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Multiplicity and Concentration of Solutions for Choquard Equation with Competing Potentials via Pseudo-Index Theory

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Abstract

In this paper, we consider the following nonlinear Choquard equation $-\varepsilon^2\Delta w+V(x)w=\varepsilon^{-\theta}\left(\mathcal{Y}_1(w)+\mathcal{Y}_2(w)\right)$, where $\varepsilon>0$, $N>2$, $\mathcal{Y}_1(w):=W_1(x)\left[I_\theta*\left(W_1|w|^p\right)\right]|w|^{p-2}w$, $\mathcal{Y}_2(w):=W_2(x)\left[I_\theta*\left(W_2|w|^q\right)\right]|w|^{q-2}w$, I_θ is the Riesz potential with order $\theta\in(0,N)$, $2\le p<q<\frac{N+\theta}{N-2}$, $\min_{\mathbb{R}^N}V>0$ and $\inf_{\mathbb{R}^N}W_i>0$, $i=1,2$. By imposing suitable assumptions to $V(x)$, $W_i(x), i=1,2$, we establish the multiplicity of semiclassical solutions by using pseudo-index theory and the existence of groundstate solutions by Nehari method. Moreover, the convergence and concentration of the positive groundstate solution are discussed.

Subject Areas

Partial Differential Equation

Keywords

Choquard Equation, Pseudo-Index, Multiplicity, Concentration

1. Introduction and Main Results

In this paper, we will study the following equation

$$-\varepsilon^2\Delta w+V(x)w=\varepsilon^{-\theta}\left(\mathcal{Y}_1(w)+\mathcal{Y}_2(w)\right), \tag{1.1}$$

where $\varepsilon>0$, $N>2$, $\theta\in(0,N)$, $\mathcal{Y}_1(w)=W_1(x)\left[I_\theta*\left(W_1|w|^p\right)\right]|w|^{p-2}w$, $\mathcal{Y}_2(w)=W_2(x)\left[I_\theta*\left(W_2|w|^q\right)\right]|w|^{q-2}w$, $2\le p<q<\frac{N+\theta}{N-2}$, $V,W_i,i=1,2$ are continuous bounded positive functions and the Riesz potential I_θ is defined as follows:

$$I_\theta := \frac{\Gamma\left(\frac{N-\theta}{2}\right)}{2^\theta \pi^{N/2}\Gamma\left(\frac{\theta}{2}\right)}|x|^{\theta-N}, \quad x \in \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

When $\varepsilon = 1$, Equation (1.1) is related to the local nonlinear perturbation of the famous Choquard equation

$$-\Delta u + u = \left(I_2 * u^2\right)u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

This equation for $N = 3$ was first proposed by Pekar [1] in quantum mechanics in 1954. In 1996, Penrose [2] [3] used this equation in a different context as a model for self-gravitating matter. In 1977, E. H. Lieb [4] proved that the existence and uniqueness of solutions to Equation (1.3) by using symmetric decreasing rearrangement inequalities. Thereafter, P. L. Lions [5] [6] further studied Equation (1.3) by means of a variational approach and obtained the multiplicity of solutions to the equation. Since then, the Choquard equation has been studied in a variety of environments and in many contexts.

J. N. Correia and C. P. Oliveira [7] considered

$$-\Delta u + \omega u = \left(\mathcal{K}_\mu * |u|^{q+1}\right)|u|^q + \epsilon\left(\mathcal{K}_\mu * |u|^{2_\mu^*}\right)|u|^{2_\mu^*-1} \quad \text{in } \mathbb{R}^N.$$

where $\omega = 1$, $2 \le q+1 < 2_\mu^* = \frac{2N-\mu}{N-2}$, ϵ is a positive parameter. They proved existence of positive solutions for a class of problems involving the Choquard term in exterior domain and the nonlinearity with critical growth by using variational method combined with Brouwer theory of degree and Deformation lemma. S. Yao, J. Sun and T. Wu [8] studied the following equation

$$-\Delta u + \lambda V(x)u = \left(I_\alpha * K|u|^p\right)K|u|^{p-2}u - |u|^{q-2}u \quad \text{in } \mathbb{R}^N.$$

When $N \ge 3$, $\lambda > 0$, $K(x) \ge 0$, $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $2 < q < 2^* = \frac{2N}{N-2}$. They proved different relationship between p and q when the competing effect of the nonlocal term with the perturbation happens.

For the semiclassical states of Choquard equation, we can refer to the following references. Y. Su a and Z. Liu [9] proved the following Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}g(u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $N \ge 5$, $\alpha \in (0, N)$, $g(u) = \left(I_\alpha * F(u)\right)F'(u)$, $F(u) = \frac{\lambda}{2_\alpha^\#}|u|^{2_\alpha^\#} + \frac{1}{2_\alpha^*}|u|^{2_\alpha^*}$, $2_\alpha^\# = \frac{N+\alpha}{N}$, $2_\alpha^* := \frac{N+\alpha}{N-2}$. Working in a variational setting, they showed the existence, multiplicity and concentration of positive solutions for such equations when the potential satisfies some suitable conditions. Y. Meng and X. He [10] considered the multiplicity and concentration phenomenon of positive solutions to Equation (1.4) in which $g(u)=Q(x)\left(I_\alpha*|u|^{2_\alpha^*}\right)|u|^{2_\alpha^*-2}u+f(u)$, $N\ge 3$, $(N-4)_+<\alpha<N$, $V(x)\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is a positive potential, $f\in C^1(\mathbb{R}^+,\mathbb{R})$ is a subcritical nonlinear term. By means of variational methods and delicate energy estimates, they established the relationship between the number of solutions and the profiles of potentials V and Q, and the concentration behavior of positive solutions is also obtained for $\varepsilon>0$ small.

Y. Ding and J. Wei [11] considered the following Schrödinger equation

$$-\varepsilon^2\Delta w+V(x)w=W(x)|w|^{p-2}w \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon>0$, $p\in\left(2,\frac{2N}{N-2}\right)$ and V,W are continuous bounded positive functions, they proved existence and concentration phenomena of semiclassical positive groundstate solutions, and multiplicity of solutions including at least one pair of sign-changing ones by pseudo-index theory and Nehari method. Later, M. Liu and Z. Tang [12] extended their research to Choquard equations.

Motivated by the above conclusions, this article mainly discusses the existence, convergence, concentration, and asymptotic property of positive groundstate solution of Equation (1.1). We also establish the multiplicity of semiclassical solutions for Equation (1.1) by pseudo-index theory which was imposed by V. Benci. The equation studied in this paper has two convolution terms and two nonlinear potentials, which bring new challenge in our arguements. Our method of proof is inspired by [11] and our conclusions extend that in [12].

Before stating the main results, we need to make some assumptions.

($\mathcal{A}1$) $V,W_i\in C^{0,\mu}(\mathbb{R}^N)$ are bounded with some $\mu\in(0,1)$, $V(x)$ achieves a global minimum on $\mathbb{R}^N$ with $\min_{\mathbb{R}^N}V(x)>0$, and $W_i(x)$ achieves a global maximum on $\mathbb{R}^N$ with $\inf_{\mathbb{R}^N}W_i(x)>0$ $\inf_{\mathbb{R}^N}W_i(x)>0$, $i=1,2$.

For $i=1,2$, we denote by

$$\tau:=\min_{\mathbb{R}^N}V,\quad \mathscr{V}:=\left\{x\in\mathbb{R}^N:V(x)=\tau\right\},\quad \tau_\infty:=\liminf_{|x|\to\infty}V(x);$$

$$k_i:=\max_{\mathbb{R}^N}W_i,\quad \mathscr{W}_i:=\left\{x\in\mathbb{R}^N:W_i(x)=k_i\right\},\quad k_{i\infty}:=\limsup_{|x|\to\infty}W_i(x).$$

($\mathcal{A}2$): $\mathscr{W}_1\cap\mathscr{W}_2\ne\varnothing$.

We set

$$x_{iv}\in\mathscr{V},\ k_{iv}:=\max_{\mathscr{V}}W_i(x)=W_i(x_{iv}),\ i=1,2;$$

$$x_w\in\mathscr{W}_1\cap\mathscr{W}_2,\ \tau_w:=\min_{\mathscr{W}_1\cap\mathscr{W}_2}V(x)=V(x_w).$$

For vector $\boldsymbol{b}=(b_1,b_2)\in\mathbb{R}^2$, we define

$$m(a,\boldsymbol{b})=\begin{cases}\left(\dfrac{\tau_\infty}{a}\right)^{\frac{\theta+2p}{2(p-1)}-\frac{N}{2}}\left(\dfrac{b_1}{k_{1\infty}}\right)^{\frac{2}{p-1}} & \text{if } \dfrac{k_{2\infty}}{b_2}\le\left(\dfrac{a}{\tau_\infty}\right)^{\frac{2+\theta}{4}\frac{q-p}{p-1}}\left(\dfrac{k_{1\infty}}{b_1}\right)^{\frac{q-1}{p-1}},\\ \left(\dfrac{\tau_\infty}{a}\right)^{\frac{\theta+2q}{2(q-1)}-\frac{N}{2}}\left(\dfrac{b_2}{k_{2\infty}}\right)^{\frac{2}{q-1}} & \text{otherwise}\end{cases}$$

and denote $\boldsymbol{k}=(k_1,k_2)$, $\boldsymbol{k}_\infty=(k_{1\infty},k_{2\infty})$, $\boldsymbol{k}_v=(k_{1v},k_{2v})$. Similarly, we set $\boldsymbol{W}_0:=(W_{10},W_{20})$, $\boldsymbol{W}(x)=(W_1(x),W_2(x))$. For $\boldsymbol{b}^i=(b_1^i,b_2^i)\in\mathbb{R}^2$, $i=1,2$, we use $\boldsymbol{b}^1\le\boldsymbol{b}^2$ to mean $\min\{b_1^2-b_1^1,b_2^2-b_2^1\}\ge 0$ and use $\boldsymbol{b}^1<\boldsymbol{b}^2$ to show $\min\{b_1^2-b_1^1,b_2^2-b_2^1\}\ge 0$ and $\max\{b_1^2-b_1^1,b_2^2-b_2^1\}>0$.

($\mathcal{A}3$): (i) $\tau<\tau_\infty$, and there exists $R_v>0$ such that $W_i(x)\le k_{iv}$, $i=1,2$ for $|x|\ge R_v$;

(ii) $\boldsymbol{k}>\boldsymbol{k}_\infty$, and there exists $R_w>0$ such that $V(x)\ge\tau_w$ for $|x|\ge R_w$.

If ($\mathcal{A}3$)-(i) holds, we let

$$\mathscr{S}_v:=\{x\in\mathscr{V}:W_i(x)=k_{iv},i=1,2\}\cup\{x\notin\mathscr{V}:W_1(x)>k_{1v}\text{ or }W_2(x)>k_{2v}\}.$$

If ($\mathcal{A}3$)-(ii) holds, we let

$$\mathscr{S}_w:=\{x\in\mathscr{W}_1\cap\mathscr{W}_2:V(x)=\tau_w\}\cup\{x\notin\mathscr{W}_1\cap\mathscr{W}_2:V(x)<\tau_w\}.$$

In the following, in the case ($\mathcal{A}3$)-(i), $\mathscr{S}$ stands for $\mathscr{S}_v$ and $\mathscr{S}$ stands for $\mathscr{S}_w$ in the case ($\mathcal{A}3$)-(ii). Clearly, $\mathscr{S}$ is bounded. Moreover, $\mathscr{S}=\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2)$, if $\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2)\ne\varnothing$.

The next theorems contain the main results of this paper.

Theorem 1.1. *Assume that* ($\mathcal{A}1$) *holds and*

$$\tau<\tau_\infty,\quad \boldsymbol{k}_v\ge\boldsymbol{k}_\infty. \tag{1.5}$$

Then there exists $m_v\ge m(\tau,\boldsymbol{k}_v)$ *such that for the maximal integer* $m\in\mathbb{N}$ *with* $m<m_v$, Equation (1.1) *possesses at least m pairs of solutions for small* $\varepsilon>0$. *Moreover*, Equation (1.1) *has a positive and a negative groundstate solution.*

Theorem 1.2. *Assume that* ($\mathcal{A}1$)-($\mathcal{A}2$) holds and

$$\tau_w\le\tau_\infty,\quad \boldsymbol{k}>\boldsymbol{k}_\infty. \tag{1.6}$$

Then there exists $m_w\ge m(\tau_w,\boldsymbol{k})$ *such that for the maximal integer* $m\in\mathbb{N}$ *with* $m<m_w$, *all the conclusions of Theorem* 1.1 *remain true.*

Theorem 1.3. *Assume that* ($\mathcal{A}1$)-($\mathcal{A}3$) *hold. Then for sufficiently small* $\varepsilon>0$, Equation (1.1) *has a positive groundstate solution* w_ε. *If* $V,W_i\in C^1(\mathbb{R}^N)$ *and* $\nabla V,\nabla W_i,i=1,2$ *are bounded additionally, then* w_ε *satisfies that*

1) *There exists a maximum point* x_ε *of* w_ε *with* $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon,\mathscr{S})=0$;

2) *There exist* $C>0$ *and sufficiently large* $R>0$ *such that*

$$w_\varepsilon(x)\le C\varepsilon^{\frac{N-1}{2}}|x-x_\varepsilon|^{\frac{1-N}{2}}\exp\left(\frac{-\sqrt{\tau}}{4\varepsilon}|x-x_\varepsilon|\right),\quad\forall|x|\ge R;$$

3) *Letting* $v_\varepsilon(x):=w_\varepsilon(\varepsilon x+x_\varepsilon)$, *then for any sequence* $x_\varepsilon\to x_0$ ($\varepsilon\to0$), *there holds* $v_\varepsilon\to v$ *in* $H^1(\mathbb{R}^N)$ *as* $\varepsilon\to0$, *where v is a least energy solution of*

$$-\Delta v+V(x_0)v=W_1^2(x_0)(I_\theta*v^p)v^{p-1}+W_2^2(x_0)(I_\theta*v^q)v^{q-1},\quad v>0. \tag{1.7}$$

If $\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2)\ne\varnothing$ *particularly, then* $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon,\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2))=0$ *and up to a sequence,* $v_\varepsilon\to v$ *in* $H^1(\mathbb{R}^N)$ *as* $\varepsilon\to0$ *with v being a least energy solution of*

$$-\Delta v+\tau v=k_1^2\left(I_\theta * v^p\right)v^{p-1}+k_2^2\left(I_\theta * v^q\right)v^{q-1},\quad v>0. \tag{1.8}$$

To prove the above results, we need the following basic conclusions.

Lemma 1.4. ([13]) *The embedding* $H^1\left(\mathbb{R}^N\right)\hookrightarrow L^{q}\left(\mathbb{R}^N\right)$ *is continuous for* $q\in\left[2,2^*\right]$, $2^*:=\frac{2N}{N-2}$, *and* $H^1\left(\mathbb{R}^N\right)\hookrightarrow L^{q}_{\text{loc}}\left(\mathbb{R}^N\right)$ *is compact for* $q\in\left[2,2^*\right)$. *Moreover,* $H_r^1\left(\mathbb{R}^N\right):=\left\{u\in H^1\left(\mathbb{R}^N\right):u(x)=u(|x|)\right\}$ *is compactly embedded into* $L^{q}\left(\mathbb{R}^N\right)$ *for* $q\in\left(2,2^*\right)$.

Lemma 1.5. ([14]) *Let* $r>0$, $q\in\left[2,2^*\right)$. *If* $\{\omega_n\}$ *is bounded in* $H^1\left(\mathbb{R}^N\right)$ *and*

$$\sup_{y\in\mathbb{R}^N}\int_{B_r(y)}|\omega_n|^{q}\,\mathrm{d}x\to 0\quad\text{as}\quad n\to\infty,$$

then $\omega_n\to 0$ *in* $L^\mu\left(\mathbb{R}^N\right)$ *for any* $\mu\in\left(2,2^*\right)$.

For simplicity, we set

$$\|w\|_1:=\|w\|_{H^1\left(\mathbb{R}^N\right)},\ |w|_q:=\|w\|_{L^q\left(\mathbb{R}^N\right)},$$
$$w^+:=\max\{0,\omega\},\ w^-:=\min\{0,w\},\ \mathbb{R}_+:=(0,\infty),$$

and use $\int_{\mathbb{R}^N}f(x)$ to denote $\int_{\mathbb{R}^N}f(x)\mathrm{d}x$ in some cases. Moreover, we use different forms of C to mean various positive constants and $o(1)$ to represent the quantities which tend to 0 as $n\to\infty$ or $j\to\infty$ in the following.

This paper is organized as follows. Section 2 is an introduction to some conclusions about the Riesz potential, which plays a very important role in the subsequent proof process. In Section 3, we provide some preliminary results for the limit equation and the auxiliary equation which are the foundation for the proof of the main theorems. Section 4 contributes to the proofs of main results. We prove the multiplicity of semiclassical solutions by Benci pseudo-index theory and show the existence of the groundstate solutions and concentration of the positive groundstate solution in Section 4.

2. Riesz Potential

The Riesz potential with order $\theta\in(0,N)$ of a function $f\in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ is defined by

$$\left(I_\theta * f\right)(x):=\int_{\mathbb{R}^N}\frac{\Gamma\left(\frac{N-\theta}{2}\right)}{2^\theta\pi^{N/2}\Gamma\left(\frac{\theta}{2}\right)}\frac{f(y)}{|x-y|^{N-\theta}}\mathrm{d}y. \tag{2.1}$$

The integral in Equation (2.1) converges in the classical Lebesgue sense for a.e. $x\in\mathbb{R}^N$ if and only if $f\in L^1\left(\mathbb{R}^N,(1+|x|)^{\theta-N}\right)$. Moreover, if $f\notin L^1\left(\mathbb{R}^N,(1+|x|)^{\theta-N}\right)$, then (1) diverges everywhere in $\mathbb{R}^N$. The Riesz potential I_θ is well-defined as an operator in $L^{q}\left(\mathbb{R}^N\right)$ if and only if $q\in\left[1,\frac{N}{\theta}\right)$. In addition, if $q\in\left(1,\frac{N}{\theta}\right)$ and $\mathfrak{r}:=\frac{Nq}{N-\theta q}$, then $I_\theta:L^{q}\left(\mathbb{R}^N\right)\to L^{\mathfrak{r}}\left(\mathbb{R}^N\right)$ is a bounded linear operator, which can be disclosed by the Hardy-Littlewood-Sobolev inequality.

Lemma 2.1. ([15]) *Let* $\theta \in (0, N)$, $\mathfrak{q} \in \left(1, \frac{N}{\theta}\right)$. *Then for any* $f \in L^{\mathfrak{q}}(\mathbb{R}^N)$, $I_\theta * f \in L^{N\mathfrak{q}/(N-\theta\mathfrak{q})}(\mathbb{R}^N)$ *and* $|I_\theta * f|_{N\mathfrak{q}/(N-\theta\mathfrak{q})} \le C_{N,\theta,\mathfrak{q}} |f|_{\mathfrak{q}}$.

Applying Lemma 2.1 to the function $f = |u|^{\mathfrak{p}} \in L^{2N/(N+\theta)}(\mathbb{R}^N)$, we obtain the following result.

Lemma 2.2. ([16]) Let $\theta \in (0, N)$. *Then for any* $\omega \in L^{2N\mathfrak{p}/(N+\theta)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left(I_\theta * |\omega|^{\mathfrak{p}}\right)|\omega|^{\mathfrak{p}} \, dx \le C_{N,\theta} |\omega|^{2\mathfrak{p}}_{2N\mathfrak{p}/(N+\theta)}.$$

In particular, if $N > 2$, $\mathfrak{p} \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2}\right]$ *and* $\omega \in H^1(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} \left(I_\theta * |\omega|^{\mathfrak{p}}\right)|\omega|^{\mathfrak{p}} \, dx \le C_{N,\theta,\mathfrak{p}} \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{\mathfrak{p}}.$$

Actually, $\mathfrak{p} \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2}\right]$ if and only if $\frac{2N\mathfrak{p}}{N+\theta} \in [2, 2^*]$. The Brézis-Lieb type lemma we use next also applies to the Riesz potential.

Lemma 2.3. ([12]) *Let* $N > 2$, $\theta \in (0, N)$, $\mathfrak{p} \in \left[2, \frac{N+\theta}{N-2}\right)$. *If* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^N)$ *as* $n \to \infty$, *then*

1) $\mathcal{B}(v_n) - \mathcal{B}(v_n - v) \to \mathcal{B}(v)$ *as* $n \to \infty$;

2) $\mathcal{B}'(v_n) - \mathcal{B}'(v_n - v) \to \mathcal{B}'(v)$ *in* $H^{-1}(\mathbb{R}^N)$ *as* $n \to \infty$,

where $\mathcal{B}(v) := \int_{\mathbb{R}^N} \left(I_\theta * |v|^{\mathfrak{p}}\right)|v|^{\mathfrak{p}} \, dx$.

Lemma 2.4. ([12]) *Let* $N > 2$, $\theta \in (0, N)$, $\mathfrak{p} \in \left[2, \frac{N+\theta}{N-2}\right)$. *If* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^N)$ *as* $n \to \infty$, *then for any* $u \in H^1(\mathbb{R}^N)$, $\langle \mathcal{B}'(v_n), u\rangle \to \langle \mathcal{B}'(v), u\rangle$ *as* $n \to \infty$, *where* $\mathcal{B}(v)$ *is defined as in* Lemma 2.3.

3. Auxiliary Problems

We consider, for $N > 2$, $\theta \in (0, N)$, $2 \le p < q < \frac{N+\theta}{N-2}$,

$$-\Delta v + av = \mathcal{Y}_1^{b_1}(v) + \mathcal{Y}_2^{b_2}(v), \quad v \in H^1(\mathbb{R}^N), \tag{3.1}$$

where $a > 0$, $b_i > 0$, $i = 1, 2$, $\mathcal{Y}_1^{b_1}(v) := b_1^2 \left(I_\theta * |v|^p\right)|v|^{p-2} v$, $\mathcal{Y}_2^{b_2}(v) := b_2^2 \left(I_\theta * |v|^q\right)|v|^{q-2} v$, and

$$-\Delta v + V_\varepsilon^a(x) v = \mathcal{Y}_{1\varepsilon}^{b_1}(v) + \mathcal{Y}_{2\varepsilon}^{b_2}(v), \quad v \in H^1(\mathbb{R}^N), \tag{3.2}$$

where $\tau \le a \le \tau_\infty$, $\boldsymbol{k}_\infty \le \boldsymbol{b} \le \boldsymbol{k}$, $\mathcal{Y}_{1\varepsilon}^{b_1}(v) := W_{1\varepsilon}^{b_1}(x)\left[I_\theta * \left(W_{1\varepsilon}^{b_1} |v|^p\right)\right]|v|^{p-2} v$, $\mathcal{Y}_{2\varepsilon}^{b_2}(v) := W_{2\varepsilon}^{b_2}(x)\left[I_\theta * \left(W_{2\varepsilon}^{b_2} |v|^q\right)\right]|v|^{q-2} v$ with

$$V^a(x) := \max\{a, V(x)\}, \quad V_\varepsilon^a(x) := V^a(\varepsilon x),$$

$$W_i^{b_i}(x) := \min\{b_i, W_i(x)\}, \quad W_{i\varepsilon}^{b_i}(x) := W_i^{b_i}(\varepsilon x), \quad i = 1, 2.$$

The solutions $v \in H^1(\mathbb{R}^N)$ of Equation (3.1) and Equation (3.2) can be obtained as critical points of the energy functionals

$$\mathcal{J}^{ab}(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + av^2\right) - \frac{1}{2p}\int_{\mathbb{R}^N}\mathcal{Y}_1(v)v - \frac{1}{2q}\int_{\mathbb{R}^N}\mathcal{Y}_2(v)v,$$

$$\mathcal{J}_\varepsilon^{ab}(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V_\varepsilon^a(x)v^2\right) - \frac{1}{2p}\int_{\mathbb{R}^N}\mathcal{Y}_{1\varepsilon}^{b_1}(v)v - \frac{1}{2q}\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon}^{b_2}(v)v,$$

respectively. And the Nehari manifolds are denoted by $\mathcal{N}^{ab}, \mathcal{N}_\varepsilon^{ab}$; the least energies by $\mathcal{E}^{ab} := \inf_{\mathcal{N}^{ab}} \mathcal{J}^{ab}$, $\mathcal{E}_\varepsilon^{ab} := \inf_{\mathcal{N}_\varepsilon^{ab}} \mathcal{J}_\varepsilon^{ab}$; and the sets of least energy solutions by $\mathcal{T}^{ab}, \mathcal{T}_\varepsilon^{ab}$, respectively. In particular, we define

$$\mathcal{J}^\infty := \mathcal{J}^{\tau_\infty \boldsymbol{k}_\infty}, \mathcal{N}^\infty := \mathcal{N}^{\tau_\infty \boldsymbol{k}_\infty}, \mathcal{E}^\infty := \mathcal{E}^{\tau_\infty \boldsymbol{k}_\infty}, V_\varepsilon^\infty := V_\varepsilon^{\tau_\infty},$$

$$\mathcal{J}_\varepsilon^\infty := \mathcal{J}_\varepsilon^{\tau_\infty \boldsymbol{k}_\infty}, \mathcal{N}_\varepsilon^\infty := \mathcal{N}_\varepsilon^{\tau_\infty \boldsymbol{k}_\infty}, \mathcal{E}_\varepsilon^\infty := \mathcal{E}_\varepsilon^{\tau_\infty \boldsymbol{k}_\infty}, W_{i\varepsilon}^\infty := W_{i\varepsilon}^{\boldsymbol{k}_\infty}, i = 1,2.$$

Lemma 3.1. *There exist* $\rho > 0$ *and* $\sigma > 0$ *such that* $\mathcal{J}^{ab}(v) > \sigma$ *for all* $\|v\|_1 = \rho$. *Moreover,* $\lim_{t\to+\infty}\mathcal{J}^{ab}(tv) = -\infty$, *if* $v \neq 0$.

Lemma 3.2. *Let* $\Psi^{ab} := \left\{\gamma \in C\left([0,1], H^1(\mathbb{R}^N)\right) : \gamma(0) = 0, \mathcal{J}^{ab}(\gamma(1)) < 0\right\}$, *then*

$$\mathcal{E}^{ab} = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t\ge 0} \mathcal{J}^{ab}(tv) = \inf_{\gamma\in\Psi^{ab}} \max_{t\in[0,1]} \mathcal{J}^{ab}(\gamma(t)) > 0.$$

Lemma 3.3. $\mathcal{E}^{ab}$ *is attained and* $\mathcal{T}^{ab}$ *is compact in* $H^1(\mathbb{R}^N)$.

Proof. We set the equivalent norm $\|v\|_1 = \left(\int_{\mathbb{R}^N}\left(|\nabla v|^2 + av^2\right)\right)^{\frac{1}{2}}$ for any $v \in H^1(\mathbb{R}^N)$. Obviously, $\mathcal{N}^{ab} \neq \varnothing$, we set $v_n \in \mathcal{N}^{ab}$ with $v_n \ge 0$ and $\mathcal{J}^{ab}(v_n) \to \mathcal{E}^{ab}$ as $n \to \infty$. On the basis of the Schwarz symmetrization and Theorem 3.1.5 in [13], there exists v_n^* as the radially symmetric decreasing rearrangment of v_n with $v_n^* \ge 0$ such that $\|v_n^*\|_1 \le \|v_n\|_1$. We can verify that $v_n^* \neq 0$. We can know that $\|v_n^*\|_1^2 \le \int_{\mathbb{R}^N}\mathcal{Y}_1^{b_1}(v_n^*)v_n^* + \mathcal{Y}_2^{b_2}(v_n^*)v_n^*$. If $\|v_n^*\|_1^2 = \int_{\mathbb{R}^N}\mathcal{Y}_1^{b_1}(v_n^*)v_n^* + \mathcal{Y}_2^{b_2}(v_n^*)v_n^*$, then $v_n^* \in \mathcal{N}^{ab}$. If $\|v_n^*\|_1^2 < \int_{\mathbb{R}^N}\mathcal{Y}_1^{b_1}(v_n^*)v_n^* + \mathcal{Y}_2^{b_2}(v_n^*)v_n^*$, then there exists $t_n \in (0,1)$ such that $t_n v_n^* \in \mathcal{N}^{ab}$ and

$$\begin{aligned}\mathcal{E}^{ab} \le \mathcal{J}^{ab}(t_n v_n^*) &< \frac{p-1}{2p}\|v_n\|_1^2 + \frac{q-p}{2pq}\int_{\mathbb{R}^N}\mathcal{Y}_2^{b_2}(v_n)v_n \\ &= \mathcal{J}^{ab}(v_n) \to \mathcal{E}^{ab} \quad \text{as } n\to\infty,\end{aligned}$$

which implies $\mathcal{J}^{ab}(t_n v_n^*) \to \mathcal{E}^{ab}$ as $n \to \infty$. Define $w_n := t_n v_n^*$, then $w_n \in \mathcal{N}^{ab}$, $w_n \ge 0$ and

$$\mathcal{J}^{ab}(w_n) \to \mathcal{E}^{ab} \quad \text{as } n \to \infty. \tag{3.3}$$

By Lemma 2.2, one can check that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Along a subsequence, we may assume $w_n \rightharpoonup w$ as $n \to \infty$. According to Lemma 1.4, $w_n \to w$ in $L^\tau(\mathbb{R}^N)$ for $\tau \in (2, 2^*)$ as $n \to \infty$. Due to $w_n \in \mathcal{N}^{ab}$ and Lemma 2.2, $\|w_n\|_1^2 \le C\left(\|w_n\|_1^{2p} + \|w_n\|_1^{2q}\right)$, which implies

$\int_{\mathbb{R}^N} \mathcal{Y}_1^{b_1}(w_n)w_n + \mathcal{Y}_2^{b_2}(w_n)w_n > C > 0$. By contradiction method, we get w ≠ 0. We can know $\|w\|_1^2 \le \liminf_{n\to\infty}\left(\int_{\mathbb{R}^N} \mathcal{Y}_1^{b_1}(w_n)w_n + \mathcal{Y}_2^{b_2}(w_n)w_n\right) = \int_{\mathbb{R}^N} \mathcal{Y}_1^{b_1}(w)w + \mathcal{Y}_2^{b_2}(w)w$ by the weakly lower semi-continuity of norm. By contradiction method, we can get $w \in \mathcal{N}^{ab}$ and by (3.3),

$$\mathcal{E}^{ab} \le \mathcal{J}^{ab}(w) \le \liminf_{n\to\infty}\left(\frac{p-1}{2p}\|w_n\|_1^2 + \frac{q-p}{2pq}\int_{\mathbb{R}^N} \mathcal{Y}_2^{b_2}(w_n)w_n\right)$$

$$= \liminf_{n\to\infty} \mathcal{J}^{ab}(w_n) = \mathcal{E}^{ab},$$

which implies $\mathcal{E}^{ab} = \mathcal{J}^{ab}(w)$ is attained. In the end, we have $\left(\mathcal{J}^{ab}\right)'(w) = 0$, where $w \in \mathcal{T}^{ab}$ is positive and radially symmetric. With similar arguments as above, $\mathcal{T}^{ab}$ is compact in $H^1(\mathbb{R}^N)$. □

In view of Theorem 3 in [17], we have the following result.

Lemma 3.4. *If there exists a least energy solution* $v \in H^1(\mathbb{R}^N)$ *for* Equation (3.1), *then* $v \in L^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, *v is either positive or negative, and v is radially symmetric up to translations.*

Lemma 3.5. *Let* $a_i > 0$ *and* $b_i^1, b_i^2 > 0$ *for* $i = 1,2$.

(i) If $\min\{a_2 - a_1, b_1^1 - b_2^1, b_1^2 - b_2^2\} \ge 0$, then $\mathcal{E}^{a_1 \boldsymbol{b}_1} \le \mathcal{E}^{a_2 \boldsymbol{b}_2}$.

(ii) If $\min\{a_2 - a_1, b_1^1 - b_2^1, b_1^2 - b_2^2\} \ge 0$ and $\max\{a_2 - a_1, b_1^1 - b_2^1, b_2^1 - b_2^2\} > 0$, then $\mathcal{E}^{a_1 \boldsymbol{b}_1} < \mathcal{E}^{a_2 \boldsymbol{b}_2}$.

Lemma 3.6. *If v is a groundstate solution of*

$$-\Delta v + \tau_\infty v = \mathcal{Y}_1^{k_{1\infty}}(v) + \mathcal{Y}_2^{k_{2\infty}}(v), \quad v \in H^1(\mathbb{R}^N), \tag{3.4}$$

with the energy $\mathcal{E}^\infty$, *where* $\mathcal{Y}_1^{k_{1\infty}}(v) := k_{1\infty}^2\left(I_\theta * |v|^p\right)|v|^{p-2}v$, $\mathcal{Y}_2^{k_{2\infty}}(v) := k_{2\infty}^2\left(I_\theta * |v|^q\right)|v|^{q-2}v$ *Letting* $u(x) := \lambda v\left(\left(\frac{a}{\tau_\infty}\right)^{\frac{1}{2}} x\right)$, *then* Equation (3.1) *is equivalent to*

$$-\Delta u + au = \left(\frac{k_{1\infty}^2}{b_1^2}\left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2}{2}}\lambda^{2-2p}\right)\mathcal{Y}_1^{b_1}(u) + \left(\frac{k_{2\infty}^2}{b_2^2}\left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2}{2}}\lambda^{2-2q}\right)\mathcal{Y}_2^{b_2}(u), \tag{3.5}$$

where $u \in H^1(\mathbb{R}^N)$, *with the energy* $\mathcal{E}_\lambda = \lambda^2\left(\frac{a}{\tau_\infty}\right)^{1-\frac{N}{2}}\mathcal{E}^\infty$.

Proof. Clearly, we can know v is a solution of Equation (3.4) if and only if u is a solution of Equation (3.5). Indeed,

$$-\Delta u + au = \frac{\lambda a}{\tau_\infty}\left(-\Delta v\left(\left(\frac{a}{\tau_\infty}\right)^{\frac{1}{2}}x\right) + \tau_\infty v\left(\left(\frac{a}{\tau_\infty}\right)^{\frac{1}{2}}x\right)\right).$$

We can verify that $v \in \mathcal{N}^\infty$ if and only if $u \in \mathcal{N}_\lambda^{ab}$, then

$$\mathcal{E}_\lambda = \lambda^2\left(\frac{a}{\tau_\infty}\right)^{1-\frac{N}{2}}\mathcal{E}^\infty.$$ □

Lemma 3.7 *Assume that* $a \le \tau_\infty, \boldsymbol{b} \ge \boldsymbol{k}_\infty$. *Then* $m(a,\boldsymbol{b})\mathcal{E}^{ab} \le \mathcal{E}^\infty$.

Proof. Noticing that if $\lambda > 0$ satisfy

$\max\left\{\frac{k_{1\infty}^2}{b_1^2}\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{2}}\lambda^{2-2p},\frac{k_{2\infty}^2}{b_2^2}\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{2}}\lambda^{2-2q}\right\}\le 1$, we can know $\mathcal{E}^{ab}\le\mathcal{E}_\lambda$.

According to the definition of $m(a,\boldsymbol{b})$, we can find two situations:

$$\frac{k_{2\infty}}{b_2}\le\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{4}\frac{q-p}{p-1}}\left(\frac{k_{1\infty}}{b_1}\right)^{\frac{q-1}{p-1}} \tag{3.6}$$

or

$$\frac{k_{1\infty}}{b_1}<\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{4}\frac{p-q}{q-1}}\left(\frac{k_{2\infty}}{b_2}\right)^{\frac{p-1}{q-1}}. \tag{3.7}$$

If (3.6) holds, let $\lambda=\left[\frac{k_{1\infty}}{b_1}\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{4}}\right]^{\frac{1}{p-1}}$, then $\mathcal{E}_\lambda=\left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2p}{2(p-1)}-\frac{N}{2}}\left(\frac{k_{1\infty}}{b_1}\right)^{\frac{2}{p-1}}\mathcal{E}^\infty$, we obtain $m(a,\boldsymbol{b})\mathcal{E}^{ab}\le\mathcal{E}^\infty$. If (3.7) holds, set $\lambda=\left[\frac{k_{2\infty}}{b_2}\left(\frac{a}{\tau_\infty}\right)^{\frac{2+\theta}{4}}\right]^{\frac{1}{q-1}}$, then $\mathcal{E}_\lambda=\left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2q}{2(q-1)}-\frac{N}{2}}\left(\frac{k_{2\infty}}{b_2}\right)^{\frac{2}{q-1}}\mathcal{E}^\infty$, we obtain $m(a,\boldsymbol{b})\mathcal{E}^{ab}\le\mathcal{E}^\infty$. □

Lemma 3.8. *If* $\tau<\tau_\infty$, $\boldsymbol{k}_v\ge\boldsymbol{k}_\infty$, *then* $m(\tau,\boldsymbol{k}_v)>1$ *and* $\mathcal{E}^{\tau\boldsymbol{k}_v}<\mathcal{E}^\infty$. *If* $\tau_w\le\tau_\infty$, $\boldsymbol{k}>\boldsymbol{k}_\infty$, *then* $m(\tau_w,\boldsymbol{k})\ge 1$ *and* $\mathcal{E}^{\tau_w\boldsymbol{k}}<\mathcal{E}^\infty$.

Proof. Set $a=\tau, b_i=k_{iv}, i=1,2$ in Equation (3.1), Equations (3.5)-(3.7), respectively. By the definition of $m(\tau,\boldsymbol{k}_v)$, we get $m(\tau,\boldsymbol{k}_v)>1$. By Lemma 3.7, we obtain $\mathcal{E}^{\tau\boldsymbol{k}_v}<\mathcal{E}^\infty$.

Similarly, we let $a=\tau_w, b_i=k_i, i=1,2$ in Equation (3.1), Equations (3.5)-(3.7), respectively. Obviously, we have $m(\tau_w,\boldsymbol{k})\ge 1$. If (3.6) holds, we pick $\lambda=\left[\frac{k_{1\infty}}{k_1}\left(\frac{\tau_w}{\tau_\infty}\right)^{\frac{2+\theta}{4}}\right]^{\frac{1}{p-1}}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}\le\mathcal{E}_\lambda\le\mathcal{E}^\infty$ by Lemmas 3.5, 3.6. If $k_1>k_{1\infty}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}\le\mathcal{E}_\lambda<\mathcal{E}^\infty$ by Lemma 3.6. If $k_2>k_{2\infty}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}<\mathcal{E}_\lambda\le\mathcal{E}^\infty$ by Lemma 3.5. Thus $\mathcal{E}^{\tau_w\boldsymbol{k}}<\mathcal{E}^\infty$. If (3.7) holds, we choose $\lambda=\left[\frac{k_{2\infty}}{k_2}\left(\frac{\tau_w}{\tau_\infty}\right)^{\frac{2+\theta}{4}}\right]^{\frac{1}{q-1}}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}\le\mathcal{E}_\lambda\le\mathcal{E}^\infty$. If $k_1>k_{1\infty}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}<\mathcal{E}_\lambda\le\mathcal{E}^\infty$. If $k_2>k_{2\infty}$, then $\mathcal{E}^{\tau_w\boldsymbol{k}}\le\mathcal{E}_\lambda<\mathcal{E}^\infty$. Thus, $\mathcal{E}^{\tau_w\boldsymbol{k}}<\mathcal{E}^\infty$. □

Now we establish some results for Equation (3.2).

Lemma 3.9. *There exist* $\rho>0,\sigma>0$ *both independent of* ε,a,b *and just dependent on* N,θ,p,τ,k, *such that* $\mathcal{J}_\varepsilon^{ab}(v)>\sigma$ *for all* $\|v\|_1=\rho$. *Moreover,* $\lim_{t\to+\infty}\mathcal{J}_\varepsilon^{ab}(tv)=-\infty$, *if* $v\ne 0$.

Lemma 3.10. *Set* $\Psi_\varepsilon^{ab}:=\left\{\gamma\in C\left([0,1],H^1(\mathbb{R}^N)\right):\gamma(0)=0,\mathcal{J}_\varepsilon^{ab}(\gamma(1))<0\right\}$, *then*

$$\mathcal{E}_\varepsilon^{ab} = \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} \mathcal{J}_\varepsilon^{ab}(tv) = \inf_{\gamma\in\Psi_\varepsilon^{ab}} \max_{t\in[0,1]} \mathcal{J}_\varepsilon^{ab}(\gamma(t)) > 0.$$

Lemma 3.11. *If* $\mathcal{J}_\varepsilon^\infty$ *possesses a* $(PS)_c$ *sequence, then either* $c=0$ *or* $c\geq\mathcal{E}_\varepsilon^\infty$. *Besides,* $\mathcal{E}_\varepsilon^\infty \geq \mathcal{E}^\infty$.

Proof. Let $\{v_n\}\subset H^1(\mathbb{R}^N)$ and $\mathcal{J}_\varepsilon^\infty(v_n)\to c$, $(\mathcal{J}_\varepsilon^\infty)'(v_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n\to\infty$. Assume $c\neq 0$, we will prove $c\geq\mathcal{E}_\varepsilon^\infty$.

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we may assume $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$ along a subsequence. Set $z_n := v_n - v$. By the Brézis-Lieb lemma, we obtain

$$\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V_\varepsilon^\infty(x)v_n^2\right) = \int_{\mathbb{R}^N}\left(|\nabla v|^2 + V_\varepsilon^\infty(x)v^2\right) + \int_{\mathbb{R}^N}\left(|\nabla z_n|^2 + V_\varepsilon^\infty(x)z_n^2\right) + o(1). \tag{3.8}$$

By the proof of Lemma 3.5 in [12], we have

$$\int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v_n)v_n = \int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v)v + \int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(z_n)z_n + o(1), \quad i=1,2, \tag{3.9}$$

where $\mathcal{Y}_{1\varepsilon}^\infty(v) := W_{1\varepsilon}^\infty(x)\left[I_\theta * \left(W_{1\varepsilon}^\infty|v|^p\right)\right]|v|^{p-2}v$,

$\mathcal{Y}_{2\varepsilon}^\infty(v) := W_{2\varepsilon}^\infty(x)\left[I_\theta * \left(W_{2\varepsilon}^\infty|v|^q\right)\right]|v|^{q-2}v$, and for any $\varphi\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v_n)\varphi = \int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v)\varphi + \int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(z_n)\varphi + o(1)\|\varphi\|_1, \quad i=1,2. \tag{3.10}$$

As the proof of Lemma 3.6 in [12], we have that for all $\varphi\in H^1(\mathbb{R}^N)$, as $n\to\infty$, $\int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v_n)\varphi \to \int_{\mathbb{R}^N}\mathcal{Y}_{i\varepsilon}^\infty(v)\varphi$, $i=1,2$, which ensures that $(\mathcal{J}_\varepsilon^\infty)'(v) = 0$. In virtue of (3.8), (3.9) and (3.10), we obtain that

$$\mathcal{J}_\varepsilon^\infty(z_n)\to c - \mathcal{J}_\varepsilon^\infty(v), \quad (\mathcal{J}_\varepsilon^\infty)'(z_n)\to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{3.11}$$

Case 1 If there exists $z_{nk}\equiv 0$, that is $v_{nk}\equiv v$, then $\mathcal{J}_\varepsilon^\infty(v) = c\neq 0$ and $v\in\mathcal{N}_\varepsilon^\infty$. Thus $c\geq\mathcal{E}_\varepsilon^\infty$.

Case 2 If $z_n\neq 0$ for all $n\in\mathbb{N}$, then there exists $t_n>0$ such that $t_nz_n\in\mathcal{N}_\varepsilon^\infty$. Hence

$$\mathcal{J}_\varepsilon^\infty(t_nz_n)\geq\mathcal{E}_\varepsilon^\infty. \tag{3.12}$$

It follows from $\left\langle(\mathcal{J}_\varepsilon^\infty)'(t_nz_n), t_nz_n\right\rangle = 0$ and $\left\langle(\mathcal{J}_\varepsilon^\infty)'(z_n), z_n\right\rangle = o(1)$ that

$$\left(1-t_n^{2p-2}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{1\varepsilon}^\infty(z_n)z_n + \left(1-t_n^{2q-2}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon}^\infty(z_n)z_n = o(1). \tag{3.13}$$

Additionally, $\|z_n\|_1^2 \leq C\int_{\mathbb{R}^N}\mathcal{Y}_{1\varepsilon}^\infty(z_n)z_n + \mathcal{Y}_{2\varepsilon}^\infty(z_n)z_n + o(1)$. If $\int_{\mathbb{R}^N}\left(I_\theta * |z_n|^p\right)|z_n|^p\to 0$ and $\int_{\mathbb{R}^N}\left(I_\theta * |z_n|^q\right)|z_n|^q\to 0$ as $n\to\infty$, then $\|z_n\|_1\to 0$ as $n\to\infty$. Thus $v_n\to v$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$ and $c = \mathcal{J}_\varepsilon^\infty(v)\geq\mathcal{E}_\varepsilon^\infty$. If $\int_{\mathbb{R}^N}\left(I_\theta * |z_n|^p\right)|z_n|^p\geq\delta>0$ or $\int_{\mathbb{R}^N}\left(I_\theta * |z_n|^q\right)|z_n|^q\geq\delta>0$, then $t_n\to 1$ as $n\to\infty$ by (3.13). Hence $\mathcal{J}_\varepsilon^\infty(t_nz_n)\to c-\mathcal{J}_\varepsilon^\infty(v)$ as $n\to\infty$

by (3.11), which implies $c \ge \mathcal{J}_\varepsilon^\infty(v) + \mathcal{E}_\varepsilon^\infty \ge \mathcal{E}_\varepsilon^\infty$ by (3.12).

Finally, it follows from $V_\varepsilon^\infty(x) \ge \tau_\infty$ and $W_{i\varepsilon}^\infty(x) \le k_{i\infty}, i=1,2$ for any $x \in \mathbb{R}^N$ that $\mathcal{J}_\varepsilon^\infty(v) \ge \mathcal{J}^\infty(v)$ for all $v \in H^1(\mathbb{R}^N)$. Thus, $\mathcal{E}_\varepsilon^\infty \ge \mathcal{E}^\infty$. □

Remark 3.12. *Similarly, if* $\mathcal{J}_\varepsilon^{ab}$ *has a* $(PS)_c$ *sequence, then either* $c=0$ *or* $c \ge \mathcal{E}_\varepsilon^\infty$.

Lemma 3.13. $\mathcal{J}_\varepsilon^{ab}$ *satisfies the* $(PS)_c$ *condition for all* $c < \mathcal{E}_\varepsilon^\infty$.

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ and $\mathcal{J}_\varepsilon^{ab}(v_n) \to c$, $(\mathcal{J}_\varepsilon^{ab})'(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$.

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we assume $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Then $(\mathcal{J}_\varepsilon^{ab})'(v) = 0$ by Lemma 2.4. Set $z_n := v_n - v$. Then $z_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and

$$z_n \to 0 \text{ in } L_{\text{loc}}^t(\mathbb{R}^N) \text{ as } n \to \infty \text{ for } t \in [2, 2^*). \tag{3.14}$$

Combine with the classical Brézis-Lieb lemma and Lemma 2.3, we have

$$\mathcal{J}_\varepsilon^{ab}(z_n) \to c - \mathcal{J}_\varepsilon^{ab}(v),\ (\mathcal{J}_\varepsilon^{ab})'(z_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N) \text{ as } n \to \infty. \tag{3.15}$$

Now we attest $\mathcal{J}_\varepsilon^\infty(z_n) \to c - \mathcal{J}_\varepsilon^{ab}(v)$, $(\mathcal{J}_\varepsilon^\infty)'(z_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$. By definition, for any $\delta > 0$, there is $R > 0$ such that $|V_\varepsilon^\infty(x) - V_\varepsilon^a(x)| \le \delta$, $|W_{i\varepsilon}^\infty(x) - W_{i\varepsilon}^{b_i}(x)| \le \delta, i=1,2$ for all $|x| > R$. Hence, according to Lemma 2.2 and the Hölder inequality, we get

$$\begin{aligned} \left|\mathcal{J}_\varepsilon^\infty(z_n) - \mathcal{J}_\varepsilon^{ab}(z_n)\right| &\le \left(\frac{p-2}{2p}|z_n|_2^2 + \frac{(q-p)k_2}{pq}|z_n|_{2Nq/(N+\theta)}^{2q}\right)\delta \\ &\quad + C\left(|z_n|_{L^2(B_R)}^2 + |z_n|_{L^{2Nq/(N+\theta)}(B_R)}^q\right), \end{aligned}$$

which together with (3.14) and (3.15), imply that

$$\mathcal{J}_\varepsilon^\infty(z_n) \to c - \mathcal{J}_\varepsilon^{ab}(v) \quad \text{as } n \to \infty. \tag{3.16}$$

For any $\varphi \in H^1(\mathbb{R}^N)$, by the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} &\left|\left\langle (\mathcal{J}_\varepsilon^\infty)'(z_n) - (\mathcal{J}_\varepsilon^{ab})'(z_n), \varphi \right\rangle\right| \\ &\le C_1\delta\left(|z_n|_2 + |z_n|_{2Np/(N+\theta)}^{2p-1} + |z_n|_{2Nq/(N+\theta)}^{2q-1}\right)\|\varphi\|_1 \\ &\quad + C_2\left(|z_n|_{L^2(B_R)} + |z_n|_{L^{2Np/(N+\theta)}(B_R)}^{2p-1} + |z_n|_{L^{2Nq/(N+\theta)}(B_R)}^{2q-1}\right)\|\varphi\|_1, \end{aligned}$$

which combining with (3.14) and (3.15), implies that

$$(\mathcal{J}_\varepsilon^\infty)'(z_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{3.17}$$

It follows from (3.16) and (3.17) that $\{z_n\}$ is a $(PS)_{c-\mathcal{J}_\varepsilon^{ab}(v)}$ sequence of $\mathcal{J}_\varepsilon^\infty$. According to Lemma 3.11, either $c = \mathcal{J}_\varepsilon^{ab}(v)$ or $c \ge \mathcal{J}_\varepsilon^{ab}(v) + \mathcal{E}_\varepsilon^\infty$. But the latter contradicts with the assumption $c < \mathcal{E}_\varepsilon^\infty$. Thus $c = \mathcal{J}_\varepsilon^{ab}(v)$ and

$$\mathcal{J}_\varepsilon^{ab}(v_n) \to \mathcal{J}_\varepsilon^{ab}(v) \quad \text{as } n \to \infty. \tag{3.18}$$

We show below that $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. According to (3.18), $\mathcal{J}_\varepsilon^{ab}(z_n) \to 0$ as $n \to \infty$. Due to

$$\mathcal{J}_\varepsilon^{ab}(z_n) = \frac{p-1}{2p}\int_{\mathbb{R}^N}\left(|\nabla z_n|^2 + V_\varepsilon^a(x) z_n^2\right) + \frac{q-p}{2pq}\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon}^{b_2}(z_n) z_n + o(1),$$

we obtain $\int_{\mathbb{R}^N}\left(|\nabla z_n|^2 + V_\varepsilon^a(x) z_n^2\right) \to 0$ as $n \to \infty$, which means that $\|z_n\|_1 \to 0$ as $n \to \infty$. By using the Brézis-Lieb lemma, $\|v_n\|_1 \to \|v\|_1$ as $n \to \infty$. Hence, $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. □

Lemma 3.14. $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{ab} \le \mathcal{E}^{\alpha\boldsymbol{\beta}}$, *where* $\alpha = V^a(0)$, $\beta_i = W_i^{b_i}(0), i = 1,2$, $\boldsymbol{\beta} := (\beta_1, \beta_2)$. *Meanwhile, if* $V(0) \le a$, $W_i(0) \ge b_i, i = 1,2$, *then* $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{ab} = \mathcal{E}^{ab}$.

Proof. Set $\bar{V}_\varepsilon(x) := V_\varepsilon^a(x) - \alpha$ and $\bar{W}_{i\varepsilon}(x) := \beta_i - W_{i\varepsilon}^{b_i}(x), i = 1,2$. Thus

$$\bar{V}_\varepsilon(x) \to 0, \bar{W}_{i\varepsilon}(x) \to 0, i = 1,2 \quad \text{a.e. on } \mathbb{R}^N \text{ as } \varepsilon \to 0. \tag{3.19}$$

Meanwhile,

$$\begin{aligned}\mathcal{J}_\varepsilon^{ab}(v) = \mathcal{J}^{\alpha\boldsymbol{\beta}}(v) &+ \frac{1}{2}\int_{\mathbb{R}^N}\bar{V}_\varepsilon(x) v^2 + \frac{\beta_1}{p}\int_{\mathbb{R}^N}\bar{W}_{1\varepsilon}(x)\left(I_\theta * |v|^p\right)|v|^p \\ &- \frac{1}{2p}\int_{\mathbb{R}^N}\bar{\mathcal{Y}}_{1\varepsilon}(v) v + \frac{\beta_2}{q}\int_{\mathbb{R}^N}\bar{W}_{2\varepsilon}(x)\left(I_\theta * |v|^q\right)|v|^q - \frac{1}{2q}\int_{\mathbb{R}^N}\bar{\mathcal{Y}}_{2\varepsilon}(v) v,\end{aligned} \tag{3.20}$$

where $\bar{\mathcal{Y}}_{1\varepsilon}(v) := \bar{W}_{1\varepsilon}(x)\left(I_\theta * \bar{W}_{1\varepsilon}|v|^p\right)|v|^{p-2} v$, $\bar{\mathcal{Y}}_{2\varepsilon}(v) := \bar{W}_{2\varepsilon}(x)\left(I_\theta * \bar{W}_{2\varepsilon}|v|^q\right)|v|^{q-2} v$. By Lemma 3.3, there is $e \in \mathcal{T}^{\alpha\boldsymbol{\beta}}$. Set $r_\varepsilon > 0$ satisfy $r_\varepsilon e \in \mathcal{N}_\varepsilon^{ab}$, we get

$$\max_{r\ge 0}\mathcal{J}_\varepsilon^{ab}(re) = \mathcal{J}_\varepsilon^{ab}(r_\varepsilon e) \ge \mathcal{E}_\varepsilon^{ab}. \tag{3.21}$$

Since $\mathcal{J}_\varepsilon^{ab}(re) \to -\infty$ as $r \to +\infty$, there exists $R_0 > 0$ such that $\mathcal{J}_\varepsilon^{ab}(re) < 0$, for all $r > R_0$. Hence we get $r_\varepsilon \le R_0$. We posit $r_\varepsilon \to r_0$ as $\varepsilon \to 0$. It follows from (3.19), (3.20), (3.21) and the Lebesgue dominated convergence theorem that

$$\begin{aligned}\mathcal{E}_\varepsilon^{ab} &\le \mathcal{J}^{\alpha\boldsymbol{\beta}}(r_\varepsilon e) + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}\bar{V}_\varepsilon(x) e^2 + \frac{\beta_1 \cdot t_\varepsilon^{2p}}{p}\int_{\mathbb{R}^N}\bar{W}_{1\varepsilon}(x)\left(I_\theta * |e|^p\right)|e|^p \\ &\quad - \frac{t_\varepsilon^{2p}}{2p}\int_{\mathbb{R}^N}\bar{\mathcal{Y}}_{1\varepsilon}(e) e + \frac{\beta_2 \cdot t_\varepsilon^{2q}}{q}\int_{\mathbb{R}^N}\bar{W}_{2\varepsilon}(x)\left[I_\theta * |e|^q\right]|e|^q - \frac{t_\varepsilon^{2q}}{2q}\int_{\mathbb{R}^N}\bar{\mathcal{Y}}_{2\varepsilon}(e) e \\ &\to \mathcal{J}^{\alpha\boldsymbol{\beta}}(r_0 e) \le \mathcal{J}^{\alpha\boldsymbol{\beta}}(e) = \mathcal{E}^{\alpha\boldsymbol{\beta}} \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Thus $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{ab} \le \mathcal{E}^{\alpha\boldsymbol{\beta}}$.

Eventually, if $V(0) \le a$, $W_i(0) \ge b_i$, then $\alpha = a$, $\beta_i = b_i$, $i = 1,2$. Hence $\bar{V}_\varepsilon(x) \ge 0$, $\bar{W}_{i\varepsilon}(x) \ge 0, i = 1,2$ for all $x \in \mathbb{R}^N$. we get $\mathcal{J}_\varepsilon^{ab}(v) \ge \mathcal{J}^{\alpha\boldsymbol{\beta}}(v)$ for all $v \in H^1(\mathbb{R}^N)$ by (3.20). Thus, $\mathcal{E}_\varepsilon^{ab} \ge \mathcal{E}^{\alpha\boldsymbol{\beta}}$. Due to $\mathcal{E}^{\alpha\boldsymbol{\beta}} \le \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{ab} \le \limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{ab} \le \mathcal{E}^{\alpha\boldsymbol{\beta}}$, we obtain $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{ab} = \mathcal{E}^{\alpha\boldsymbol{\beta}} = \mathcal{E}^{ab}$.

Lemma 3.15. *If* $\tau \le a < \tau_\infty, \boldsymbol{k} \ge \boldsymbol{b} \ge \boldsymbol{k}_\infty$ *or* $\tau \le a \le \tau_\infty, \boldsymbol{k} \ge \boldsymbol{b} > \boldsymbol{k}_\infty$, *then there exists* $\varepsilon^{ab} > 0$ *such that for all* $\varepsilon \le \varepsilon^{ab}, \mathcal{E}_\varepsilon^{ab}$ *is attained at* $v_\varepsilon^{ab} > 0$.

Proof. Noting Lemma 3.8, we have $\mathcal{E}^{\alpha\boldsymbol{\beta}} < \mathcal{E}^\infty$, where $\alpha = V^a(0)$ and $\beta_i = W_i^{b_i}(0), i = 1,2$. By Lemmas 3.14 and 3.11, there exists $\varepsilon^{ab} > 0$ such that

$\mathcal{E}_\varepsilon^{ab} < \mathcal{E}^\infty \le \mathcal{E}_\varepsilon^\infty$ for all $\varepsilon \le \varepsilon^{ab}$. By Lemma 3.13, $\mathcal{J}_\varepsilon^{ab}$ satisfies the $(PS)_{\mathcal{E}_\varepsilon^{ab}}$ condition for all $\varepsilon \le \varepsilon^{ab}$, which together with Lemmas 3.9 and 3.10 imply that $\mathcal{E}_\varepsilon^{ab}$ is attained at $v_\varepsilon^{ab} \in H^1(\mathbb{R}^N)$. Since $\mathcal{J}_\varepsilon^{ab}(v) = \mathcal{J}_\varepsilon^{ab}(|v|)$ for any $v \in H^1(\mathbb{R}^N)$, we may assume that $v^{ab} \ge 0$. By bootstrap method and elliptic regularity theory, $v_\varepsilon^{ab} \in C^2(\mathbb{R}^N)$. By strong maximum principle, $v_\varepsilon^{ab} > 0$. □

4. Proof of the Main Results

Setting $v(x) := w(\varepsilon x)$, the Equation (1.1) is equivalent to

$$-\Delta v + V(\varepsilon x) v = \mathcal{Y}_1(v) + \mathcal{Y}_2(v), \quad v \in H^1(\mathbb{R}^N), \tag{4.1}$$

where $\mathcal{Y}_1(v) := W_1(\varepsilon x)\left[I_\theta * \left(W_1(\varepsilon x)|v|^p\right)\right]|v|^{p-2} v$,

$\mathcal{Y}_2(v) := W_2(\varepsilon x)\left[I_\theta * \left(W_2(\varepsilon x)|v|^q\right)\right]|v|^{q-2} v$. If $v_\varepsilon(x)$ is a solution of Equation (4.1), then $w_\varepsilon(x) = v_\varepsilon\left(\frac{x}{\varepsilon}\right)$ is a solution of Equation (1.1).

Noting $V(\varepsilon x) = V_\varepsilon^\tau(x), W_i(\varepsilon x) = W_{i\varepsilon}^{k_i}(x), i = 1,2$, we find that Equation (4.1) is particular form of Equation (3.2). We set

$$\mathcal{J}_\varepsilon := \mathcal{J}_\varepsilon^{\tau\boldsymbol{k}}, \mathcal{N}_\varepsilon := \mathcal{N}_\varepsilon^{\tau\boldsymbol{k}}, \mathcal{E}_\varepsilon := \mathcal{E}_\varepsilon^{\tau\boldsymbol{k}}, \mathcal{T}_\varepsilon := \mathcal{T}_\varepsilon^{\tau\boldsymbol{k}}, V_\varepsilon := V_\varepsilon^\tau, W_{i\varepsilon} := W_{i\varepsilon}^{k_i}, i = 1,2.$$

4.1. Proof of Theorem 1.1

Without loss of generality, we assume $x_{iv} = 0$. Then $V(0) = \tau$, $W_i(0) = k_{iv}$, $i = 1,2$.

Lemma 4.1. *There exists an m-dimensional subspace* $\mathcal{D}_{rm}$ *of* $H^1(\mathbb{R}^N)$ *such that* $\sup_{v\in\mathcal{D}_{rm}} \mathcal{J}_\varepsilon(v) < \mathcal{E}^\infty$, *for all* $r \ge r_m$, $\varepsilon \le \varepsilon_m$, *where* r_m *and* ε_m *are existing constants depending on m.*

Proof. Choose $a = \tau$, $b_i = k_{iv}, i = 1,2$, in Equation (3.1). By Lemma 3.3, there exists $v \in \mathcal{T}^{\tau\boldsymbol{k}_v}$ and $v(x) = v(|x|) > 0$. Let $r > 0$, $\chi_r \in C_0^\infty(\mathbb{R}_+)$ satisfy $\chi_r(t) = 1$ for $t \le r$ and $\chi_r(t) = 0$ for $t \ge r+1$ with $|\chi_r'(t)| \le 2$. Set $v_r(x) := \chi_r(|x|) v(x)$ for $x \in \mathbb{R}^N$. It follows from $\|v_r - v\|_1^2 \le \bar{C}\left(\int_{|x|>r} |\nabla v|^2 + v^2\right) \to 0$ as $r \to \infty$, that $v_r \to v$ in $H^1(\mathbb{R}^N)$, $v_r \to v$ in $L^{\frac{2Ns}{N+\theta}}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}\left(I_\theta * v_r^s\right)v_r^s \to \int_{\mathbb{R}^N}\left(I_\theta * v^s\right)v^s$ for $s = p,q$ as $r \to \infty$. There exists $d_r > 0$ such that $d_r v_r \in \mathcal{N}^{\tau\boldsymbol{k}_v}$ and $d_r \to 1$ as $r \to \infty$. Hence

$$\begin{aligned}\max_{d\ge0} \mathcal{J}^{\tau\boldsymbol{k}_v}(dv_r) &= \left(\frac{d_r^2}{2} - \frac{d_r^{2p}}{2p}\right)\int_{\mathbb{R}^N} \mathcal{Y}_1^{k_{1v}}(v_r)v_r + \left(\frac{d_r^2}{2} - \frac{d_r^{2q}}{2q}\right)\int_{\mathbb{R}^N} \mathcal{Y}_2^{k_{2v}}(v_r)v_r \\ &\to \frac{p-1}{2p}\int_{\mathbb{R}^N} \mathcal{Y}_1^{k_{1v}}(v)v + \frac{q-1}{2q}\int_{\mathbb{R}^N} \mathcal{Y}_2^{k_{2v}}(v)v\,(r \to \infty) \\ &= \max_{d\ge0} \mathcal{J}^{\tau\boldsymbol{k}_v}(dv) = \mathcal{J}^{\tau\boldsymbol{k}_v}(v) = \mathcal{E}^{\tau\boldsymbol{k}_v} = \mathcal{E}^{\tau\boldsymbol{k}_v}.\end{aligned} \tag{4.2}$$

Additionally,

$$V_\varepsilon(x) \to V(0) = \tau, \quad W_{i\varepsilon}(x) \to W_i(0) = k_{iv}, \quad i = 1,2 \quad \text{as } \varepsilon \to 0 \tag{4.3}$$

uniformly on any bounded set of *x*. There exists $\hat{d}_r > 0$ such that $\hat{d}_r v_r \in \mathcal{N}_\varepsilon$ and $\hat{d}_r \to 1$ as $r \to \infty$. Therefore, (4.2) and (4.3) mean that

$$\begin{aligned}\max_{d\geq 0}\mathcal{J}_\varepsilon(dv_r) &= \left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2p}}{2p}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{1\varepsilon}(v_r)v_r+\left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2q}}{2q}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon}(v_r)v_r\\ &\to \left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2p}}{2p}\right)\int_{\mathbb{R}^N}\mathcal{Y}_1^{k_{1v}}(v_r)v_r+\left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2q}}{2q}\right)\int_{\mathbb{R}^N}\mathcal{Y}_2^{k_{2v}}(v_r)v_r\,(\varepsilon\to 0)\\ &\to \max_{d\geq 0}\mathcal{J}^{\tau\boldsymbol{k}_v}(dv_r)\to\mathcal{E}^{\tau\boldsymbol{k}_v}(r\to\infty).\end{aligned} \tag{4.4}$$

According to lemma 3.8, we get $m(\tau,\boldsymbol{k}_v)>1$. We let $m_v = m(\tau,\boldsymbol{k}_v)$. For the maximal integer $m\in\mathbb{Z}_+$ with $m<m_v$, we have $m\geq 1$. Define $\eta_{rj}(x) := v_r(x_1-2j(x+1),x_2,\cdots,x_N)$ for $j=0,1,\cdots,m-1$ and set $\mathcal{D}_{rm} := span\{\eta_{rj}(x): j=0,1,\cdots,m-1\}$. We can get $(\eta_{ri},\eta_{rj})_1 = 0$ if $i\neq j$. Hence $\dim\mathcal{D}_{rm}=m$. Similarly as (4.4), for all $j=1,2,\cdots,m-1$, we get

$$\begin{aligned}\max_{d\geq 0}\mathcal{J}_\varepsilon(d\psi_{rj}) &= \left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2p}}{2p}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{1\varepsilon}(\psi_{rj})\psi_{rj}+\left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2q}}{2q}\right)\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon}(\psi_{rj})\psi_{rj}\\ &\to \left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2p}}{2p}\right)\int_{\mathbb{R}^N}\mathcal{Y}_1^{k_{1v}}(v_r)v_r+\left(\frac{\hat{d}_r^2}{2}-\frac{\hat{d}_r^{2q}}{2q}\right)\int_{\mathbb{R}^N}\mathcal{Y}_2^{k_{2v}}(v_r)v_r\,(\varepsilon\to 0)\\ &\to \max_{d\geq 0}\mathcal{J}^{\tau\boldsymbol{k}_v}(dv_r)\to\mathcal{E}^{\tau\boldsymbol{k}_v}(r\to\infty).\end{aligned}$$

Thus, for all $\delta>0$, there exist $r_\delta>0,\varepsilon_\delta>0$ such that $\max_{d\geq 0}\mathcal{J}_\varepsilon(d\psi_{rj})\leq\mathcal{E}^{\tau\boldsymbol{k}_v}+\delta$, for all $r\geq r_\delta$ and $\varepsilon\leq\varepsilon_\delta, j=0,1,\cdots,m-1$. For any $v\in\mathcal{D}_{rm}$, we posit $v=\sum_{j=0}^{m-1}d_j\psi_{rj}$, where $d_j\in\mathbb{R}$ for $j=0,1,\cdots,m-1$. Thus, we have $\mathcal{J}_\varepsilon(v)\leq\sum_{j=0}^{m-1}\mathcal{J}_\varepsilon(d_j\psi_{rj})\leq\sum_{j=0}^{m-1}\max_{d\geq 0}\mathcal{J}_\varepsilon(d\psi_{r_j})\leq m(\mathcal{E}^{\tau\boldsymbol{k}_v}+\delta)$ for all $r\geq r_\delta$ and $\varepsilon\leq\varepsilon_\delta$, which implies that $\sup_{v\in\mathcal{D}_{rm}}\mathcal{J}_\varepsilon(v)\leq m(\mathcal{E}^{\tau\boldsymbol{k}_v}+\delta)$. Due to Lemma 3.7, we set $0<\delta<\frac{\mathcal{E}^\infty}{m}-\mathcal{E}^{\tau\boldsymbol{k}_v}$, then there is $r_m>0$, $\varepsilon_m>0$ such that $\sup_{v\in\mathcal{D}_{rm}}\mathcal{J}_\varepsilon(v)<\mathcal{E}^\infty$, for all $r\geq r_m$, $\varepsilon\leq\varepsilon_m$. □

Lemma 4.2. Equation (4.1) *has at least m pairs of semiclassical solutions.*

Proof. Let us consider the symmetric group $\mathbb{Z}_2=\{id,-id\}$ and set $\Sigma := \{\mathcal{T}\subset\mathcal{D}:\mathcal{T}$ is closed and $\mathcal{T}=-\mathcal{T}\}$. For any $\mathcal{T}\in\Sigma$, the Krasnoselskii genus of $\mathcal{T}$ is denoted by

$$\text{gen}(\mathcal{T}) := \inf\{n:\text{there exists } g\in C(\mathcal{T},\mathbb{R}^n\setminus\{0\})\text{ and } g \text{ is odd}\}.$$

Set $\mathcal{H} := \{h\in C(\mathcal{D},\mathcal{D}): h \text{ is an odd home omorphism}\}$ and for any $\mathcal{T}\in\Sigma$, define Benci pseudo-index of $\mathcal{T}$ by

$$i(\mathcal{T}) := \min_{h\in\mathcal{H}}\text{gen}(h(\mathcal{T})\cap\partial B\rho),$$

where $\rho > 0$ is a constant defined in Lemma 3.9. Let $\varsigma_j := \inf_{i(\mathcal{T}) \geq j} \sup_{v \in \mathcal{T}} \mathcal{J}_\varepsilon(v)$, $j = 1, 2, \cdots, m$. We can easily to verify that $\varsigma_1 \leq \varsigma_2 \leq \cdots \leq \varsigma_m$.

When $j = 1$, for any $\mathcal{T} \in \Sigma$ and $i(\mathcal{T}) \geq 1$, we have $\text{gen}(\mathcal{T} \cap \partial B\rho) \geq 1$, which means $\mathcal{T} \cap \partial B\rho \neq \varnothing$. By Lemma 3.9 that $\sup_{v \in \mathcal{T}} \mathcal{J}_\varepsilon(v) > \sigma$ and $\varsigma_1 \geq \sigma$.

When $j = m$, taking into account that the Krasnoselskii genus satisfies the dimension property [18], we have $\text{gen}(h(\mathscr{D}_{rm}) \cap \partial B\rho) = \dim \mathscr{D}_{rm} = m$ for all $h \in \mathcal{H}$, which implies $i(\mathscr{D}_{rm}) = m$. Hence $\varsigma_m \leq \sup_{v \in \mathscr{D}_{rm}} \mathcal{J}_\varepsilon(v)$. Due to Lemmas 4.4, 3.11, we have that for any $r \geq r_m$, $\varepsilon \leq \varepsilon_m$,

$$\sigma \leq \varsigma_1 \leq \varsigma_2 \leq \cdots \leq \varsigma_m \leq \sup_{v \in \mathscr{D}_{rm}} \mathcal{J}_\varepsilon(v) < \mathcal{E}^\infty \leq \mathcal{E}_\varepsilon^\infty. \tag{4.5}$$

Next we are going to prove $\varsigma_j\ (j = 1, 2, \cdots, m)$ are critical values of $\mathcal{J}_\varepsilon$ by using Theorem 1.4 in [18]. Set $\varsigma_0 := \sigma$, $\varsigma_\infty := \sup_{v \in \mathscr{D}_{rm}} \mathcal{J}_\varepsilon(v)$

$$(\mathcal{J}_\varepsilon)^c := \left\{ v \in H^1(\mathbb{R}^N) : \mathcal{J}_\varepsilon(v) \leq c \right\}, \quad \mathcal{K}_c := \left\{ v \in H^1(\mathbb{R}^N) : \mathcal{J}_\varepsilon(v) = c, (\mathcal{J}_\varepsilon)'(v) = 0 \right\}.$$

Since $\mathcal{J}_\varepsilon$ is an even fuctional, $(\mathcal{J}_\varepsilon)^c \in \Sigma$, $\mathcal{K}_c \in \Sigma$, for all $c \in [\varsigma_0, \varsigma_\infty]$. According to (4.5) and Lemma 3.13, $\mathcal{J}_\varepsilon$ satisfies the $(PS)_c$ condition for any $c \in [\varsigma_0, \varsigma_\infty]$, which means that $\mathcal{K}_c$ is compact in $H^1(\mathbb{R}^N)$, for any $c \in [\varsigma_0, \varsigma_\infty]$. For any $c \in [\varsigma_0, \varsigma_\infty]$, $d > 0$ and $(\mathcal{K}_c)_d := \left\{ v \in H^1(\mathbb{R}^N) : \text{dist}(v, \mathcal{K}_c) < d \right\}$, choose $\delta = \frac{d}{4}$, then by the contradiction method we can get that there exists $\tilde{\varepsilon} > 0$ such that $\left\| (\mathcal{J}_\varepsilon)'(v) \right\| \geq \frac{8\tilde{\varepsilon}}{\delta}$, for all $v \in \mathcal{J}_\varepsilon^{-1}([c - 2\tilde{\varepsilon}, c + 2\tilde{\varepsilon}]) \setminus \overline{(\mathcal{K}_c)_{d/2}}$.

On the basis of Lemma 2.3 in [14], we choose $\mathcal{S} := H^1(\mathbb{R}^N) \setminus (\mathcal{K}_c)_d$, there exists $\tilde{\mu} \in C([0,1] \times H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$ such that $\tilde{\mu}\left(1, (\mathcal{J}_\varepsilon)^{c+\tilde{\varepsilon}} \cap \mathcal{S}\right) \subset (\mathcal{J}_\varepsilon)^{c-\tilde{\varepsilon}}$ and $\tilde{\mu}(t, \cdot)$ is an odd homeomorphism on $H^1(\mathbb{R}^N)$ for any $t \in [0,1]$. Set $\mu(\cdot) := \tilde{\mu}(1, \cdot)$, then μ is an odd homeomorphism on $H^1(\mathbb{R}^N)$ and

$$\mu\left((\mathcal{J}_\varepsilon)^{c+\tilde{\varepsilon}} \setminus (\mathcal{K}_c)_d\right) \subset (\mathcal{J}_\varepsilon)^{c-\tilde{\varepsilon}}. \tag{4.6}$$

For any $\mathcal{T} \in \Sigma$ and $\mathcal{T} \subset (\mathcal{J}_\varepsilon)^{\varsigma_0} = (\mathcal{J}_\varepsilon)^\sigma$, then $\mathcal{J}_\varepsilon(v) \leq \sigma$ for any $v \in \mathcal{T}$. By Lemma 3.9, we have $\mathcal{T} \cap \partial B\rho = \varnothing$. As a result, $\text{gen}(\mathcal{T} \cap \partial B\rho) = 0$ and

$$i(\mathcal{T}) = \min_{h \in \mathcal{H}} \text{gen}(h(\mathcal{T}) \cap \partial B\rho) = 0. \tag{4.7}$$

Then, we get

$$\mathscr{D}_{rm} \subset (\mathcal{J}_\varepsilon)^{\varsigma_\infty} \text{ and } i(\mathscr{D}_{rm}) = m \geq 1. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we have that $\varsigma_1, \varsigma_2, \cdots, \varsigma_m$ are critical values of $\mathcal{J}_\varepsilon$, and $\text{gen}(\mathcal{K}_c) \geq r + 1$ if $c := \varsigma_j = \varsigma_{j+1} = \cdots = \varsigma_{j+r}$ with $j \geq 1$ and $j + r \leq m$. Since $\mathcal{J}_\varepsilon$ is even, we infer that $\mathcal{J}_\varepsilon$ has at least m pairs of critical points which are also solutions of Equation (4.1). □

Lemma 4.3. Equation (4.1) *has at least one positive and one negative least* *energy solution for* $m \geq 1$.

Proof. Choose $a = \tau, b_i = k_i, i = 1,2$ in Equation (3.1), then $\alpha = V^\tau(0) = V(0) = \tau$, $\beta_i = W_i^{k_i}(0) = W_i(0) = k_i$, $i = 1,2$. Due to Lemmas 3.7, 3.11, 3.14, 3.13, $\mathcal{J}_\varepsilon$ has a $(PS)_{\mathcal{E}_\varepsilon}$ sequence and satisfies $(PS)_{\mathcal{E}_\varepsilon}$ condition. According to Lemma 3.15, there exists $\varepsilon_0 > 0$ such that $\mathcal{E}_\varepsilon$ is attained at $v_\varepsilon > 0$ for all $\varepsilon \leq \varepsilon_0$. Hence, v_ε and $-v_\varepsilon$ are positive and negative least energy solutions of Equation (4.1), respectively. □

This completes the proof.

4.2. Proof of Theorem 1.2

We can assume without loss of generality that $x_w = 0$. Then $V(0) = \tau_\omega$, $W_i(0) = k_i, i = 1,2$. Setting $a = \tau_\omega, b_i = k_i, i = 1,2$ in Equation (3.1), there is $v \in \mathcal{T}^{\tau_\omega \boldsymbol{k}}$. Due to Lemma 3.8, $m(\tau_\omega, \boldsymbol{k}) \geq 1$. We set

$$m_w = \begin{cases} m(\tau_\omega, \boldsymbol{k}) & \text{if } m(\tau_\omega, \boldsymbol{k}) > 1, \\ \dfrac{3}{2} & \text{if } m(\tau_\omega, \boldsymbol{k}) = 1. \end{cases}$$

For the maximal integer $m < m_w$, we get $m \geq 1$. Because of Lemma 3.7, $m\mathcal{E}^{\tau_\omega \boldsymbol{k}} < \mathcal{E}^\infty$. The remaining proof of this theorem is similar to the proof of Theorem 1.1 and other details are omitted.

4.3. Proof of Theorem 1.3

In general, we assume $x_{iv} = 0$. Then $V(0) = \tau$, $W_i(0) = k_{iv}, i = 1,2$. We can verify that the condition of ($\mathcal{A}3$)(i) implies that (1.5) holds. It follows from Theorem 1.1 that Equation (1.1) has a positive groundstate solution $w_\varepsilon(x)$ and Equation (4.1) has a positive least energy solution $v_\varepsilon(x) = w_\varepsilon(\varepsilon x)$. Next, we will prove the case ($\mathcal{A}3$)(i), the other case can be handled similarly.

Lemma 4.4. $v_\varepsilon \to v$ *as* $\varepsilon \to 0$ *in the sence of sequence after translations.*

Proof. Set $\varepsilon_j \to 0$ as $j \to \infty$, $v_j := v_{\varepsilon_j} \in \mathcal{T}_{\varepsilon_j}$ with $v_j > 0$. Thus, we have

$$\mathcal{E}_{\varepsilon_j} = \mathcal{J}_{\varepsilon_j}(v_j) = \frac{p-1}{2p}\int_{\mathbb{R}^N}\left(|\nabla v_j|^2 + V_{\varepsilon_j}(x)v_j^2\right) + \frac{q-p}{2pq}\int_{\mathbb{R}^N}\mathcal{Y}_{2\varepsilon_j}(v_j)v_j \geq C\|v_j\|_1^2,$$

due to Lemma 3.14, we know that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^N)$. Let $\limsup\limits_{j\to\infty} \sup\limits_{y\in\mathbb{R}^N}\int_{B_1(y)} v_j^2 = 0$, by Lemmas 1.5, 2.1, we obtain $v_j \to 0$ in $L^{2Nr/(N+\theta)}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}(I_\theta * v_j^r)v_j^r \to 0$ as $j \to \infty$ for $r = p,q$, which together with $v_j \in \mathcal{N}_{\varepsilon_j}$ imply that $\|v_j\|_1 \to 0$ as $j \to \infty$. It is a contradiction with $\|v_j\|_1 \geq C > 0$. Thus, there is $\delta > 0$ and $y_j' \in \mathbb{R}^N$ such that

$$\int_{B_1(y_j')} v_j^2 \geq \delta. \tag{4.9}$$

Define $\hat{v}_j(x) := v_j(x + y_j')$, $\hat{V}_{\varepsilon_j}(x) := V_{\varepsilon_j}(x + y_j')$, $\hat{W}_{i\varepsilon_j}(x) := W_{i\varepsilon_j}(x + y_j')$, $i = 1,2$. Thus, $\hat{v}_j$ is the solution of

$$-\Delta\hat{v}_j + \hat{V}_{\varepsilon_j}(x)\hat{v}_j = \hat{\mathcal{Y}}_{1\varepsilon_j}(\hat{v}_j) + \hat{\mathcal{Y}}_{2\varepsilon_j}(\hat{v}_j), \quad \hat{v}_j > 0 \tag{4.10}$$

with least energy

$$\hat{\mathcal{E}}_{\varepsilon_j} = \hat{\mathcal{J}}_{\varepsilon_j}\left(\hat{v}_j\right) := \frac{p-1}{2p}\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{1\varepsilon_j}\left(\hat{v}_j\right)\hat{v}_j + \frac{q-1}{2q}\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{2\varepsilon_j}\left(\hat{v}_j\right)\hat{v}_j. \quad (4.11)$$

where $\hat{\mathcal{Y}}_{1\varepsilon_j}\left(\hat{v}_j\right) := \hat{W}_{1\varepsilon_j}(x)\left[I_\theta * \left(\hat{W}_{1\varepsilon_j}\hat{v}_j^p\right)\right]\hat{v}_j^{p-1}$,
$\hat{\mathcal{Y}}_{2\varepsilon_j}\left(\hat{v}_j\right) := \hat{W}_{2\varepsilon_j}(x)\left[I_\theta * \left(\hat{W}_{2\varepsilon_j}\hat{v}_j^q\right)\right]\hat{v}_j^{q-1}$. Additonally,
$\left[I_\theta * \left(W_{1\varepsilon_j}v_j^p\right)\right]\left(x+y_j'\right) = \left[I_\theta * \left(\hat{W}_{1\varepsilon_j}\hat{v}_j^p\right)\right](x)$,
$\left[I_\theta * \left(W_{2\varepsilon_j}v_j^q\right)\right]\left(x+y_j'\right) = \left[I_\theta * \left(\hat{W}_{2\varepsilon_j}\hat{v}_j^q\right)\right](x)$ for any $x \in \mathbb{R}^N$, which imply that

$$\hat{\mathcal{E}}_{\varepsilon_j} = \hat{\mathcal{J}}_{\varepsilon_j}\left(\hat{v}_j\right) = \mathcal{J}_{\varepsilon_j}\left(v_j\right) = \mathcal{E}_{\varepsilon_j}. \quad (4.12)$$

Due to the boundedness of $\left\{\hat{v}_j\right\}$, we can suppose without loss of generality that

$$\hat{v}_j \rightharpoonup v \quad \text{in } H^1\left(\mathbb{R}^N\right) \quad \text{as } j \to \infty, \quad (4.13)$$

$$\hat{v}_j \to v \quad \text{in } L_{loc}^r\left(\mathbb{R}^N\right) \quad \text{as } j \to \infty \quad \text{for } r \in \left[2, 2^*\right), \quad (4.14)$$

which combine with (4.9) imply that $v \neq 0$.

According to V and $W_i, i = 1,2$ are bounded, we posit

$$V_{\varepsilon_j}\left(y_j'\right) \to V_0 \quad \text{and} \quad W_{i\varepsilon_j}\left(y_j'\right) \to W_{i0}, \quad i = 1,2 \quad \text{as } j \to \infty. \quad (4.15)$$

Because of $\nabla V : \left|\nabla V(x)\right| \le M$ for all $x \in \mathbb{R}^N$, we have that for any $r > 0$, $\left|\hat{V}_{\varepsilon_j}(x) - V_{\varepsilon_j}\left(y_j'\right)\right| \le \varepsilon_j M r$, for all $x \in B_r(0)$. Hence $\hat{V}_{\varepsilon_j} \to V_0$, $\hat{W}_{i\varepsilon_j} \to W_{i0}, i = 1,2$ as $j \to \infty$ uniformly on any bounded set of *x*. Using the proof of Lemma 3.14, we have

$$\limsup_{j\to\infty} \hat{\mathcal{E}}_{\varepsilon_j} \le \mathcal{E}^{V_0 W_0}. \quad (4.16)$$

Uniting (4.10), (4.13), (4.15), we get that for any $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$,

$$0 = \lim_{j\to\infty}\int_{\mathbb{R}^N}\left[\nabla\hat{v}_j\nabla\varphi + \hat{V}_{\varepsilon_j}(x)\hat{v}_j\varphi - \hat{\mathcal{Y}}_{1\varepsilon_j}\left(\hat{v}_j\right)\varphi - \hat{\mathcal{Y}}_{2\varepsilon_j}\left(\hat{v}_j\right)\varphi\right]$$
$$= \int_{\mathbb{R}^N}\left[\nabla v\nabla\varphi + V_0 v\varphi - \mathcal{Y}_{10}(v)\varphi - \mathcal{Y}_{20}(v)\varphi\right],$$

with $\mathcal{Y}_{10}(v) := W_{10}\left[I_\theta * \left(W_{10}v^p\right)\right]v^{p-2}v$, $\mathcal{Y}_{20}(u) := W_{20}\left[I_\theta * \left(W_{20}v^q\right)\right]v^{q-2}v$, which means v solves

$$-\Delta v + V_0 v = \mathcal{Y}_{10}(v) + \mathcal{Y}_{20}(v), \quad v > 0 \quad (4.17)$$

with energy

$$\mathcal{J}^{V_0 W_0}(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla v\right|^2 + V_0 v^2\right) - \frac{1}{2p}\int_{\mathbb{R}^N}\mathcal{Y}_{10}(v)v - \frac{1}{2q}\int_{\mathbb{R}^N}\mathcal{Y}_{20}(v)v \ge \mathcal{E}^{V_0 W_0}. \quad (4.18)$$

Due to Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^N}\mathcal{Y}_{i0}(v)v \le \liminf_{j\to\infty}\int_{\mathbb{R}^N}\hat{\mathcal{Y}}_{i\varepsilon_j}\left(\hat{v}_j\right)\hat{v}_j, \quad i = 1,2. \quad (4.19)$$

Combining (4.11), (4.16), (4.18) and (4.19),
$\mathcal{E}^{V_0 W_0} \le \mathcal{J}^{V_0 W_0}(v) \le \liminf_{j\to\infty}\hat{\mathcal{J}}_{\varepsilon_j}\left(\hat{v}_j\right) \le \limsup_{j\to\infty}\hat{\mathcal{E}}_{\varepsilon_j} \le \mathcal{E}^{V_0 W_0}$. Hence,

$$\lim_{j\to\infty}\hat{\mathcal{E}}_{\varepsilon_j} = \mathcal{E}^{V_0 W_0} = \mathcal{J}^{V_0 W_0}(v). \quad (4.20)$$

Choose $\xi \in C_0^\infty(\mathbb{R}_+)$ satisfy $\operatorname{supp}\xi(t) \subset B_2$ and $\xi \equiv 1$ on B_1 with $|\xi'(t)| \le 2$. Define $\tilde{\mu}_j(x) := \xi\left(\frac{x}{j}\right)v(x)$ and $z_j(x,y) := \hat{v}_j(x) - \tilde{\mu}_j(x)$ for $x \in \mathbb{R}^N$. Thus as $j \to \infty$, $\tilde{\mu}_j \to v$ in $H^1(\mathbb{R}^N)$, $\tilde{\mu}_j \to v$ in $L^r(\mathbb{R}^N)$ for $r \in \left[2, \frac{N+\theta}{N-2}\right]$, $\tilde{\mu}_j \to v$ a.e. on $\mathbb{R}^N$ and $z_j \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, $z_j \to 0$ in $L_{loc}^r(\mathbb{R}^N)$ for $r \in \left[2, \frac{N+\theta}{N-2}\right)$, $z_j \to 0$ a.e. on $\mathbb{R}^N$.

Next, our main goal is to obtain $\hat{\mathcal{J}}_{\varepsilon_j}(z_j) \to 0$ and $\left\langle \left(\hat{\mathcal{J}}_{\varepsilon_j}\right)'(z_j), z_j \right\rangle \to 0$ as $j \to \infty$, where

$$\hat{\mathcal{J}}_{\varepsilon_j}(z_j) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla z_j|^2 + \hat{V}_{\varepsilon_j}(x)|z_j|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{1\varepsilon_j}(z_j)z_j - \frac{1}{2q}\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{2\varepsilon_j}(z_j)z_j.$$

Indeed, similar to the proof of Theorem 1.3 in [12], we can obtain

$$\|z_j\|_1^2 = \|\hat{v}_j\|_1^2 - \|\tilde{\mu}_j\|_1^2 + o(1), \tag{4.21}$$

$$\int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)|z_j|^2 = \int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)|\hat{v}_j|^2 - \int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)|\tilde{\mu}_j|^2 + o(1), \tag{4.22}$$

$$\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{1\varepsilon_j}(z_j)z_j = \int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{i\varepsilon_j}(\hat{v}_j)\hat{v}_j - \int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{i\varepsilon_j}(\tilde{\mu}_j)\tilde{\mu}_j + o(1), \quad i = 1,2. \tag{4.23}$$

According to the Lebesgue dominated convergence theorem, we get that

$$\int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)\tilde{\mu}_j^2 = \int_{\mathbb{R}^N} V_0 v^2 + o(1), \tag{4.24}$$

$$\int_{\mathbb{R}^N} \hat{\mathcal{Y}}_{i\varepsilon_j}(\tilde{\mu}_j)\tilde{\mu}_j = \int_{\mathbb{R}^N} \mathcal{Y}_{i0}(v)v + o(1), \quad i = 1,2. \tag{4.25}$$

Additionally,

$$|\nabla \tilde{\mu}_j|_2^2 = |\nabla v|_2^2 + o(1). \tag{4.26}$$

By (4.21), (4.22), (4.23), (4.24), (4.25), (4.26), (4.20), (4.10) and (4.17), we have

$$\begin{aligned} &\hat{\mathcal{J}}_{\varepsilon_j}(z_j) = \hat{\mathcal{E}}_{\varepsilon_j} - \mathcal{J}^{V_0W_0}(v) + o(1) = o(1), \\ &\left\langle \left(\hat{\mathcal{J}}_{\varepsilon_j}\right)'(z_j), z_j \right\rangle = \left\langle \left(\hat{\mathcal{J}}_{\varepsilon_j}\right)'(v_j), v_j \right\rangle - \left\langle \left(\mathcal{J}^{V_0W_0}\right)'(v), v \right\rangle + o(1) = o(1). \end{aligned} \tag{4.27}$$

Due to (4.27), we get that $o(1) = \hat{\mathcal{J}}_{\varepsilon_j}(z_j) - \frac{1}{2p}\left\langle \left(\hat{\mathcal{J}}_{\varepsilon_j}\right)'(z_j), z_j \right\rangle \ge C\|z_j\|_1^2$, which means $z_j \to 0$ in $H^1(\mathbb{R}^N)$ as $j \to \infty$. Hence $\|\hat{v}_j - v\|_1 \le \|z_j\|_1 + \|\tilde{\mu}_j - v\|_1$ as $j \to \infty$. □

Lemma 4.5. *$\hat{v}_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$.*

Proof. We have that there are $\delta > 0$, $x_n \in \mathbb{R}^N$, $|x_n| \to \infty$ as $n \to \infty$ such that $|\hat{v}_{j_n}(x_n)| \ge \delta$ by contradiction method. Meanwhile, there exists $C_0 > 0$ which independent of j such that $|\hat{v}_{j_n}(x_n)| \le C_0\left(\int_{B_1(x_n)} \hat{v}_{j_n}^2\right)^{\frac{1}{2}}$. Thus by applying the Minkowski inequality, we have

$$\delta \le \left|\hat{v}_{j_n}(x_n)\right| \le C_0\left(\int_{\mathbb{R}^N}\left|\hat{v}_{j_n}-v\right|^2\right)^{\frac{1}{2}} + C_0\left(\int_{B_1(x_n)}|v|^2\right)^{\frac{1}{2}} \to 0 \quad \text{as } n\to\infty,$$

which is impossible. □

Lemma 4.6. $\{\xi_j y_j'\}_j$ *is bounded on* $\mathbb{R}^N$.

Proof. Assume by contradiction that there is $|\varepsilon_j y_j'| \to \infty$ as $j\to\infty$ along a subsequence. Therefore $V_0 \ge \tau_\infty > \tau$ and $W_{i0} \le k_{i\infty} \le k_{iv}, i=1,2$, which together with Lemma 3.5, imply that $\mathcal{E}^{V_0 \boldsymbol{W}_0} > \mathcal{E}^{\tau \boldsymbol{k}_v}$. However, due to (4.12), (4.20) and Lemma 3.14, we have $\mathcal{E}^{V_0 \boldsymbol{W}_0} = \lim_{j\to\infty}\mathcal{E}_{\varepsilon_j} \le \limsup_{j\to\infty}\mathcal{E}_{\varepsilon_j} \le \mathcal{E}^{\tau \boldsymbol{k}_v}$, which is a contradiction.

Hence, without loss of generality we may posit

$$\varepsilon_j y_j' \to x_0 \quad \text{as } j\to\infty. \tag{4.28}$$

By (4.15), we obtain

$$V_0 = V(x_0), \quad W_{i0} = W_i(x_0), \quad i=1,2. \tag{4.29}$$

Noticing (4.17), we claim v is a least energy solution of Equation (1.7). □

Lemma 4.7. $\{\varepsilon y_\varepsilon\}_\varepsilon$ *is bounded, where* $y_\varepsilon \in \mathbb{R}^N$ *is a maximum point of* v_ε.

Proof. Suppose there exists $\varepsilon_j \to 0$ with $|\varepsilon_j y_j| \to \infty$ as $j\to\infty$ where $y_j := y_{\varepsilon_j}$ is a maximum point of $v_j := v_{\varepsilon_j}$. By Lemmas 4.4, 4.5, 4.6, we can obtain that there is $y_j' \in \mathbb{R}^N$ such that $\hat{v}_j = v_j(\cdot + y_j') \to v \ne 0$ in $H^1(\mathbb{R}^N)$ as $j\to\infty$ and $\hat{v}_j(x) \to 0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$, $\{\varepsilon_j y_j'\}_j$ is bounded on $\mathbb{R}^N$. Hence $|\varepsilon_j y_j - \varepsilon_j y_j'| \ge |\varepsilon_j y_j| - |\varepsilon_j y_j'| \to \infty$ as $j\to\infty$, which means that $|y_j - y_j'| \to \infty$ as $j\to\infty$. Therefore $\max_{\mathbb{R}^N} v_j = v_j(y_j) = \hat{v}_j(y_j - y_j') \to 0$ as $j\to\infty$. Due to $\hat{v}_j > 0$, we get $\hat{v}_j \to 0$ as $j\to\infty$ uinformly in $x\in\mathbb{R}^N$, which contradicts with $v\ne 0$.

Lemma 4.8. $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{S}_v) = 0$.

Proof. According to Lemma 4.7, we get there is $\varepsilon_j \to 0$ with $\varepsilon_j y_j \to y_0$ as $j\to\infty$, where $y_j := y_{\varepsilon_j}$ is the maximum point of $v_j := v_{\varepsilon_j}$. We just require to attest $y_0 \in \mathscr{S}_v$. By Lemmas 4.4, 4.6, there exists $y_j' \in \mathbb{R}^N$ satisfying $\hat{v}_j(x) = v_j(x + y_j')$ and (4.28). Due to Lemma 4.5, we can suppose $\hat{v}_j(x_j') = \max_{\mathbb{R}^N}\hat{v}_j$ and $\{x_j'\}_j$ is bounded on $\mathbb{R}^N$. Hence $y_j = x_j' + y_j'$ and $\varepsilon_j y_j - \varepsilon_j y_j' = \varepsilon_j x_j' \to 0$ as $j\to\infty$. And combining with (4.28), (4.29), mean that

$$y_0 = x_0, V(y_0) = V_0, W_i(y_0) = W_{i0}, i=1,2. \tag{4.30}$$

Assume by contradiction that $y_0 \notin \mathscr{S}_v$, then we have $V(y_0) = \tau$, $W_1(y_0) < k_{1v}$, $W_2(y_0) = k_{2v}$ or $V(y_0) = \tau$, $W_1(y_0) = k_{1v}$, $W_2(y_0) < k_{2v}$ or $V(y_0) > \tau$, $W_i(y_0) \le k_{iv}$, $i=1,2$. Due to Lemma 3.5, $\mathcal{E}^{V(y_0)\boldsymbol{W}(y_0)} > \mathcal{E}^{\tau \boldsymbol{k}_v}$. Combining (4.12), (4.20), (4.30), and Lemma 3.14, we have $\lim_{j\to\infty}\mathcal{E}_{\varepsilon_j} = \lim_{j\to\infty}\hat{\mathcal{E}}_{\varepsilon_j} = \mathcal{E}^{V_0 \boldsymbol{W}_0} = \mathcal{E}^{V(y_0)\boldsymbol{W}(y_0)} > \mathcal{E}^{\tau \boldsymbol{k}_v} \ge \limsup_{j\to\infty}\mathcal{E}_{\varepsilon_j}$, which is a contradiction.

Particularly, if $\mathscr{V} \cap (\mathscr{W}_1 \cap \mathscr{W}_2) \ne \varnothing$, then $x_0 \in \mathscr{S}_v = \mathscr{V} \cap (\mathscr{W}_1 \cap \mathscr{W}_2)$, we can get $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{V} \cap (\mathscr{W}_1 \cap \mathscr{W}_2)) = 0$ and $V(x_0) = \tau$, $W_i(x_0) = k_i$, $i=1,2$, which combine with Equation (1.7) mean that v is a least energy solution of Equation (1.8). □

Lemma 4.9. *For* $p,q\in\left(2,\frac{N+\theta}{N-2}\right)$, *there is* $C>0$ *and* $\hat{R}>0$ *such that for all small* $\varepsilon>0$, $v_\varepsilon(x)\le C|x|^{\frac{1-N}{2}}\exp\left(\frac{-\sqrt{\tau}}{2}|x|\right)$ *for all* $|x|\ge\hat{R}$.

Proof. We check its correctness for any sequence. By Lemma 4.5, we obtain

$$\lim_{|x|\to\infty}\hat{W}_{1\varepsilon_j}(x)\left(I_\theta*\left(\hat{W}_{1\varepsilon_j}\hat{v}_j^p\right)\right)(x)\left(\hat{v}_j(x)\right)^{p-2}$$
$$+\hat{W}_{2\varepsilon_j}(x)\left(I_\theta*\left(\hat{W}_{2\varepsilon_j}\hat{v}_j^q\right)\right)(x)\left(\hat{v}_j(x)\right)^{q-2}=0$$

uniformly in $j\in\mathbb{N}$, which means that there exists $\hat{R}>0$ such that for any $|x|\ge\hat{R}$ and $j\in\mathbb{N}$,

$$\hat{W}_{1\varepsilon_j}(x)\left(I_\theta*\left(\hat{W}_{1\varepsilon_j}\hat{v}_j^p\right)\right)(x)\left(\hat{v}_j(x)\right)^{p-2}+\hat{W}_{2\varepsilon_j}(x)\left(I_\theta*\left(\hat{W}_{2\varepsilon_j}\hat{v}_j^q\right)\right)(x)\left(\hat{v}_j(x)\right)^{q-2}\le\frac{3}{4}\tau. \tag{4.31}$$

Thus, by (4.10) and (4.31), we have $-\Delta\hat{v}_j+\frac{\tau}{4}\hat{v}_j\le 0$ for any $|x|\ge\hat{R}$ and $j\in\mathbb{N}$.

Similar to the proof of Theorem 1.3 in [12] we can know that for any $|x|\ge\hat{R}$ and $j\in\mathbb{N}$, $\hat{v}_j(x)\le C|x|^{\frac{1-N}{2}}\exp\left(\frac{-\sqrt{\tau}}{2}|x|\right)$. □

Set $x_\varepsilon=\varepsilon y_\varepsilon$. Then $w_\varepsilon(x_\varepsilon)=v_\varepsilon(y_\varepsilon)$. Due to Lemma 4.7, x_ε is a maximum point of w_ε and $\{x_\varepsilon\}_\varepsilon$ is bounded on $\mathbb{R}^N$. According to Lemma 4.8, $\lim_{\varepsilon\to 0}\mathrm{dist}(x_\varepsilon,\mathscr{S}_v)=0$. On the basis of Lemmas 4.4, 4.5, $\hat{v}_\varepsilon(x)=v_\varepsilon(x+y'_\varepsilon)=w_\varepsilon(\varepsilon x+x_\varepsilon-\varepsilon x'_\varepsilon)$, where $x'_\varepsilon=y_\varepsilon-y'_\varepsilon$ is a maximum point of $\hat{v}_\varepsilon$ with $\varepsilon x'_\varepsilon\to 0$ as $\varepsilon\to 0$. Finally, we obtain that, $w_\varepsilon(x)\le C\varepsilon^{\frac{N-1}{2}}|x-x_\varepsilon|^{\frac{1-N}{2}}\exp\left(\frac{-\sqrt{\tau}}{4\varepsilon}|x-x_\varepsilon|\right)$, for all $|x|\ge R$, by Lemma 4.9, where $R:=\hat{R}+\sup_\varepsilon|x_\varepsilon|$.

The proof of Theorem 1.3 is completed.

By making reasonable assumptions about potentials, we use pseudo-index theory to prove the multiplicity of semiclassical solutions to Equation (1.1). The existence of groundstate solutions are proved using Nehari method. In addition, we also demonstrate the concentration and convergence of the positive groundstate solution.

Availability of Data and Material

All of the data and material is owned by the authors.

Competing Interests

We declare that there are no competing interests that might be perceived to influence the results reported in this paper.

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