Open Access Library Journal
2024, Volume 11, e11241
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Tremendous Development of Functional Inequalities and Cauchy-Jensen Functional Equations with 3*k*-Variables on Banach Space and Stability Derivation on Fuzzy-Algebras

Ly Van An

Faculty of Mathematics Teacher Education, Tay Ninh University, Tay Ninh, Vietnam
Email: lyvanan145@gmail.com, lyvananvietnam@gmail.com

How to cite this paper: An, L.V. (2024) Tremendous Development of Functional Inequalities and Cauchy-Jensen Functional Equations with 3*k*-Variables on Banach Space and Stability Derivation on Fuzzy-Algebras. *Open Access Library Journal*, **11**: e11241.
https://doi.org/10.4236/oalib.1111241

Received: January 19, 2024
Accepted: February 25, 2024
Published: February 28, 2024

Abstract

In this paper, I study to solve functional inequalities and equations of type Cauchy-Jensen with 3*k*-variables in a general form. I first introduce the concept of the general Cauchy-Jensen equation and next, I use the direct method of proving the solutions of the Jensen-Cauchy functional inequalities relative to the general Cauchy-Jensen equations and then I show that their solutions are mappings that are additive mappings calculated and finally apply the derivative setup on fuzzy algebra also the results of the paper.

Subject Areas

Mathematics

Keywords

Functional Equation, Functional Inequality Additivity, Banach Space, Derivation on Fuzzy-Algebras

1. Introduction

Let $\mathbf{G}$ be an m-divisible group where $m \in \mathbb{N} \setminus \{0\}$ and $\mathbf{X}$, $\mathbf{Y}$ be a normed space on the same field $\mathbb{K}$, and $f : \mathbf{G} \to \mathbf{X}$ ($f : \mathbf{G} \to \mathbf{Y}$) be a mapping. I use the notation $\|\cdot\|_{\mathbf{X}}$ ($\|\cdot\|_{\mathbf{Y}}$) for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, I investigate functional inequalities and equations when when $\mathbf{G}$ be an m-divisible group where $m \in \mathbb{N}$ and $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a Banach space with norm $\|\cdot\|_{\mathbf{Y}}$.

In fact, when $\mathbf{G}$ be an m-divisible group where $m \in \mathbb{N}$ and $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a Banach space with norm $\|\cdot\|_{\mathbf{Y}}$

I solve and prove the Hyers-Ulam-Rassias type stability of following functional inequalities and equations.

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+2k\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} \quad (1)$$

and

$$\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+2k\sum_{j=1}^{k} f\left(z_j\right)=2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right) \quad (2)$$

Where k is a positive integer.

The study of the functional equation stability originated from a question of S. M. Ulam [1], concerning the stability of group homomorphisms. Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{G}\to\mathbf{G}'$ satisfies:

$$d\left(f\left(x*y\right),f\left(x\right)\circ f\left(y\right)\right)<\delta$$

for all $x,y\in\mathbf{G}$ then there is a homomorphism $h:\mathbf{G}\to\mathbf{G}'$ with

$$d\left(f\left(x\right),h\left(x\right)\right)<\varepsilon,$$

for all $x\in\mathbf{G}$, if the answer, is affirmative, I would say that equation of homomophism $h(x*y)=h(y)\circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace a functional equation with an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is how the solutions of the inequality differ from those of the given function equation. Hyers gave a first affirmative answer the question Ulam as follows:

In 1941 D. H. Hyers [2] Let $\varepsilon\ge 0$ and let $f:\mathbf{E}_1\to\mathbf{E}_2$ be a mapping between *Banach* space such that

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\|\le\varepsilon,$$

for all $x,y\in\mathbf{E}_1$ and some $\varepsilon\ge 0$. It was shown that the limit

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x\in\mathbf{E}_1$ and that $T:\mathbf{E}_1\to\mathbf{E}_2$ is that unique additive mapping satisfying

$$\left\|f\left(x\right)-T\left(x\right)\right\|\le\varepsilon,\forall x\in\mathbf{E}_1.$$

Next in 1978 Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:

Consider $\mathbf{E},\mathbf{E}'$ to be two Banach spaces, and let $f:\mathbf{E}\to\mathbf{E}'$ be a mapping such that $f(tx)$ is continous in t for each fixed x. Assume that there exist $\theta\ge 0$ and $p\in[0,1)$ such that

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\|\le\varepsilon\left(\|x\|^p+\|y\|^p\right),\forall x,y\in\mathbb{E}.$$

then there exists a unique linear $L:\mathbf{E}\to\mathbf{E}'$ satifies

$$\left\| f(x) - L(x) \right\| \le \frac{2\theta}{2-2^p}\|x\|, x \in \mathbf{E}.$$

Next J. M. Rassias [4] following the spirit of the innovative approach of Th. M. Rassias for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \ne 1$.

Next in 1992, a generalized of Rassias' Theorem was obtained by Găvruta [5].

Let $(\mathbf{G}, +)$ be a group Abelian and $\mathbf{E}$ a *Banach* space.

Denote by $\phi : \mathbf{G} \times \mathbf{G} \to [0, \infty)$ a function such that

$$\tilde{\phi}(x, y) = \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x, 2^n y) < \infty$$

for all $x, y \in \mathbf{G}$. Suppose that $f : \mathbf{G} \to \mathbf{E}$ is a mapping satisfying

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \varepsilon, \quad \forall x, y \in G.$$

There exists a unique additive mapping $T : \mathbf{G} \to \mathbf{E}$ such that

$$\left\| f(x) - T(x) \right\| \le \tilde{\phi}(x, x), \quad \forall x, y \in G.$$

Generally speaking for a more specific problem, when considering this famous result, the additive Cauchy equation

$$f(x+y) = f(x) + f(y)$$

is said to have the Hyers-Ulam stability on $(\mathbf{E}_1, \mathbf{E}_2)$ with $\mathbf{E}_1$ and $\mathbf{E}_2$ are Banach spaces if for each $f : \mathbf{E}_1 \to \mathbf{E}_2$ satisfying

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \varepsilon$$

for all $x, y \in \mathbf{E}_1$ for some $\varepsilon > 0$, there exists an additive $h : \mathbf{E}_1 \to \mathbf{E}_2$ such that $f - h$ is bounded on $\mathbf{E}_1$. The method which was provided by Hyers, and which produces the additive *h*, was called a direct method.

Afterward, Gilány showed that if satisfies the functional inequality

$$\left\| 2f(x) + 2f(y) - f(xy^{-1}) \right\| \le \left\| f(xy) \right\| \tag{3}$$

Then *f* satisfies the Jordan-von Newman functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}) \tag{4}$$

Gilányi [6] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality.

Recently, the authors studied the Hyers-Ulam stability for the following functional inequalities and equation

$$\left\| f(x) + f(y) + 2f(y) \right\| \le \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| \tag{5}$$

$$f(x) + f(y) + 2f(y) = 2f\left(\frac{x+y}{2} + z\right) \tag{6}$$

in Banach spaces.

In this paper, I solve and prove the Hyers-Ulam stability for inequality (1.1) is related to Equation (1.2), ie the functional inequalities and equation with $3k$ variables. Under suitable assumptions on spaces $\mathbf{G}$ and $\mathbf{X}$ or $\mathbf{G}$ and $\mathbf{Y}$, I will prove that the mappings satisfy the (1.1) - (1.2). Thus, the results in this paper are generalization of those in [1]-[33] for inequality (1.1) is related to Equation (1.2) with $3k$ variables.

The paper is organized as follows:

In the section preliminary, I remind some basic notations such as:

Concept of the divisible group, definition of the stability of Cauchy-Jenen functional inequalities and functional equation, Solutions of the equation, functional inequalities and functional equation, the crucial problem when constructing solutions for Cauchy-Jensen inequalities.

Section 3: Establish a solution to the generalized Cauchy-Jensen functional inequalities (2.2) when I assume that G be a *m-divisible* abelian group and X is a normed space.

Section 4: Stability of functional inequalities (1.1) related to the Cauchy-Jensen equation when I assume that G be a *m-divisible* abelian group and Y is a Banach space.

Section 5: Establish solutions to functional inequalities (1.1) based on the definition when I assume that G be a *m-divisible* abelian group and Y is a Banach space.

Section 6: The stability of derivation on fuzzy-algebras.

2. Preliminaries

2.1. Concept of Divisible Group

A group $\mathbf{G}$ is called divisible if for every $x \in \mathbf{G}$ and every positive integer n there is a $y \in \mathbf{G}$ so that $ny = x$, *i.e.*, every element of $\mathbf{G}$ is divisible by every positive integer. A abelian group $\mathbf{G}$ is called divisible if for every $x \in \mathbf{G}$ and every $n \in \mathbb{N}$ there is some $y \in \mathbf{G}$ so that $x = ny$. divisible by every positive integer. Let $\mathbf{G}$ be an n-divisible abelian group where $n \in \mathbb{N}$ (*i.e.*, $a \to na : \mathbf{G} \to \mathbf{G}$ is a surjection).

Denote by

$$M(\mathbf{G},\mathbf{X}) = \{ f \mid f : \mathbf{G} \to \mathbf{X} \}$$

$$L^{\infty}(\mathbf{G},\mathbf{X}) = \left\{ f : \mathbf{G} \to \mathbf{X} \mid \|f\|_{\infty} := \sup_{x \in \mathbf{G}} \|f\|_{\mathbf{X}} < \infty \right\}$$

The sets $M(\mathbf{G},\mathbf{Y}), M(\mathbf{G}^r,\mathbf{X})$ and $M(\mathbf{G}^r,\mathbb{R}^+)$ can be defined similarly where

$$\mathbf{G}^r = \left\{ (x_1, x_2, \cdots, x_r) : x_j \in \mathbf{G}, j = 1, \cdots, k \right\}$$

2.2. Definition of the Stability of Functional Inequalities and Functional Equation

Given mappings $E : M(\mathbf{G},\mathbf{X}) \to M(\mathbf{G}^r,\mathbb{R}^+)$, $\varphi : \mathbf{G}^r \to \mathbb{R}$ and $\psi : \mathbf{G} \to \mathbb{R}^+$. If

$$E(f)(x_1,x_2,\cdots,x_r)\le\varphi(x_1,x_2,\cdots,x_r)$$

for all $x_1,x_2,\cdots,x_r\in\mathbf{G}$ implies that there exists $g\in M(\mathbf{G},\mathbf{X})$ such that $E(g)\le 0$ and $\|f(x)-g(x)\|_\infty\le\psi(x)$, for all $x\in\mathbf{G}$, then we say that the inequality $E(f)\le 0$ is (φ,ψ)-stable in $M(\mathbf{G},\mathbf{X})$. In this case, we also say that the solutions of the inequality $E(f)\le 0$ is (φ,ψ)-stable in $M(\mathbf{G},\mathbf{X})$. Given mappings $E:M(\mathbf{G},\mathbf{X})\to M(\mathbf{G}^r,\mathbb{R}^+)$, $\varphi:\mathbf{G}^r\to\mathbb{R}$ and $\psi:\mathbf{G}\to\mathbb{R}^+$ if

$$\|E(f)(x_1,x_2,\cdots,x_r)\|_\infty\le\varphi(x_1,x_2,\cdots,x_r)$$

for all $x_1,x_2,\cdots,x_r\in\mathbf{G}$, implies that there exists $g\in M(\mathbf{G},\mathbf{X})$ such that $E(g)=0$ and $\|f(x)-g(x)\|_\infty\le\psi(x)$, for all $x\in\mathbf{G}$, then we say that the inequality $E(f)\le 0$ is (φ,ψ)-stable in $M(\mathbf{G},\mathbf{X})$. In this case, we also say that the solutions of the inequality $E(f)=0$ is (φ,ψ)-stable in $M(\mathbf{G},\mathbf{X})$.

It is well known that if an additive function $f:\mathbb{R}\to\mathbb{R}$ satisfies one of the following conditions:

1) *f* is continuous at a point;
2) *f* is monotonic on an interval of positive length;
3) *f* is bounded on an interval of positive length;
4) *f* is integrable;
5) *f* is measurable;

then *f* is of the form $f(x)=cx$ with a real constant *c*.

2.3. Solutions of the Equation

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive *mapping*.

The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be an Jensen additive *mapping*.

The functional equation

$$f(x)+f(y)+2f(z)=2f\left(\frac{x+y}{2}+z\right)$$

is called the Cauchy-Jensen equation. In particular, every solution of the equation is said to be an additive *mapping*.

2.4. Solutions of the Functional Inequalities

The functional inequalities

$$\|f(x)+f(y)+2f(z)\|\le\left\|2f\left(\frac{x+y}{2}+z\right)\right\|$$

is called the Cauchy-Jensen inequalities. In particular, every solution of the inequalities is said to be an additive *mapping*

2.5. The Crucial Problem When Constructing Solutions for Cauchy-Jensen Inequalities

Suppose a mapping $f:\mathbf{G}\to\mathbf{X}$, the equation

$$\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)=mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right) \tag{7}$$

is said to a generalized Cauchy-Jensen equation.

And function inequalities

$$\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)\le mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right) \tag{8}$$

is said to a generalized Cauchy-Jensen function inequalitiess Note: case $m=2$ and $k=1$ so (7) it is called a classical Cauchy-Jensen equation, (8) it is called a Cauchy-Jensen function inequalities.

3. Establish a Solution to the Generalized Cauchy-Jensen Functional Inequality

Now, I first study the solutions of (8). Note that for inequalities, $\mathbf{G}$ be a m-divisible group where $m\in\mathbb{N}\setminus\{0\}$ and $\mathbf{X}$ be a normed spaces. Under this setting, I can show that the mapping satisfying (8) is additive. These results are give in the following.

Lemma 1. Let $f:\mathbf{G}\to\mathbf{X}$ be a mapping such that satisfies

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{X}}\le\left\|mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{X}} \tag{9}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$ if and only if $f:\mathbf{G}\to\mathbf{X}$ is additive.

Proof. Prerequisites

Assume that $f:\mathbf{G}\to\mathbf{Y}$ satisfies (9) Replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (9), I get

$$|2k+m|\left\|f\left(0\right)\right\|_X\le|m|\left\|f\left(0\right)\right\|_X$$

$$\left(|2k+m|-|m|\right)\left\|f\left(0\right)\right\|_X\le 0$$

So $f\left(0\right)=0$.

Next I replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(-mz,\cdots,-mz,0,\cdots,0,z,\cdots,z\right)$ in (9), I get $\left\|kf\left(-mz\right)+kmf\left(z\right)\right\|\le 0$ and so

$$f\left(-mz\right)=-mf\left(z\right) \tag{10}$$

for all $z\in\mathbf{G}$.

Next I replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,-\frac{x_j+y_j}{m},\cdots,-\frac{x_j+y_j}{m}\right)$ in (9) and (10) I have

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{X}} \\
&=\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)-\sum_{j=1}^{k} f\left(x_j+y_j\right)\right\|_{\mathbf{X}} \\
&\leq\left\|mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}-\sum_{j=1}^{k}\frac{x_j+y_j}{m}\right)\right\|_{\mathbf{X}}=\left\|f(0)\right\|_{\mathbf{X}}=0
\end{aligned} \tag{11}$$

Therefore

$$\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)=\sum_{j=1}^{k} f\left(x_j+y_j\right) \tag{12}$$

Finally we replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(u,\cdots,u,v,\cdots,v)$ in (12) so

$$f(u)+f(v)=f(u+v).$$

Sufficient conditions:

Suppose $f:\mathbf{G}\to\mathbf{Y}$ is additive. Then

$$f\left(\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j\right)=\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right) \tag{13}$$

and so

$$f\left(p\sum_{j=1}^{k} x_j\right)=p\sum_{j=1}^{k} f\left(x_j\right)$$

for all $p\in\mathbb{Q}$ and $x_1,x_2,\cdots,x_r\in\mathbf{G}$.

Therefore

$$\begin{aligned}
&\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right) \\
&=mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}\right)+m\sum_{j=1}^{k} f\left(z_j\right)=mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right)
\end{aligned} \tag{14}$$

So I have something to prove

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}}\leq\left\|mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} \tag{15}$$

□

From the proof of the lemma 2, I get the following corollary:

Corollary 1. Suppose a mapping $f:\mathbf{G}\to\mathbf{X}$, The following clauses are equivalent

1) *f* is additive.

2) $\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)=mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right)$,

$$\forall x_j,y_j,z_j\in\mathbf{G},\quad j=1,\cdots,k.$$

3) $\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+m\sum_{j=1}^{k} f\left(z_j\right)\right\|\leq\left\|mf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{m}+\sum_{j=1}^{k} z_j\right)\right\|$

$$\forall x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}.$$

Note: Clearly, a vector space is a *m-divisible* abelian group, so Corollary 3.2 is right when $\mathbf{G}$ is a vector space.

Through the Lemma 2 proof, I have the remark:

Remark: When the letting $m = 2k$ (means that m is always even) and $\mathbf{G}$ is an m-divisible abelian gourp then $\mathbf{G}$ must be a 2-divisible abelian gourp.

4. Stability of Functional Inequalities Related to the Cauchy-Jensen Equation

Now, I first study the solutions of (1.1). Note that for inequalities, $\mathbf{G}$ be a m-divisible group where $m \in \mathbb{N} \setminus \{0\}$ and $\mathbf{Y}$ be a Banach spaces. Under this setting, I can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Theorem 2. For $\phi : \mathbf{G}^{3k} \to \mathbb{R}^+$ be a function such that

$$\lim_{n\to\infty} \frac{1}{(2k)^n} \phi\left((2k)^n x_1, \cdots, (2k)^n x_k, (2k)^n y_1, \cdots, (2k)^n y_k, \cdots, (2k)^n z_1, \cdots, (2k)^n z_k\right) = 0 \quad (16)$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$.

And

$$\tilde{\phi}(x_1, \cdots, x_k, z_1, \cdots, z_k) = \sum_{n=0}^{\infty} \frac{1}{(2k)^{n+1}} \phi\left((2k)^{n+1} x_1, \cdots, (2k)^{n+1} x_k, 0, \cdots, 0, (2k)^n z_1, \cdots, (2k)^n z_k\right) < \infty \quad (17)$$

for all $x_1, \cdots, x_k, z_1, \cdots, z_k, z_j \in \mathbf{G}$. Suppose that an odd mapping $f : \mathbf{G} \to \mathbf{Y}$ satisfies

$$\begin{aligned} &\left\| \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + 2k \sum_{j=1}^{k} f(z_j) \right\|_{\mathbf{Y}} \\ &\le \left\| 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} + \phi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) \end{aligned} \quad (18)$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$.

Then there exists a unique additive mapping $\psi : \mathbf{G} \to \mathbf{Y}$ such that

$$\|f(x) - \psi(x)\|_{\mathbf{Y}} \le \tilde{\phi}(x, \cdots, x, x, \cdots, x) \quad (19)$$

for all $x \in \mathbf{G}$.

Proof. Replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (18), we get

$$\left(\left|2k^2 + k\right| - |2k|\right) \|f(0)\|_{\mathbf{Y}} \le 0. \quad (20)$$

so $f(0) = 0$.

Next I replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(2kx, \cdots, 2kx, 0, \cdots, 0, -x, \cdots, -x)$ in (18), I get

$$\left\|kf(2kx) - 2k^2 f(x)\right\|_{\mathbf{Y}} \le \phi(2kx, 2kx, \cdots, 2kx, 0, 0, \cdots, 0, -x, -x, \cdots, -x) \quad (21)$$

$$\left\|f(x) - \frac{1}{2k} f(2kx)\right\|_{\mathbf{Y}} \le \frac{1}{2k^2} \phi(2kx, 2kx, \cdots, 2kx, 0, 0, \cdots, 0, -x, -x, \cdots, -x)$$

Hence

$$\left\| \frac{1}{(2k)^l} f\left((2k)^l x\right) - \frac{1}{(2k)^m} f\left((2k)^m x\right) \right\|_Y$$

$$\leq \sum_{j=l}^{m-1} \left\| \frac{1}{(2k)^j} f\left((2k)^j x\right) - \frac{1}{(2k)^{j+1}} f\left((2k)^{j+1} x\right) \right\|_Y \tag{22}$$

$$\leq \frac{1}{2k^2} \sum_{j=l+1}^{m} \frac{1}{(2k)^j} \phi\left((2k)^{j+1} x, \cdots, (2k)^{j+1} x, 0, 0, \cdots, 0, -(2k)^j x, \cdots, -(2k)^j x\right)$$

$$= 0$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{G}$. It follows from (22) that the sequence $\left\{ \frac{1}{(2k)^n} f\left((2k)^n x\right) \right\}$ is a cauchy sequence for all $x \in \mathbf{G}$. Since $\mathbf{Y}$ is complete space, the sequence $\left\{ \frac{1}{(2k)^n} f\left((2k)^n x\right) \right\}$ coverges.

So one can define the mapping $\psi : \mathbf{G} \to \mathbf{Y}$ by

$$\psi(x) := \lim_{n\to\infty} \frac{1}{(2k)^n} f\left((2k)^n x\right)$$

for all $x \in \mathbf{G}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (22), I get (19).

Now, It follows from (18) I have

$$\left\| \sum_{j=1}^{k} \psi(x_j) + \sum_{j=1}^{k} \psi(y_j) + 2k \sum_{j=1}^{k} \psi(z_j) \right\|_{\mathbf{Y}}$$

$$= \lim_{n\to\infty} \left\| \frac{1}{(2k)^n} \sum_{j=1}^{k} f\left((2k)^n x_j\right) + \frac{1}{(2k)^n} \sum_{j=1}^{k} f\left((2k)^n y_j\right) + 2k \frac{1}{(2k)^n} \sum_{j=1}^{k} f\left((2k)^n z_j\right) \right\|_{\mathbf{Y}}$$

$$= \lim_{n\to\infty} \frac{1}{(2k)^n} \left\| \sum_{j=1}^{k} f\left((2k)^n x_j\right) + \sum_{j=1}^{k} f\left((2k)^n y_j\right) + 2k \sum_{j=1}^{k} f\left((2k)^n z_j\right) \right\|_{\mathbf{Y}}$$

$$\leq \lim_{n\to\infty} \frac{1}{(2k)^n} \left(\left\| 2kf\left((2k)^n \sum_{j=1}^{k} \frac{x_j + y_j}{2k} + (2k)^n \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} \right.$$

$$\left. + \phi\left((2k)^n x_1, \cdots, (2k)^n x_k, (2k)^n y_1, \cdots, (2k)^n y_k, (2k)^n z_1, \cdots, (2k)^n z_k\right) \right)$$

$$= \left\| 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} \tag{23}$$

So I have

$$\left\| \sum_{j=1}^{k} \psi(x_j) + \sum_{j=1}^{k} \psi(y_j) + 2k \sum_{j=1}^{k} \psi(z_j) \right\|_{\mathbf{Y}} \leq \left\| 2k\psi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} \tag{24}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$.

Hence from Lemma 1 and corollary 1 it follows that ψ is an additive mapping.

Finally I have to prove that ψ is a unique additive mapping.

Now, let $\psi':\mathbf{G}\to\mathbf{Y}$ be another generalized *Cauchy-Jensen* additive mapping satisfying (19). Then I have

$$\begin{aligned}
\|\psi(x)-\psi'(x)\|_{\mathbf{Y}} &= \frac{1}{(2k)^n}\left\|\psi\left((2k)^n x\right)-\psi'\left((2k)^n x\right)\right\|_{\mathbf{Y}} \\
&\le \frac{1}{(2k)^n}\left(\left\|f\left((2k)^n x\right)-\psi\left((2k)^n x\right)\right\|_{\mathbf{Y}}+\left\|f\left((2k)^n x\right)-\psi'\left(\frac{x}{2^n}\right)\right\|_{\mathbf{Y}}\right) \\
&\le 2\frac{1}{(2k)^n}\tilde{\phi}\left((2k)^n x,\cdots,(2k)^n x,0,\cdots,0,(2k)^n x,\cdots,(2k)^n x\right)
\end{aligned} \tag{25}$$

which tends to zero as $n\to\infty$ for all $x\in X$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbf{G}$. This proves the uniqueness of ψ'.

□

From Theorem 2 I have the following corollarys.

Corollary 2. For $\mathbf{G}$ is a normed space and $p,r\neq 0, q>0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \\
&\le \left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}}+\theta\cdot\prod_{j=1}^{k}\|x_j\|^p\cdot\prod_{j=1}^{k}\|y_j\|^q\cdot\prod_{j=1}^{k}\|z_j\|^r
\end{aligned} \tag{26}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$ then fí a additive mapping.

Corollary 3. For $\mathbf{G}$ is a normed space and $0<p,r<1, q\neq 0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \\
&\le \left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\|x_j\|^p+\sum_{j=1}^{k}\|y_j\|^q+\sum_{j=1}^{k}\|z_j\|^r\right)
\end{aligned} \tag{27}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$. Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}}\le\theta k\left(\frac{(2k)^p}{2k-(2k)^p}\|x\|^p+\frac{1}{2k-(2k)^k}\|x\|^r\right) \tag{28}$$

for all $x\in\mathbf{G}$.

Theorem 3. For $\phi:\mathbf{G}^{3k}\to\mathbb{R}^+$ be a function such that

$$\lim_{n\to\infty}(2k)^n\phi\left(\frac{1}{(2k)^n}x_1,\cdots,\frac{1}{(2k)^n}x_k,\frac{1}{(2k)^n}y_1,\cdots,\frac{1}{(2k)^n}y_k,-\frac{1}{(2k)^n}z_1,\cdots,-\frac{1}{(2k)^n}z_k\right)=0 \tag{29}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$, and

$$\begin{aligned}
&\tilde{\phi}(x_1,\cdots,x_k,z_1,\cdots,z_k) \\
&=\sum_{n=0}^{\infty}\phi(2k)^n\phi\left(\frac{1}{(2k)^n}x_1,\cdots,\frac{1}{(2k)^n}x_k,0,0,\cdots,0,\cdots,\frac{1}{(2k)^{n+1}}z_1,\cdots,\frac{1}{(2k)^{n+1}}z_k\right)<\infty
\end{aligned} \tag{30}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$.

Suppose that be an odd mapping $f:\mathbf{G}\to\mathbf{Y}$ satisfies

$$\begin{aligned}&\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}}\\&\le\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}+\phi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned} \tag{31}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$.

Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}}\le\tilde{\phi}(x,\cdots,x,x,\cdots,x) \tag{32}$$

for all $x\in\mathbf{G}$.

Proof. Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (31), I get

$$\left(\left|2k^2+k\right|-\left|2k\right|\right)\|f(0)\|_{\mathbf{Y}}\le 0. \tag{33}$$

so $f(0)=0$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,2kx,0,\cdots,0,-x,\cdots,-x)$ in (31), I get

$$\left\|kf(2kx)-2k^2f(x)\right\|_{\mathbf{Y}}\le\phi(2kx,2kx,\cdots,2kx,0,0,\cdots,0,-x,-x,\cdots,-x) \tag{34}$$

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\|_{\mathbf{Y}}\le\frac{1}{k}\phi\left(x,x,\cdots,x,0,0,\cdots,0,-\frac{x}{2k},-\frac{x}{2k},\cdots,-\frac{x}{2k}\right)$$

The remainder is similar to the proof of Theorem 2. This completes the proof. □

From Theorem 2 andTheorem 2. I have the following corollarys.

Corollary 4. For $\mathbf{G}$ is a normed space and $p,r\ne 0, q>0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that

$$\begin{aligned}&\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}}\\&\le\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}+\theta\cdot\prod_{j=1}^{k}\|x_j\|^p\cdot\prod_{j=1}^{k}\|y_j\|^q\cdot\prod_{j=1}^{k}\|z_j\|^r\end{aligned} \tag{35}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$, then *f* is a additive mapping.

Corollary 5. For $\mathbf{G}$ is a normed space and $0<p,r<1, q\ne 0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that

$$\begin{aligned}&\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+2k\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}}\\&\le\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\|x_j\|^p+\sum_{j=1}^{k}\|y_j\|^q+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \tag{36}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$. Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}}\le\theta k\left(\frac{(2k)^p}{(2k)^p-2k}\|x\|^p+\frac{1}{(2k)^k-2k}\|x\|^r\right) \tag{37}$$

for all $x \in \mathbf{G}$.

5. Establish Solutions to Functional Inequalities Based on the Definition

Now, I first study the solutions of (1). We first consider the mapping

$$E : M(\mathbf{G}, \mathbf{Y}) \to M(\mathbf{G}^r, \mathbb{R}^*)$$

as

$$E(f)(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) = \left\| \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + 2k \sum_{j=1}^{k} f(z_j) \right\| - \left\| 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|$$

then the inequalities $Ef \le 0$ is $(\phi, \tilde{\phi})$-*stable* in $M(\mathbf{G}, \mathbf{Y})$ where $(\phi, \tilde{\phi})$ is as Theorem 2 and Theorem 3.

Note that for inequalities, $\mathbf{G}$ be a m-divisible group where $m \in \mathbb{N} \setminus \{0\}$ and $\mathbf{Y}$ be a Banach spaces. Under this setting, we can show that the mapping satisfying (1) is additive. These results are give in the following.

Theorem 4. For $\phi : \mathbf{G}^{3k} \to \mathbb{R}^+$ be a function such that

$$\lim_{n \to \infty} \frac{1}{(2k)^n} \phi\left((2k)^n x_1, \cdots, (2k)^n x_k, (2k)^n y_1, \cdots, (2k)^n y_k, \cdots, (2k)^n z_1, \cdots, (2k)^n z_k \right) = 0 \quad (38)$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$, and

$$\begin{aligned} &\tilde{\phi}(x_1, \cdots, x_k, z_1, \cdots, z_k) \\ &= \sum_{n=0}^{\infty} \frac{1}{(2k)^{n+1}} \Big(\phi\left((2k)^{n+1} x_1, \cdots, (2k)^{n+1} x_k, 0, \cdots, 0, -(2k)^n z_1, \cdots, -(2k)^n z_k \right) \quad (39) \\ &+ \phi\left(-(2k)^{n+1} x_1, \cdots, -(2k)^{n+1} x_k, 0, \cdots, 0, (2k)^n z_1, \cdots, (2k)^n z_k \right) \Big) < \infty \end{aligned}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$.

Suppose that a mapping $f : \mathbf{G} \to \mathbf{Y}$ satisfies $f(0) = 0$ for all $x \in \mathbf{G}$, and

$$\begin{aligned} &\left\| \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + 2k \sum_{j=1}^{k} f(z_j) - 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} \quad (40) \\ &\le \phi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) \end{aligned}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{G}$.

Then there exists a unique additive mapping $\psi : \mathbf{G} \to \mathbf{Y}$ such that

$$\| f(x) - \psi(x) \|_{\mathbf{Y}} \le \tilde{\phi}(x, \cdots, x, x, \cdots, x) \quad (41)$$

for all $x \in \mathbf{G}$.

Proof. I replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(2kx, \cdots, 2kx, 0, \cdots, 0, -x, \cdots, -x)$ in (40), I get

$$\left\| kf(2kx) + 2k^2 f(-x) \right\|_{\mathbf{Y}} \le \phi(2kx, 2kx, \cdots, 2kx, 0, 0, \cdots, 0, -x, -x, \cdots, -x) \quad (42)$$

continue I replace x by $-x$ in (42), I have

$$\left\| kf(-2kx) + 2k^2 f(x) \right\|_{\mathbf{Y}} \le \phi(-2kx, -2kx, \cdots, -2kx, 0, 0, \cdots, 0, x, x, \cdots, x) \quad (43)$$

put

$$g(x)=\frac{f(x)-f(-x)}{2} \tag{44}$$

So since (45), (43) and (44), I have

$$\left\|f(x)-\frac{1}{2k}f(2kx)\right\|_{\mathbf{Y}} \le \frac{1}{2k^2}\big(\phi(2kx,2kx,\cdots,2kx,0,0,\cdots,0,-x,-x,\cdots,-x) + \phi(-2kx,-2kx,\cdots,-2kx,0,0,\cdots,0,x,x,\cdots,x)\big) \tag{45}$$

Hence

$$\begin{aligned}
&\left\|\frac{1}{(2k)^l}f\left((2k)^l x\right)-\frac{1}{(2k)^m}f\left((2k)^m x\right)\right\|_{\mathbf{Y}} \\
&\le \sum_{j=l}^{m-1}\left\|\frac{1}{(2k)^j}f\left((2k)^j x\right)-\frac{1}{(2k)^{j+1}}f\left((2k)^{j+1} x\right)\right\|_{\mathbf{Y}} \\
&\le \frac{1}{2k^2}\sum_{j=l+1}^{m}\frac{1}{(2k)^j}\Big(\phi\left((2k)^{j+1}x,\cdots,(2k)^{j+1}x,0,0,\cdots,0,-(2k)^j x,\cdots,-(2k)^j x\right) \\
&+\phi(-2kx,-2kx,\cdots,-2kx,0,0,\cdots,0,x,x,\cdots,x)\Big) \\
&=0
\end{aligned} \tag{46}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbf{G}$. It follows from (46) that the sequence $\left\{\frac{1}{(2k)^n}f\left((2k)^n x\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{G}$. Since $\mathbf{Y}$ is complete space, the sequence $\left\{\frac{1}{(2k)^n}f\left((2k)^n x\right)\right\}$ coverges.

So one can define the mapping $\psi:\mathbf{G}\to\mathbf{Y}$ by

$$\psi(x):=\lim_{n\to\infty}\frac{1}{(2k)^n}f\left((2k)^n x\right)$$

for all $x\in\mathbf{G}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (46), I get (41).

Now, It follows from (40)we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{k}\psi(x_j)+\sum_{j=1}^{k}\psi(y_j)+2k\sum_{j=1}^{k}\psi(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&=\lim_{n\to\infty}\left\|\frac{1}{(2k)^n}\sum_{j=1}^{k}f\left((2k)^n x_j\right)+\frac{1}{(2k)^n}\sum_{j=1}^{k}f\left((2k)^n y_j\right)\right. \\
&\left.+2k\frac{1}{(2k)^n}\sum_{j=1}^{k}f\left((2k)^n z_j\right)-2kf\left((2k)^n\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+(2k)^n\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&=\lim_{n\to\infty}\frac{1}{(2k)^n}\left\|\sum_{j=1}^{k}f\left((2k)^n x_j\right)+\sum_{j=1}^{k}f\left((2k)^n y_j\right)+2k\sum_{j=1}^{k}f\left((2k)^n z_j\right)\right. \\
&\left.-2kf\left((2k)^n\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+(2k)^n\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&\le\phi\left((2k)^n x_1,\cdots,(2k)^n x_k,(2k)^n y_1,\cdots,(2k)^n y_k,(2k)^n z_1,\cdots,(2k)^n z_k\right)=0
\end{aligned} \tag{47}$$

So I have

$$\sum_{j=1}^{k}\psi(x_j)+\sum_{j=1}^{k}\psi(y_j)+2k\sum_{j=1}^{k}\psi(z_j)=2k\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right) \tag{48}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$.

Hence from Lemma 2 and corollary 1, it follows that ψ is an additive mapping.

Finally I have to prove that ψ is a unique additive mapping.

Now, let $\psi':\mathbf{G}\to\mathbf{Y}$ be another generalized *Cauchy-Jensen* additive mapping satisfying (41). Then we have

$$\begin{aligned}
&\|\psi(x)-\psi'(x)\|_{\mathbf{Y}}=\frac{1}{(2k)^n}\left\|\psi\left((2k)^n x\right)-\psi'\left((2k)^n x\right)\right\|_{\mathbf{Y}}\\
&\le\frac{1}{(2k)^n}\left(\left\|f\left((2k)^n x\right)-\psi\left((2k)^n x\right)\right\|_{\mathbf{Y}}+\left\|f\left((2k)^n x\right)-\psi'\left(\frac{x}{2^n}\right)\right\|_{\mathbf{Y}}\right)\\
&\le 2\frac{1}{(2k)^n}\tilde{\phi}\left((2k)^n x,\cdots,(2k)^n x,0,\cdots,0,(2k)^n x,\cdots,(2k)^n x\right)\\
&=\sum_{n=0}^{\infty}\frac{1}{(2k)^{n+1}}\Big(\phi\left((2k)^{n+1}x_1,\cdots,(2k)^{n+1}x_k,0,\cdots,0,-(2k)^n z_1,\cdots,-(2k)^n z_k\right)\\
&\quad+\phi\left(-(2k)^{n+1}x_1,\cdots,-(2k)^{n+1}x_k,0,\cdots,0,(2k)^n z_1,\cdots,(2k)^n z_k\right)\Big)<\infty
\end{aligned} \tag{49}$$

which tends to zero as $n\to\infty$ for all $x\in X$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbf{X}$. This proves the uniquence of ψ'.

From Theorem 4 I have the following corollarys.

Corollary 6. For $\mathbf{G}$ is a normed space and $p,r\neq 0,q>0,\theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that $f(0)=0$ and

$$\begin{aligned}
&\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\theta\cdot\prod_{j=1}^{k}\|x_j\|^p\cdot\prod_{j=1}^{k}\|y_j\|^q\cdot\prod_{j=1}^{k}\|z_j\|^r
\end{aligned} \tag{50}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$ then *f* is an additive mapping.

Corollary 7. For $\mathbf{G}$ is a normed space and $0<p,r<1,q\neq 0,\theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that $f(0)=0$ and

$$\begin{aligned}
&\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\theta\left(\sum_{j=1}^{k}\|x_j\|^p+\sum_{j=1}^{k}\|y_j\|^q+\sum_{j=1}^{k}\|z_j\|^r\right)
\end{aligned} \tag{51}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{G}$. Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbf{Y}}\le\theta k\left(\frac{(2k)^p}{2k-(2k)^p}\|x\|^p+\frac{1}{2k-(2k)^k}\|x\|^r\right) \tag{52}$$

for all $x\in\mathbf{G}$.

Theorem 5. For $\phi:\mathbf{G}^{3k}\to\mathbb{R}^+$ be a function such that

$$\lim_{n\to\infty}(2k)^n \phi\left(\frac{1}{(2k)^n}x_1,\cdots,\frac{1}{(2k)^n}x_k,\frac{1}{(2k)^n}y_1,\cdots,\frac{1}{(2k)^n}y_k,\cdots,\frac{1}{(2k)^n}z_1,\cdots,\frac{1}{(2k)^n}z_k\right)=0 \quad (53)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$.

And

$$\begin{aligned}&\tilde{\phi}(x_1,\cdots,x_k,z_1,\cdots,z_k)\\&=\sum_{n=0}^{\infty}(2k)^{n-1}\left(\phi\left((2k)^{-n}x_1,\cdots,(2k)^{-(n+1)}x_k,0,\cdots,0,-(2k)^{n+1}z_1,\cdots,-(2k)^{n+1}z_k\right)\right.\\&\left.+\phi\left(-(2k)^{-n}x_1,\cdots,-(2k)^{-n}x_k,0,\cdots,0,(2k)^{n+1}z_1,\cdots,(2k)^{n+1}z_k\right)\right)<\infty\end{aligned} \quad (54)$$

for all $x_1,\cdots,x_k,z_1,\cdots,z_k \in \mathbf{G}$.

Suppose that a mapping $f:\mathbf{G}\to\mathbf{Y}$ satisfies $f(0)=0$ for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$.

And

$$\begin{aligned}&\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\phi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned} \quad (55)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$.

Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}}\le\tilde{\phi}(x,\cdots,x,x,\cdots,x) \quad (56)$$

for all $x\in\mathbf{G}$.

The proof is similar to theorem 4.

Corollary 8. For $\mathbf{G}$ is a normed space and $p,r\neq 0, q>0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that $f(0)=0$ and

$$\begin{aligned}&\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\theta\cdot\prod_{j=1}^{k}\|x_j\|^p\cdot\prod_{j=1}^{k}\|y_j\|^q\cdot\prod_{j=1}^{k}\|z_j\|^r\end{aligned} \quad (57)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$ then f í a additive mapping.

Corollary 9. For $\mathbf{G}$ is a normed space and $0<p,r<1, q\neq 0, \theta>0$. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a function such that $f(0)=0$ and

$$\begin{aligned}&\left\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+2k\sum_{j=1}^{k}f(z_j)-2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\theta\left(\sum_{j=1}^{k}\|x_j\|^p+\sum_{j=1}^{k}\|y_j\|^q+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \quad (58)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{G}$. Then there exists a unique additive mapping $\psi:\mathbf{G}\to\mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}}\le\theta k\left(\frac{(2k)^p}{2k-(2k)^p}\|x\|^p+\frac{1}{2k-(2k)^k}\|x\|^r\right) \quad (59)$$

for all $x\in\mathbf{G}$.

6. The Stability of Derivation on Fuzzy-Algebras

Lemma 6. Let $(\mathbf{Y},\mathbb{N})$ be a fuzzy normed vector space and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$N\left(\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+2k\sum_{j=1}^{k} f\left(z_j\right),t\right)\geq N\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right),\frac{t}{2k}\right) \quad (60)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{Y}$ and all $t>0$. Then f is Cauchy additive.

Proof. I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (60), I have

$$N\left(\left(2k^2+2k\right)f(0),t\right)=N\left(f(0),\frac{t}{2k^2+2k}\right)\geq N\left(2kf(0),\frac{t}{2k}\right)=1 \quad (61)$$

for all $t>0$. By N_5 and N_6, $N\left(f(0),\frac{t}{2k}\right)=1$. It follows N_2 that $f(0)=0$.

Next I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(-y,\cdots,-y,y,\cdots,y,0,\cdots,0)$ in (60), I have

$$N\left(kf(-y)+kf(y),t\right)=N\left(f(-y)+f(y),\frac{t}{k}\right)\geq N\left(2kf(0),\frac{t}{2k^2+2k}\right) \quad (62)$$

It follows N_2 that $f(-y)+f(y)=0$.

So

$$f(-y)=-f(y)$$

Next I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(-2z,\cdots,-2z,0,\cdots,0,z,0,\cdots,0)$ in (60), we have

$$N\left(-kf(2z)+2kf(z),t\right)=N\left(f(-2z)+2f(z),\frac{t}{k}\right)\geq N\left(2kf(0),\frac{t}{2k^2+2k}\right) \quad (63)$$

It follows N_2 that $f(-2z)+2f(z)=0$.

So

$$f(2z)=2f(z)$$

for all $t>0$ and for all $z\in\mathbf{X}$.

Next I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $\left(x,\cdots,x,y,\cdots,y,z_1=-\frac{x+y}{2},z_2=0,\cdots,0\right)$ in (60), we have

$$\begin{aligned}N\left(f(x)+f(y)-f(x+y),\frac{t}{k}\right)&=N\left(f(x)+f(y)+2f\left(-\frac{x+y}{2}\right),\frac{t}{k}\right)\\&\geq N\left(2kf(0),\frac{t}{2k^2+2k}\right)\end{aligned} \quad (64)$$

for all $t>0$. and for all $x,y\in\mathbf{X}$ Thus

$$f(x)+f(y)=f(x+y)$$

for all $x,y\in\mathbf{X}$, as desired.

□

Theorem 7. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an

$L < \frac{1}{2k}$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) \leq \frac{L}{2k}\psi(2kx_1,\cdots,2kx_k,2ky_1,\cdots,2ky_k,2kz_1,\cdots,2kz_k) \tag{65}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$ and $f(0)=0$.

Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping sattisfying

$$N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qf(x_j)-\sum_{j=1}^{k}qf(y_j)-2k\sum_{j=1}^{k}qf(z_j),t\right) \geq \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)} \tag{66}$$

$$N\left(f\left(\prod_{j=1}^{k}x_j\cdot y_j\right)-\prod_{j=1}^{k}f(x_j)\cdot\prod_{j=1}^{k}y_j-\prod_{j=1}^{k}x_j\cdot\prod_{j=1}^{k}f(y_j),t\right) \geq \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)} \tag{67}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$ and for all $q>0$. Then

$$H(x) = N-\lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right) \tag{68}$$

exists each $x\in\mathbf{X}$ and defines a fuzzy derivation $H:\mathbf{X}\to\mathbf{X}$, such that

$$N(f(x)-H(x),t) \geq \frac{(1-L)t}{(1-L)+L\psi(x_1,\cdots,x_k,0,\cdots,0)} \tag{69}$$

for all $t>0$ and for all $q>0$.

Proof. Letting $q=1$ and I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (84), I get

$$N\left(2kf\left(\frac{x}{2k}\right)-f(x),t\right) \geq \frac{t}{1+\varphi(x,\cdots,0,0,\cdots,0,0,\cdots,0)} \tag{70}$$

for all $x\in\mathbf{X}$. Now I consider the set

$$\mathbb{M} := \{h:\mathbf{X}\to\mathbf{Y}\}$$

and introduce the generalized metric on S as follows:

$$d(g,h) := \inf\left\{\beta\in\mathbb{R}_+ : N(g(x)-h(x),\beta t) \geq \frac{t}{t+\varphi(x,0,\cdots,0,0,\cdots,0,0,\cdots,0)}, \forall x\in\mathbf{X}, \forall t>0\right\}, \tag{71}$$

where, as usual, $\inf\phi=+\infty$. That has been proven by mathematicians $(\mathbb{M},d)$ is complete (see [32]).

Now I cosider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x) := 2kg\left(\frac{x}{2k}\right)$$

for all $x\in\mathbf{X}$. Let $g,h\in\mathbb{M}$ be given such that $d(g,h)=\varepsilon$ then

$$N\left(g(x)-h(x),\varepsilon t\right)\geq\frac{t}{t+\varphi\left(x,0,\cdots,0,0,\cdots,0,0,\cdots,0\right)},\forall x\in\mathbf{X},\forall t>0.$$

Hence

$$\begin{aligned}N\left(g(x)-h(x),\varepsilon t\right)&=N\left(2kg\left(\frac{x}{2k}\right)-2kh\left(\frac{x}{2k}\right),L\varepsilon t\right)\\&=N\left(g\left(\frac{x}{2k}x\right)-h\left(\frac{x}{2k}x\right),\frac{L}{2k}\varepsilon t\right)\\&\geq\frac{\frac{Lt}{2k}}{\frac{Lt}{2k}+\varphi\left(\frac{x}{2k},\cdots,0,0,\cdots,0,0,\cdots,0\right)}\\&\geq\frac{\frac{Lt}{2k}}{\frac{Lt}{2k}+\frac{L}{2k}\varphi\left(x,0,\cdots,0,\cdots,0,0,\cdots,0\right)}\\&=\frac{t}{t+\varphi\left(x,x,\cdots,x,x,\cdots,x\right)},\forall x\in\mathbf{X},\forall t>0.\end{aligned}\tag{72}$$

So $d(g,h)=\varepsilon$ implies that $d(Tg,Th)\leq L\cdot\varepsilon$. This means that

$$d(Tg,Th)\leq Ld(g,h)$$

for all $g,h\in\mathbb{M}$. It folows from (70) that I have.

For all $x\in\mathbb{X}$. So $d(f,Tf)\leq 1$. By Theorem 1.2, there exists a mapping $H:\mathbf{X}\to\mathbf{Y}$ satisfying the fllowing:

1) H is a fixed point of T, *i.e.*,

$$H\left(\frac{x}{2k}\right)=\frac{1}{2k}H(x)\tag{73}$$

for all $x\in\mathbf{X}$. The mapping H is a unique fixed point T in the set

$$\mathbb{Q}=\left\{g\in\mathbb{M}:d(f,g)<\infty\right\}.$$

This implies that H is a unique mapping satisfying (73) such that there exists a $\beta\in(0,\infty)$ satisfying

$$N\left(f(x)-H(x),\beta t\right)\geq\frac{t}{t+\varphi\left(x,0,\cdots,0,0,\cdots,0,0,\cdots,0\right)},\forall x\in\mathbf{X}.$$

2) $d\left(T^{l}f,H\right)\to 0$ as $l\to\infty$. This implies equality

$$N-\lim_{l\to\infty}(2k)^{l}f\left(\frac{x}{(2k)^{l}}\right)=H(x)$$

for all $x\in\mathbb{X}$.

3) $d(f,H)\leq\frac{1}{1-L}d(f,Tf)$. which implies the inequality.

4) $d(f,H)\leq\frac{1}{1-L}$.

This follows that the inequality (70) is satisfied.

By (85)

$$N\left((2k)^{p+1} f\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{(2k)^{p+1}}+\sum_{j=1}^{k}\frac{qz_j}{(2k)^p}\right)-(2k)^p\sum_{j=1}^{k}qf\left(\frac{x_j}{(2k)^p}\right)\right.$$
$$\left.-(2k)^p\sum_{j=1}^{k}qf\left(\frac{y_j}{(2k)^p}\right)-(2k)^p\,2k\sum_{j=1}^{k}qf\left(\frac{z_j}{(2k)^p}\right),t\right) \tag{74}$$
$$\geq\frac{t}{t+\psi\left(\frac{x_1}{(2k)^p},\cdots,\frac{x_k}{(2k)^p},\frac{y_1}{(2k)^p},\cdots,\frac{y_k}{(2k)^p},\frac{z_1}{(2k)^p},\cdots,\frac{z_k}{(2k)^p}\right)}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, for all $t>0$ and for all $q\in\mathbb{R}$. So

$$N\left((2k)^{p+1} f\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{(2k)^{p+1}}+\sum_{j=1}^{k}\frac{qz_j}{(2k)^p}\right)-(2k)^p\sum_{j=1}^{k}qf\left(\frac{x_j}{(2k)^p}\right)\right.$$
$$\left.-(2k)^p\sum_{j=1}^{k}qf\left(\frac{y_j}{(2k)^p}\right)-(2k)^p\,2k\sum_{j=1}^{k}qf\left(\frac{z_j}{(2k)^p}\right),t\right) \tag{75}$$
$$\geq\frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p}+\frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, for all $t>0$ and for all $q\in\mathbb{R}$.

Since

$$\lim_{n\to\infty}\frac{\frac{t}{(2k)^p}}{\frac{t}{(2k)^p}+\frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)}=1$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{X}$, $\forall t>0$, $q\in\mathbb{R}$. So

$$N\left(2kH\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qH(x_j)-\sum_{j=1}^{k}qH(y_j)-2k\sum_{j=1}^{k}qH(z_j),t\right)=1 \tag{76}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, $\forall t>0$, $q\in\mathbb{R}$. So

$$2kH\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k}qz_j\right)-\sum_{j=1}^{k}qH(x_j)-\sum_{j=1}^{k}qH(y_j)-2k\sum_{j=1}^{k}qH(z_j)=0 \tag{77}$$

Thus the mapping

$$H:\mathbf{X}\to\mathbf{X}$$

is additive and $\mathbf{R}$-linear by (85) I have

$$N\left((2k)^{2p} f\left(\prod_{j=1}^{k}\frac{x_j\cdot y_j}{(2k)^{2p}}\right)-(2k)^p\prod_{j=1}^{k}f\left(\frac{x_j}{(2k)^p}\right)\cdot\prod_{j=1}^{k}y_j\right.$$
$$\left.-\prod_{j=1}^{k}x_j\cdot(2k)^p\prod_{j=1}^{k}f\left(\frac{y_j}{(2k)^p}\right),t\right) \tag{78}$$
$$\geq\frac{t}{t+\psi\left(\frac{x_1}{(2k)^p},\cdots,\frac{x_k}{(2k)^p},\frac{y_1}{(2k)^p},\cdots,\frac{y_k}{(2k)^p},0,\cdots,0\right)}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$.

$$N\left((2k)^{2p} f\left(\prod_{j=1}^{k}\frac{x_j\cdot y_j}{(2k)^{2p}}\right)-(2k)^p\prod_{j=1}^{k} f\left(\frac{x_j}{(2k)^p}\right)\cdot\prod_{j=1}^{k} y_j -\prod_{j=1}^{k} x_j\cdot(2k)^p\prod_{j=1}^{k} f\left(\frac{y_j}{(2k)^p}\right),t\right) \geq \frac{\frac{t}{(2k)^{2p}}}{\frac{t}{(2k)^{2p}}+\frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)} \tag{79}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$ Since

$$\lim_{p\to\infty}\frac{\frac{t}{(2k)^{2p}}}{\frac{t}{(2k)^{2p}}+\frac{L^p}{(2k)^p}\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)}=1 \tag{80}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$ Thus

$$N\left(f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f(x_j)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f(y_j),t\right)=1 \tag{81}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$ Thus

$$f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f(x_j)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f(y_j)=0 \tag{82}$$

So the mapping $H:\mathbf{X}\to\mathbf{X}$ is a fuzzy derivation, as desired.

□

Theorem 8. Let $\psi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<1$

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\leq 2k\psi\left(\frac{x_1}{2k},\cdots,\frac{x_k}{2k},\frac{y_1}{2k},\cdots,\frac{y_k}{2k},\frac{z_1}{2k},\cdots,\frac{z_k}{2k}\right) \tag{83}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$ and $f(0)=0$.

Let $f:\mathbf{X}\to\mathbf{X}$ be a mapping sattisfying

$$N\left(2kf\left(\sum_{j=1}^{k}\frac{qx_j+qy_j}{2k}+\sum_{j=1}^{k} qz_j\right)-\sum_{j=1}^{k} qf(x_j)-\sum_{j=1}^{k} qf(y_j)-2k\sum_{j=1}^{k} qf(z_j),t\right) \geq \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)} \tag{84}$$

$$N\left(f\left(\prod_{j=1}^{k} x_j\cdot y_j\right)-\prod_{j=1}^{k} f(x_j)\cdot\prod_{j=1}^{k} y_j-\prod_{j=1}^{k} x_j\cdot\prod_{j=1}^{k} f(y_j),t\right) \geq \frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0)} \tag{85}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, for all $t>0$ and for all $q>0$. Then

$$\beta(x) = N - \lim_{n\to\infty} \frac{1}{(2k)^n} f\left((2k)^n x\right) \tag{86}$$

exists each $x \in \mathbf{X}$ and defines a fuzzy derivation $H : \mathbf{X} \to \mathbf{X}$.

Such that

$$N\left(f(x) - H(x), t\right) \geq \frac{(1-L)t}{(1-L) + L\psi(x_1, \cdots, x_k, 0, \cdots, 0)} \tag{87}$$

for all $t > 0$ and for all $q > 0$.

7. Conclusion

In this article, I introduced the concept of the general Jensen Cauchy functional equation, then I used a direct method to show that the solutions of the Jensen-Cauchy functional inequality are additive maps related to the functional equation, Jensen-Cauchy. Then apply the derivative setup on fuzzy algebra.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Ulam, S.M. (1960) A Collection of the Mathematical Problems. Interscience Publishers, New York.

[2] Hyers, S.D.H. (1941) On the Stability of the Linear Functional Equation. *Proceedings of the National Academy of Sciences of the United States of America*, **27**, 222-224. https://doi.org/10.1073/pnas.27.4.222

[3] Rassias, T.M. (1978) On the Stability of the Linear Mapping in Banach Spaces. *Proceedings of the AMS*, **72**, 297-300. https://doi.org/10.1090/S0002-9939-1978-0507327-1

[4] Rassias, J.M. (1984) On Approximation of Approximately Linear Mappings by Linear Mappings. *Bulletin des Sciences Mathématiques*, **108**, 445-446.

[5] Găvruta, P. (1994) A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings. *Journal of Mathematical Analysis and Applications*, **184**, 431-436. https://doi.org/10.1006/jmaa.1994.1211

[6] Gilányi, A. (2002) On a Problem by K. Nikodem. *Mathematical Inequalities & Applications*, **5**, 707-710. https://doi.org/10.7153/mia-05-71

[7] Fechner, W. (2006) Stability of a Functional Inequality Associated with the Jordan-Von Neumann Functional Equation. *Aequationes Mathematicae*, **71**, 149-161. https://doi.org/10.1007/s00010-005-2775-9

[8] Rassias, T.M. (1990) Problem 16; 2, In: Report of the 27th International Symposium on Functional Equations. *Aequationes Mathematicae*, **39**, 292-293.

[9] Gajda, Z. (1991) On Stability of Additive Mappings. *International Journal of Mathematics and Mathematical Sciences*, **14**, 431-434. https://doi.org/10.1155/S016117129100056X

[10] Rassias, T.M. and Emrl, P.S. (1992) On the Behaviour of Mappings Which Do Not Satisfy Hyers-Ulam Stability. *Proceedings of the AMS*, **114**, 989-993. https://doi.org/10.1090/S0002-9939-1992-1059634-1

[11] Gavruta, P. (1994) A Generalization of the Hyers-Ulam-Rassias Stability of Ap-

proximately Additive Mappings. *Journal of Mathematical Analysis and Applications*, **184**, 431-436. https://doi.org/10.1006/jmaa.1994.1211

[12] Jung, S. (1996) On the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings. *Journal of Mathematical Analysis and Applications*, **204**, 221-226. https://doi.org/10.1006/jmaa.1996.0433

[13] Czerwik, P. (2002) Functional Equations and Inequalities in Several Variables. World Scientific, Singapore. https://doi.org/10.1142/9789812778116

[14] Hyers, D.H., Isac, G. and Rassias, T.M. (1998) Stability of Functional Equation in Several Variables. Rirkhäuser, Basel. https://doi.org/10.1007/978-1-4612-1790-9

[15] Rassias, J.M. (1984) On Approximation of Approximately Linear Mappings by Linear Mappings. *Bulletin des Sciences Mathématiques*, **108**, 445-446.

[16] Isac, G. and Rassias, T.M. (1996) Stability of Additive Mappings: Applications to Nonlinear Analysis. *International Journal of Mathematics and Mathematical Sciences*, **19**, 219-228. https://doi.org/10.1155/S0161171296000324

[17] Hyers, D.H., Isac, G. and Rassias, T.M. (1998) On the Asymptoticity Aspect of Hyers-Ulam Stability of Mappings. *Proceedings of the AMS*, **126**, 425-420. https://doi.org/10.1090/S0002-9939-98-04060-X

[18] Park, C. (2002) On the Stability of the Linear Mapping in Banach Modules. *Journal of Mathematical Analysis and Applications*, **275**, 711-720. https://doi.org/10.1016/S0022-247X(02)00386-4

[19] Park, C. (2005) Isomorphisms between Unital C*-Algebras. *Journal of Mathematical Analysis and Applications*, **307**, 753-762. https://doi.org/10.1016/j.jmaa.2005.01.059

[20] Park, C. (2008) Hyers-Ulam-Rassias Stability of Homomorphisms in Quasi-Banach Algebras. *Bulletin des Sciences Mathématiques*, **132**, 87-96. https://doi.org/10.1016/j.bulsci.2006.07.004

[21] Rassias, T.M. (2000) The Problem of S. M. Ulam for Approximately Multiplicative Mappings. *Journal of Mathematical Analysis and Applications*, **246**, 352-378. https://doi.org/10.1006/jmaa.2000.6788

[22] Rassias, T.M. (2000) On the Stability of Functional Equations in Banach Spaces. *Journal of Mathematical Analysis and Applications*, **251**, 264-284. https://doi.org/10.1006/jmaa.2000.7046

[23] Rassias, T.M. (2003) Functional Equations, Inequalities and Applications. Kluwer Academic, Dordrecht. https://doi.org/10.1007/978-94-017-0225-6

[24] Skof, F. (1983) Proprieta localie approssimazione di operatori. *Rendiconti del Seminario Matematico e Fisico di Milano*, **53**, 113-129. https://doi.org/10.1007/BF02924890

[25] Rassias, J.M. (1982) On Approximation of Approximately Linear Mappings by Linear Mappings. *Journal of Functional Analysis*, **46**, 126-130. https://doi.org/10.1016/0022-1236(82)90048-9

[26] Rassias, J.M. (1989) Solution of a Problem of Ulam. *Journal of Approximation Theory*, **57**, 268-273. https://doi.org/10.1016/0021-9045(89)90041-5

[27] Rassias, J.M. (1994) Complete Solution of the Multi-Dimensional Problem of Ulam. *Discussiones Mathematicae*, **14**, Article ID: 101107.

[28] Rassias, J.M. (2002) On Some Approximately Quadratic Mappings Being Exactly Quadratic. *The Journal of the Indian Mathematical Society*, **69**, 155-160.

[29] Baak, C., Boo, D. and Rassias, T.M. (2006) Generalized Additive Mapping in Banach Modules and Isomorphisms between C*-Algebras. *Journal of Mathematical*

Analysis and Applications, **314**, Article ID: 150161. https://doi.org/10.1016/j.jmaa.2005.03.099

[30] Ng, C.T. (1990) Jensens Functional Equation on Groups. *Aequationes Mathematicae*, **39**, 85-90. https://doi.org/10.1007/BF01833945

[31] Parnami, J.C. and Vasudeva, H.L. (1992) On Jensens Functional Equation. *Aequationes Mathematicae*, **43**, 211-218. https://doi.org/10.1007/BF01835703

[32] Haruki, H. and Rassias, T.M. (1995) New Generalizations of Jensens Functional Equation. *Proceedings of the AMS*, **123**, 495-503. https://doi.org/10.2307/2160907

[33] Van, L. (2023) An Exploiting Quadratic Exploiting Quadratic $\varphi(\delta_1, \delta_2)$-Function Inequalities on Fuzzy Banach Spaces Based on General Quadratic Equations with $2k$-Variables. *Open Journal of Mathematical Sciences*, **7**, 287-298. https://doi.org/10.30538/oms2023.0212