



# An Inverse Eigenproblems for Pseudo-Symmetric Matrices

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## Abstract

In this paper, we explore the inverse spectral problem for a kind of pseudo-symmetric matrix  $\mathcal{A}(\epsilon, c, s)$  that corresponds to a graph of a banana tree when all  $\epsilon = 1$ . The problem of reconstructing pseudo-symmetric matrices from their spectra was considered. The sufficient and necessary conditions for the existence of these matrices are provided, and an algorithm and numerical example were provided to illustrate their characteristics. In addition, the effectiveness of the solution was validated using MatlabR2021b.

## Subject Areas

Algebra, Applied Statistical Mathematics

## Keywords

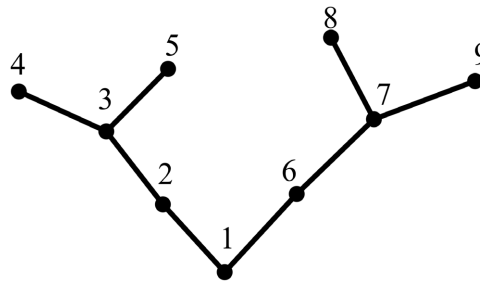
Inverse Spectral Problem, Pseudo-Symmetric, A Banana Tree, Spectral Data

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## 1. Introduction

The inverse eigenvalue problem for pseudo-symmetric matrices (PIEP) is the issue of reconstructing a matrix with a prespecified structure from given eigen-data. For example, Xu [1] reconstructed the pseudo-Jacobi matrix using its spectrum and the spectra of the two complementary principal matrices. Following this, Xu [2] [3] reconstructed both the modified pseudo-Jacobi matrix and the doubly periodic pseudo-Jacobi matrix. Su [4] [5] reconstructed the pseudo-Jacobi matrix and a special acyclic matrix using the spectral data obtained from its matrix. In addition to the previously discussed inverse spectral problem for pseudo-symmetric matrices, we know that one means of describing the structuring of a matrix is to denote it by its corresponding graph, and several authors have investigated the problem of describing the inverse of a constructed matrix in terms of a graph. As in references [6] [7] [8] [9] has investigated inverse eigenvalue





**Figure 1.** An (2, 4)-banana tree.

where main diagonal  $a = (a_1, a_2, \dots, a_9) \in \mathbb{R}$  and the two first lower and upper off-diagonals  $b_{-1} = (b_{1,2}, \dots, b_{7,9})$  and  $b_1 = (\epsilon_{1,2}b_{1,2}, \dots, \epsilon_{7,9}b_{7,9})$ , with all  $b > 0$  and all  $\epsilon \in \{1, -1\}$ . If all  $\epsilon = 1$ , the pseudo-symmetric matrix  $\mathcal{A}(\epsilon, c, s)$  changed to the symmetric matrix  $\mathcal{A}(c, s)$ .

**PIEPB**  $(\epsilon, c, s)$ . Given different numbers

$$\{\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)}\} \in \mathbb{R},$$

construct an  $n \times n$  matrix  $A_n \in \mathcal{A}(\epsilon, c, s)$  such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are the eigenvalues of  $A_j, j = 1, 2, \dots, n$ , the leading principal submatrix of  $A$ , respectively.

The structure of this work is organized as follows. In Sect.2, some properties of pseudo-symmetric matrix  $\mathcal{A}(\epsilon, c, s)$  were given. In Sect.3, the necessary and sufficient conditions for the solvability of an inverse eigenvalue problems of pseudo-symmetric matrix  $\mathcal{A}(\epsilon, c, s)$ . In Sect.4, the evolutionary plots of the distribution of eigenvalues of low-order leading principal submatrix were drawn through algorithm and numerical simulation, indicating the accuracy of the result. Finally, in Sect.5, the conclusions are presented.

## 2. Preliminaries

**Lemma 2.1.** Let  $A_n \in \mathcal{A}(\epsilon, c, s)$  and let  $A_j$  be the  $j \times j$  leading principal submatrix of  $A$ , with characteristic polynomial

$\varphi_j(\lambda) = \det(\lambda I_j - A_j), j = 1, \dots, n$ . Then the sequence  $\{\varphi_j(\lambda)\}_{j=1}^n$  satisfies the recurrence formulae

$$\varphi_1(\lambda) = \lambda - a_1,$$

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - \epsilon_{1,j}b_{1,j}^2 \det(B_j^\lambda), j = (i-1)s + 2, i = 1, 2, \dots, c,$$

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - \epsilon_{(j-1),j}b_{(j-1),j}^2\varphi_{j-2}(\lambda), j = (i-1)s + 3, i = 1, 2, \dots, c,$$

$$\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - \epsilon_{(i-1)s+3,j}b_{(i-1)s+3,j}^2\varphi_{(i-1)s+2}(\lambda) \prod_{k=(i-1)s+4}^{j-1} (\lambda - a_k),$$

$$j = (i-1)s + 4, \dots, is + 1, i = 1, 2, \dots, c.$$

where  $B_j^\lambda$  is the submatrix row and columns 2, 3, ..., n-1 of the matrix

$$\lambda I_n - A_n, \varphi_0(\lambda) = 1 \text{ and } \prod_{k=(i-1)s+4}^{j-1} (\lambda - a_k) = 1 \text{ when } j = (i-1)s + 4.$$

**Lemma 2.2.** Let  $A_n \in \mathcal{A}(\epsilon, c, s)$ , then  $\varphi_j(\lambda)$  and  $\varphi_{j-1}(\lambda)$  have no common root.

**Proof.** Supposing that  $\lambda_0$  is a common root of  $\varphi_j(\lambda)$  and  $\varphi_{j-1}(\lambda)$ , by Lemma 2.1, we get that  $\lambda_0$  is also a root of  $\varphi_{j-2}(\lambda)$ , and so we have  $\varphi_0(\lambda) = 0 \neq 1$ , Which is a contradiction. Hence  $\varphi_j(\lambda)$  and  $\varphi_{j-1}(\lambda)$  have no common root.

**Lemma 2.3. ([8])** Let  $P(\lambda)$  be a monic polynomial of degree  $n$ , with all real zeroes. If  $\lambda_1$  and  $\lambda_n$  are, respectively, the smallest and largest zero of  $P(\lambda)$ , then

- (1) If  $\mu < \lambda_1$ , we have that  $(-1)^n \varphi(\mu) > 0$ ,
- (2) If  $\mu > \lambda_n$ , we have that  $\varphi(\mu) > 0$ .

**Lemma 2.4. ([15])** Let all  $\epsilon = 1$  for  $A_n \in \mathcal{A}(\epsilon, c, s)$ ,  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$ , respectively, be the smallest and largest eigenvalues of its  $j \times j$  leading principal submatrix  $A_j$ ,  $j = 1, \dots, n$ . Then we have

$$\lambda_n^{(1)} < \lambda_{n-1}^{(1)} < \dots < \lambda_2^{(1)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \dots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)},$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, i = 2, \dots, j; j = 2, \dots, n.$$

**Lemma 2.5.** The sufficient and necessary conditions for the existence of symmetric matrix  $A_n \in \mathcal{A}(c, s)$  whose graph is an  $(c, s)$ -banana tree, such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are the smallest and largest eigenvalue of  $A_j$ , the leading principal submatrix of  $A$ , respectively, is

$$\lambda_n^{(1)} < \lambda_{n-1}^{(1)} < \dots < \lambda_3^{(1)} < \lambda_2^{(1)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \dots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}. \tag{1}$$

**Proof.** Sufficiency: suppose that  $\{\lambda_j^{(1)}, \lambda_j^{(j)}\}_{j=1}^n$  satisfies (1) for  $j = 1, \dots, n$ . To prove the existence of a symmetric matrix  $A_n \in \mathcal{A}(c, s)$  with the required properties is equivalent to going to prove that the system

$$\begin{cases} P_j(\lambda_j^{(1)}) = 0 \\ P_j(\lambda_j^{(j)}) = 0 \end{cases}, j = 1, \dots, n, \tag{2}$$

For  $j = 1$ , it is immediate that  $a_1 = \lambda_1^{(1)}$ .

According to Lemma 2.1,  $\epsilon_{1,j} = 1$  for  $j = (i-1)s + 2, i = 1, 2, \dots, c$ , system (2) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j)\varphi_{j-1}(\lambda_j^{(1)}) - b_{1,j}^2 \det(B_j^{\lambda_j^{(1)}}) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j)\varphi_{j-1}(\lambda_j^{(j)}) - b_{1,j}^2 \det(B_j^{\lambda_j^{(j)}}) = 0 \end{cases}, j = (i-1)s + 2, i = 1, 2, \dots, c,$$

by condition (1) and Lemma 2.3, we get  $(-1)^{j-1} \varphi_{j-1}(\lambda_j^{(1)}) > 0$ ,  $\det(B_j^{\lambda_j^{(1)}}) > 0$ ,  $\varphi_{j-1}(\lambda_j^{(j)}) > 0$ , and  $(-1)^{j-2} \det(B_j^{\lambda_j^{(j)}}) > 0$ , then

$$a_j = \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})}{\varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})},$$

$$b_{1,j}^2 = \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})} > 0.$$

According to Lemma 2.1,  $\epsilon_{(j-1),j} = 1$  for  $j = (i-1)s + 3, i = 1, 2, \dots, c$ , system (2) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j) \varphi_{j-1}(\lambda_j^{(1)}) - b_{(j-1),j}^2 \varphi_{j-2}(\lambda_j^{(1)}) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j) \varphi_{j-1}(\lambda_j^{(j)}) - b_{(j-1),j}^2 \varphi_{j-2}(\lambda_j^{(j)}) = 0 \end{cases}, j = (i-1)s + 3, i = 1, 2, \dots, c$$

by condition (1) and Lemma 2.3, we have  $(-1)^{j-1} \varphi_{j-1}(\lambda_j^{(1)}) > 0$  and  $\varphi_{j-1}(\lambda_j^{(j)}) > 0$  then

$$a_j = \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})},$$

$$b_{(j-1),j}^2 = \frac{\lambda_j^{(j)} - \lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})} > 0.$$

According to Lemma 2.1,  $\epsilon_{(i-1)s+3,j} = 1$  for  $j = (i-1)s + 4, \dots, is + 1, i = 1, 2, \dots, c$ , system (2) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j) \varphi_{j-1}(\lambda_j^{(1)}) - b_{(i-1)s+3,j}^2 \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j) \varphi_{j-1}(\lambda_j^{(j)}) - b_{(i-1)s+3,j}^2 \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) = 0 \end{cases},$$

by condition (1), Lemma 2.3 and Lemma 2.4, we get  $(-1)^{j-1} \varphi_{j-1}(\lambda_j^{(1)}) > 0$ ,

$$\varphi_{j-1}(\lambda_j^{(j)}) > 0, (-1)^{(i-1)s+2} \varphi_{(i-1)s+2}(\lambda_j^{(1)}) > 0, \varphi_{(i-1)s+2}(\lambda_j^{(j)}) > 0,$$

$$(-1)^{j-(i-1)s-4} \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k) > 0, \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) > 0, \text{ then}$$

$$a_j = \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)},$$

$$b_{(i-1)s+3,j}^2 = \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)} > 0.$$

**Necessity:** Assuming that there exists a unique matrix  $A_n \in \mathcal{A}(c, s)$  such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are the smallest and maximal eigenvalues of their corres-

ponding principal submatrices, respectively, and by Cauchy's alternating theorem and Lemma 2.4, (1) holds.

### 3. Results

#### 3.1. The Solution of PIEPB $(\epsilon, c, s)$

Theorem 3.1. Let different numbers  $\{\lambda_n^{(1)}, \lambda_{n-1}^{(1)}, \dots, \lambda_1^{(1)}, \lambda_2^{(2)}, \dots, \lambda_n^{(n)}\} \in \mathbb{R}$ , if

$$\varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}}) \neq 0, \tag{3}$$

$$\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)}) \neq 0, \tag{4}$$

$$\begin{aligned} &\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) \\ &- \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k) \neq 0, \end{aligned} \tag{5}$$

where  $j = 2, 3, \dots, n$ , then there exists a matrix  $A_n \in \mathcal{A}(\epsilon, c, s)$ , such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the eigenvalues of  $A_j, j = 1, \dots, n$ .

**Proof.** Sufficiency: suppose the conditions (3) (4) and (5) hold. To prove the existence of a pseudo-symmetric matrix  $A_n \in \mathcal{A}(\epsilon, c, s)$  is equivalent to proving that

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = 0 \\ \varphi_j(\lambda_j^{(j)}) = 0 \end{cases}, j = 1, \dots, n, \tag{6}$$

When  $j = 1$ , it is immediate that  $a_1 = \lambda_1^{(1)}$ .

According to Lemma 2.1 for  $j = (i-1)s + 2, i = 1, 2, \dots, c$ , system (6) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j) \varphi_{j-1}(\lambda_j^{(1)}) - \epsilon_{1,j} b_{1,j}^2 \det(B_j^{\lambda_j^{(1)}}) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j) \varphi_{j-1}(\lambda_j^{(j)}) - \epsilon_{1,j} b_{1,j}^2 \det(B_j^{\lambda_j^{(j)}}) = 0 \end{cases}, j = (i-1)s + 2, i = 1, 2, \dots, c,$$

by Lemma 2.3, we have

$$\begin{aligned} a_j &= \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})}{\varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})}, \tag{7} \\ \epsilon_{1,j} b_{1,j}^2 &= \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)}) \det(B_j^{\lambda_j^{(1)}})} = \alpha. \end{aligned}$$

according to Lemma 2.2,  $(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)}) \neq 0$ , added to condition (3), we get  $\alpha \neq 0$ , and

$$b_{1,j} = \sqrt{|\alpha|} > 0, \quad \epsilon_{1,j} = \frac{\alpha}{|\alpha|}. \tag{8}$$

According to Lemma 2.1 for  $j = (i-1)s + 3, i = 1, 2, \dots, c$ , system (6) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j)\varphi_{j-1}(\lambda_j^{(1)}) - \epsilon_{(j-1),j} b_{(j-1),j}^2 \varphi_{j-2}(\lambda_j^{(1)}) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j)\varphi_{j-1}(\lambda_j^{(j)}) - \epsilon_{(j-1),j} b_{(j-1),j}^2 \varphi_{j-2}(\lambda_j^{(j)}) = 0 \end{cases}, j = (i-1)s + 3, i = 1, 2, \dots, c,$$

by Lemma 2.3, we get

$$a_j = \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})}, \tag{9}$$

$$\epsilon_{(j-1),j} b_{(j-1),j}^2 = \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{j-2}(\lambda_j^{(1)})} = \beta.$$

according to Lemma 2.2,  $(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)}) \neq 0$ , added to condition (4), we get  $\beta \neq 0$ , and

$$b_{(j-1),j} = \sqrt{|\beta|} > 0, \quad \epsilon_{(j-1),j} = \frac{\beta}{|\beta|}. \tag{10}$$

According to Lemma 2.1 for  $j = (i-1)s + 4, \dots, is + 1, i = 1, 2, \dots, c$ , system (6) can be written as

$$\begin{cases} \varphi_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j)\varphi_{j-1}(\lambda_j^{(1)}) - \epsilon_{(i-1)s+3,j} b_{(i-1)s+3,j}^2 \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k) = 0 \\ \varphi_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j)\varphi_{j-1}(\lambda_j^{(j)}) - \epsilon_{(i-1)s+3,j} b_{(i-1)s+3,j}^2 \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) = 0 \end{cases},$$

by Lemma 2.3, we have

$$a_j = \frac{\lambda_j^{(1)} \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \lambda_j^{(j)} \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)}, \tag{11}$$

$$\epsilon_{(i-1)s+3,j} b_{(i-1)s+3,j}^2 = \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)})}{\varphi_{j-1}(\lambda_j^{(1)}) \varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)}) \varphi_{(i-1)s+2}(\lambda_j^{(1)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(1)} - a_k)} = \theta.$$

according to Lemma 2.2,  $(\lambda_j^{(j)} - \lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(1)}) \varphi_{j-1}(\lambda_j^{(j)}) \neq 0$ , added to condition (5), we get  $\theta \neq 0$ , and

$$b_{(i-1)s+3,j} = \sqrt{|\theta|} > 0, \quad \epsilon_{(i-1)s+3,j} = \frac{\theta}{|\theta|}. \tag{12}$$

**Conversely.** Suppose that PIEPB  $(\epsilon, c, s)$  has a unique solution, i.e., the system (6) has solutions  $a_j, b$  and satisfies all  $b > 0$ . In the same way that we proved the sufficiency, we get (7)-(12). Thus (3), (4) and (5) hold.

### 3.2. Algorithm 1

Algorithm1 (Unique solution of the **PIEPB**  $(\epsilon, c, s)$ ).

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1: **Input:**  $c, s, \{\lambda_j^{(l)}, \lambda_j^{(j)}\}_{j=1}^n$  . where  $n = cs + 1$  .

2:  $a_1 = \lambda_1^{(l)}$  .

3: **For**  $j = (i-1)s + 2, i = 1, 2, \dots, c$  **do**

4:   **if**  $\varphi_{j-1}(\lambda_j^{(l)})\det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)})\det(B_j^{\lambda_j^{(l)}}) = 0$  **then**

5:         ending the algorithm 1.

6:   **end if**

7:   **if**  $\varphi_{j-1}(\lambda_j^{(l)})\det(B_j^{\lambda_j^{(j)}}) - \varphi_{j-1}(\lambda_j^{(j)})\det(B_j^{\lambda_j^{(l)}}) \neq 0$  **then**

8:         computing  $a_j, b_{1,j}$  and  $\epsilon_{1,j}$  by (7) and (8).

9:   **end if**

10: **end for**

11: **For**  $j = (i-1)s + 3, i = 1, 2, \dots, c$  **do**

12:   **if**  $\varphi_{j-1}(\lambda_j^{(l)})\varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)})\varphi_{j-2}(\lambda_j^{(l)}) = 0$  **then**

13:         ending the algorithm 1.

14:   **end if**

15:   **if**  $\varphi_{j-1}(\lambda_j^{(l)})\varphi_{j-2}(\lambda_j^{(j)}) - \varphi_{j-1}(\lambda_j^{(j)})\varphi_{j-2}(\lambda_j^{(l)}) \neq 0$  **then**

16:         computing  $a_j, b_{(j-1),j}$  and  $\epsilon_{(j-1),j}$  by (9) and (10).

17:   **end if**

18: **end for**

19: **For**  $j = (i-1)s + 4, \dots, is + 1, i = 1, 2, \dots, c$  **do**

20: **if**  $\varphi_{j-1}(\lambda_j^{(l)})\varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)})\varphi_{(i-1)s+2}(\lambda_j^{(l)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(l)} - a_k) = 0$

**then**

21:         ending the algorithm 1.

22:   **end if**

23: **if**  $\varphi_{j-1}(\lambda_j^{(l)})\varphi_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(j)} - a_k) - \varphi_{j-1}(\lambda_j^{(j)})\varphi_{(i-1)s+2}(\lambda_j^{(l)}) \prod_{k=(i-1)s+4}^{j-1} (\lambda_j^{(l)} - a_k) \neq 0$

**then**

24:         computing  $a_j, b_{(i-1)s+3,j}$  and  $\epsilon_{(i-1)s+3,j}$  by (11) and (12).

25:   **end if**

26: **end for**

---

## 4. Numerical Examples

### 4.1. Example 1

**Table 1**, which satisfy conditions (1). By applying MatlabR2021b, we compute the following matrix.



**Table 1.** Spectral data for Lemma 2.5.

$j$	1	2	3	4	5	6	7	8	9
$\lambda_j^{(1)}$	1.0000	-2.0000	-3.0000	-4.0000	-5.0000	-6.0000	-7.0000	-8.0000	-9.0000
$\lambda_j^{(j)}$	1.0000	2.5000	4.0000	4.5000	5.0000	6.0000	7.0000	8.0000	9.0000

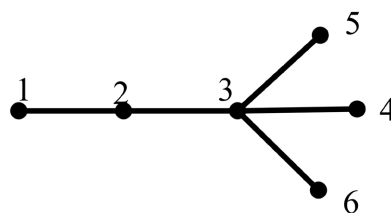
$$A_9 = \begin{bmatrix} 1.0000 & 1.9364 & 0 & 0 & 0 & 5.7852 & 0 & 0 & 0 \\ 1.9364 & 0 & 2.8722 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.8722 & 1.0000 & 3.1378 & 3.5966 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.1378 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.5966 & 0 & -1.2364 & 0 & 0 & 0 & 0 \\ 5.7852 & 0 & 0 & 0 & 0 & -1.4279 & 4.1556 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.1556 & 2.0115 & 5.7833 & 4.9181 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.7833 & -2.2485 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.9181 & 0 & -2.0778 \end{bmatrix}.$$

Recompute that: the eigenvalues of its the leading principal submatrices  $A_j$ ,  $j = 1, \dots, 9$ , are:

- $\Lambda(A_1) = \{1.0000\}$ ,
- $\Lambda(A_2) = \{-1.5000, 2.5000\}$ ,
- $\Lambda(A_3) = \{-3.0000, 1.0000, 4.0000\}$ ,
- $\Lambda(A_4) = \{-4.0000, -0.9478, 1.9478, 5.0000\}$ ,
- $\Lambda(A_5) = \{-5.5000, -1.4288, -0.4516, 2.1439, 6.0000\}$ ,
- $\Lambda(A_6) = \{-6.5000, -5.2765, -0.7170, -0.0754, 5.4045, 6.5000\}$ ,
- $\Lambda(A_7) = \{-7.5000, -5.3941, -0.8648, -0.2904, 2.0710, 5.8255, 7.5000\}$ ,
- $\Lambda(A_8) = \{-8.5000, -5.5735, -4.7300, -0.7334, -0.1191, 4.2519, 6.0028, 8.5000\}$ ,
- $\Lambda(A_9) = \{-9.5000, -5.8133, -5.0707, -2.1488, -0.7265, -0.1021, 4.7635, 6.1190, 9.5000\}$ .

**4.1. Example 2**

Given different real number  $\lambda_6^{(1)} = 4$ ,  $\lambda_5^{(1)} = 5$ ,  $\lambda_4^{(1)} = -0.1$ ,  $\lambda_3^{(1)} = 2$ ,  $\lambda_2^{(1)} = 0.1$ ,  $\lambda_1^{(1)} = 1$ ,  $\lambda_2^{(2)} = -8$ ,  $\lambda_3^{(3)} = 3$ ,  $\lambda_4^{(4)} = 0.6$ ,  $\lambda_5^{(5)} = -7$ ,  $\lambda_6^{(6)} = -9$ . construct a matrix  $A_6 \in \mathcal{A}(\epsilon, 1, 5)$ , such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are the eigenvalues of  $A_j, j = 1, 2, \dots, 6$ , respectively. In particular, this matrix corresponds to the graph is an (1, 5)-banana tree when all  $\epsilon = 1$  (Figure 2).



**Figure 2.** An (1, 5)-banana tree.

By applying algorithm 1 and matlabR2021b, we compute the following matrix

$$A_6 = \begin{bmatrix} 1.0000 & 2.8460 & 0 & 0 & 0 & 0 \\ -2.8460 & -8.9000 & 9.9680 & 0 & 0 & 0 \\ 0 & 9.9680 & -3.2295 & 2.7638 & 2.3389 & 4.4658 \\ 0 & 0 & 2.7638 & 0.0157 & 0 & 0 \\ 0 & 0 & 2.3389 & 0 & -6.9523 & 0 \\ 0 & 0 & -4.4658 & 0 & 0 & -8.8135 \end{bmatrix}.$$

Recompute that: the eigenvalues of its the leading principal submatrices  $A_j$ ,  $j = 1, \dots, 6$ , are:

$$\Lambda(A_1) = \{1.0000\}.$$

$$\Lambda(A_2) = \{0.1000, -8.0000\}.$$

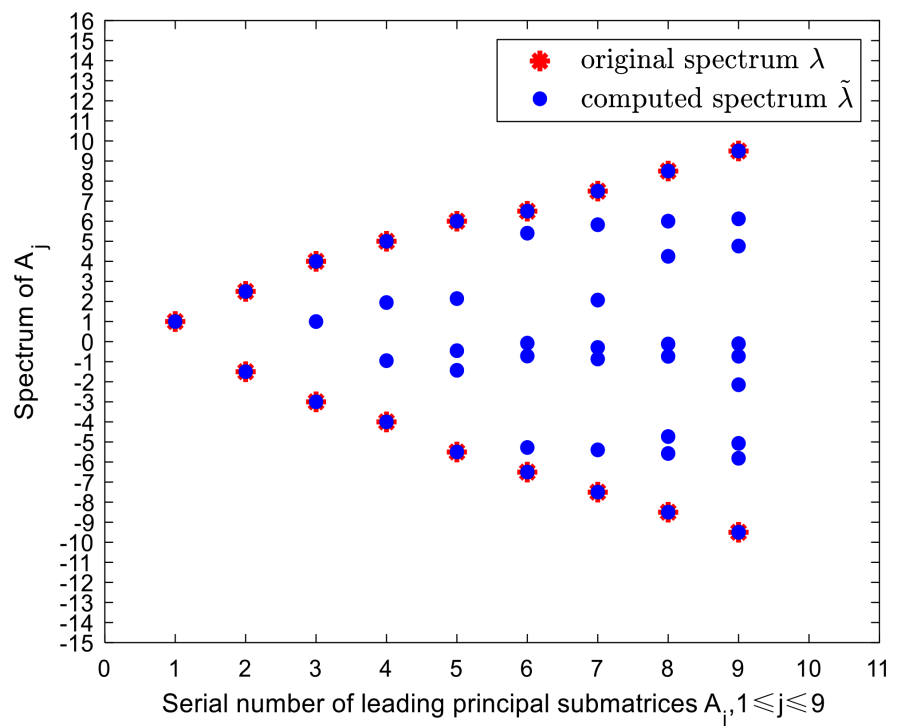
$$\Lambda(A_3) = \{3.0000, 2.0000, -16.1295\}.$$

$$\Lambda(A_4) = \{4.6961, 0.6000, -0.1000, -16.3100\}.$$

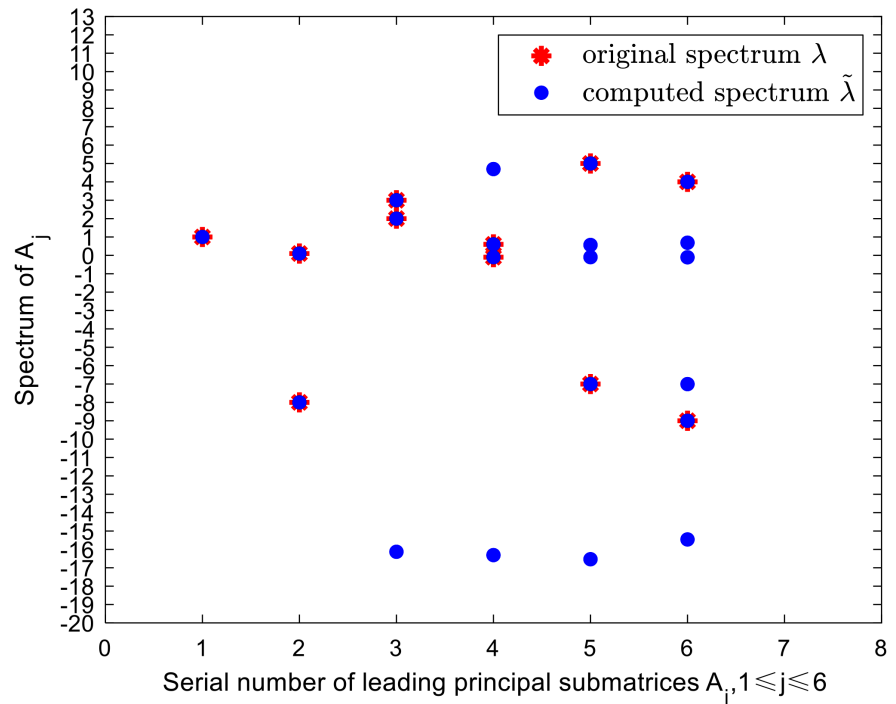
$$\Lambda(A_5) = \{5.0000, 0.5656, -0.0974, -7.0000, -16.5343\}.$$

$$\Lambda(A_6) = \{4.0000, 0.6886, -0.1053, -7.0048, -9.0000, -15.4581\}.$$

**Figure 3** displays the extreme eigenvalues of  $A_j$ ,  $j = 1, \dots, 9$ , as presented in example 1. These are in line with the recomputed eigenvalues of  $A_j$ ,  $j = 1, \dots, 9$ . Likewise, **Figure 4** presents the eigenvalues of  $A_j$ ,  $j = 1, \dots, 6$ , as featured in example 4.2, which are consistent with the recomputed eigenvalues of  $A_j$ ,  $j = 1, \dots, 6$ .



**Figure 3.** Distribution of eigenvalues of  $A_j$  in example 1.



**Figure 4.** Distribution of eigenvalues of  $A_j$  in example 2.

## 5. Conclusion

Based on matrices whose graph is a banana tree, we study the inverse spectral problem for pseudo-symmetric matrices. It is obviously difficult to reconstruct the pseudo-symmetric matrix given the eigenvalues, and in order to achieve this deficiency, other constraints are added on top of the eigendata, which in turn enables the reconstruction of the pseudo-symmetric matrix. We obtained sufficient necessary conditions to solve the problem.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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