, 0\right) \in \tilde{T}\left(u_n, s_n\right)$. And by Point (3) of Proposition 2.2, if $\mathrm{e}^{-2s_n} \le 2$, as n is sufficiently large we get that

$$
\begin{aligned}
\left|\left\langle I'\left(v_n\right), h_n\right\rangle_{E^* \times E}\right| &\le \frac{2}{\sqrt{n}} \left\|\left(\tilde{h}_n, 0\right)\right\|_{H^s} \\
&= \frac{2}{\sqrt{n}} \left\|\tilde{h}_n\right\|_{H^s}^2 \\
&= \frac{2}{\sqrt{n}} \left\{\int_{\mathbb{R}^N} \left|\tilde{h}_n(x)\right|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \left\|\tilde{h}_n(x)\right\|_{D^s}^2 \mathrm{d}x\right\} \\
&= \frac{2}{\sqrt{n}} \left\{\int_{\mathbb{R}^N} \left|h_n(x)\right|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \left\|h_n(x)\right\|_{D^s}^2 \mathrm{d}x\right\} \\
&\le \frac{4}{\sqrt{n}} \left\|h_n\right\|_{H^s}^2.
\end{aligned}
$$

By Point (2) of Proposition 2.2, for $n \in \mathbf{N}$ large since

$$
\left|s_n\right| = \left|s_n - 0\right| \le \min_{t \in [0,1]} \left\|\left(u_n, s_n\right) - \varepsilon_n\right\|_{H^s} \le \frac{1}{\sqrt{n}}
$$

Let the minimizing sequence $\{\varepsilon_n\} \subset \tilde{\Gamma}(c)$, substitute into the formula. Notice that the particular choice of the minimizing sequence $\{\varepsilon_n\} \subset \tilde{\Gamma}(c)$ is used here.

□

3. Proof of Theorem 2.1.

Lemma 3.1. *let* $\{v_n\} \subset S(a)$ *be the PS sequence obtained in Lemma, there exists* $\{\lambda_n\} \subset \mathbb{R}$ *such that, up to a subsequence:*

$$
\int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{a_i}{\sigma_i + 2} \left|v_n(x)\right|^{\sigma_i + 2} \mathrm{d}x \to c_1 > 0
$$

and

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i}v_n(x)\mathrm{d}x\to c_2>0.$$

Proof. By $\{v_n\}$ is bounded in H^s, there exists $v_c\in H^s$ such that $v_n\rightharpoonup v_c$ weakly in H^s. If $\{v_n\}$ converge strongly to v_c, by Lemma 2.7, v_c is a critical point for F restricted to $S(a)$. By $\partial_s\tilde{I}(u_n,s_n)\to 0$ and (H2) there exists $\varepsilon_n\to 0$ such that $-\varepsilon_n\le\partial_s\tilde{I}(u_n,s_n)$, we get

$$\begin{aligned}\gamma(c)+\frac{1}{2}\varepsilon_n&=I(v_n)-\frac{1}{2}\partial_s\tilde{I}(v_n,v_n)\\&=\frac{N}{4}\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x-\left(1+\frac{N}{2}\right)\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\\&\le\left(\frac{(\beta-2)N}{4}-1\right)\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\end{aligned}\tag{3.1}$$

By $\gamma(c)>0$ and $\frac{(\beta-2)N}{4}-1\ge 0$, such that $\int_{\mathrm{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x$ is bounded and strictly greater than zero, that is, there exists $c_1>0$ such that, up to a subsequence

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\to c_1,\tag{3.2}$$

otherwise, if $c_1=0$, then we get $\left(\frac{(\beta-2)N}{4}-1\right)\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\to 0$, as $n\to\infty$, contradiction with $\gamma(c)+\frac{1}{2}\varepsilon_n\to\gamma(c)>0$.

Similarly, from (H2), we conclude that $\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x$ is also bounded and by (17) and (18), we get

$$\begin{aligned}\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x&=\frac{4}{N}\gamma(c)+\frac{2}{N}\varepsilon_n+\left(2+\frac{4}{N}\right)\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\\&\to\frac{4}{N}\gamma(c)+\left(2+\frac{4}{N}\right)c_1\end{aligned}$$

by calculation as $n\to\infty$, we can be expressed as the existence of $c_2>0$, such that

$$\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i+2}\mathrm{d}x\to c_2.\tag{3.3}$$

□

Lemma 3.2. $\lambda_n\to\lambda_c<0$ *in* $\mathbb{R}$.

Proof. Next, we're going to prove that $\lambda_n\to\lambda_c$ in $\mathbb{R}$, by calculation we get

$$\begin{aligned}\left\langle I'(v_n),z\right\rangle_{(H^s)^*\times H^s}&=\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v_n(-\Delta)^{\frac{s}{2}}z\mathrm{d}x-\int_{\mathbb{R}^N}\sum_{i=1}^{m}a_i\left|v_n(x)\right|^{\sigma_i}v_n(x)z(x)\\&\quad-\lambda_n\int_{\mathbb{R}^N}v_n(x)z(x)\mathrm{d}x\end{aligned}$$

with

$$\lambda_n = \frac{1}{|v_n|_2}\left\{\|v_n\|_{D^s}^2 - \int_{\mathbb{R}^N}\sum_{i=1}^m a_i |v_n(x)|^{\sigma_i+2}\,\mathrm{d}x\right\}.$$

Up to a subsequence, by the definition of S(a), lemma and (3.1), we have that $\{\lambda_n\}$ is bounded away from zero for $n \to +\infty$. Thus, there exists λ_c such that $\lambda_n \to \lambda_c$.□

Lemma 3.3. $-\Delta v_n - \lambda_n v_n - g(v_n) \to 0$ *in* E^*.

Proof. The next thing we have to prove is the compactness result for $g: E \to E^*$, $u \to g(u)$ in the subspace $H_r^s(\mathbb{R}^N)$.

Next, the function $u \mapsto \int_{\mathbb{R}^v}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|u(x)|^{\sigma_i+2}\,\mathrm{d}x$ is weakly continuous. Then for any weakly convergent sequence u_n, when $u_n \rightharpoonup u$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|u_n(x)|^{\sigma_i+2}\,\mathrm{d}x - \int_{\mathbb{R}^N}\sum_{i=1}^m \frac{a_i}{\sigma_i+2}|u(x)|^{\sigma_i+2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_0^1\sum_{i=1}^m a_i |tu_n(x)+(1-t)u(x)|^{\sigma_i}(tu_n(x)+(1-t)u(x))(u_n(x)-u(x))\,\mathrm{d}t\mathrm{d}x \\
&\le \int_{\mathbb{R}^N}\max_{t\in[0,1]}\sum_{i=1}^m a_i |tu_n(x)+(1-t)u(x)|^{\sigma_i}|tu_n(x)+(1-t)u(x)||u_n(x)-u(x)|\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^N}\max_{t\in[0,1]}\left\{|tu_n(x)+(1-t)u(x)|^{\alpha-1}+|tu_n(x)+(1-t)u(x)|^{\beta-1}\right\}|u_n(x)-u(x)|\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^N}\left\{(|u_n(x)|+|u(x)|)^{\alpha-1}+(|u_n(x)|+|u(x)|)^{\beta-1}\right\}|u_n(x)-u(x)|\,\mathrm{d}x \\
&\le C\left|(|u_n|+|u|)^{\alpha}\right|_{\frac{\alpha}{\alpha-1}}|u_n-u|_{\alpha}+\left|(|u_n|+|u|)^{\beta}\right|_{\frac{\beta}{\beta-1}}|u_n-u|_{\beta}.
\end{aligned}$$

using (H2) and (2.1). Now from the compactness of the inclusion $H_r^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2s}$ if $N \ge 3$ or $p > 2$ if $N = 2$ (see [13]), we see that $\|u_n - u\|_{L^\alpha} \to 0$ and $\|u_n - u\|_{L^\beta} \to 0$.

From the previous steps 1 to 3, we obtain that there is $-\Delta v_n - \lambda_c v_n \to g(v_c)$. By $\lambda_c < 0$, and we deduce that $v_n \to (-\Delta^s - \lambda_c)^{-1} g(v_c)$ in $H^s(\mathbb{R}^N)$. And by $v_n \to v_c \in S(a)$ in H^s, so we prove the Theorem 1.1.□

Conflicts of Interest

The author declares no conflicts of interest.

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