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Indefinite Dot Quotients in 3-Dimensional Space

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Abstract

Up to now, there is a long time problem that dot product has no corresponding division. In order to solve this problem, in this paper, indefinite dot quotients are introduced as extensive inverse operations of dot products, which solve the problem in 3-dimensional space that the quotient of a number and a vector on dot product does not exist from another angle. Some basic properties, and some expected operation properties, and two forms of geometric expressions and six coordinate formulas of indefinite dot quotients are presented.

Subject Areas

Vector Analysis, Analytic Geometry, Mechanics

Keywords

Indefinite Dot Quotient, Indefinite Dot Division, Dot Product, Dot Quotient, Dot Division, Vector Division, Vector Quotient, Cross Product

1. Introduction

The dot product (also scalar product) is one of the most important multiplications of vectors. It is not only widely used in analytic geometry, vector analysis, linear algebra, mechanics [1] [2] [3]. but also widely used in other fields such as engineering, computer graphs [4] [5] [6]. But there is something imperfect, since dot product does not have corresponding division. Almost everyone spends some time to consider this problem when starting to learn dot product. Then obtains a sad result: the division on dot product does not exist. As a result, there are no papers which successfully present the quotient of a number and a vector as the inverse operation of a dot product.

Our purpose of this paper is to set up a theory to solve the problem that dot product has no corresponding division in 3-dimensional space. Dot product is quite different to cross product. The former exits in each vector space, but the latter just exists in some special vector space [7] [8]. Fortunately, they all exist in three dimensional space. As most of people think the divisions on cross products do not exist, in 2022, Mr. Wang and Ms. Chen successfully built the theory of indefinite cross divisions in 3-dimensional space to have solved the problem that cross product has no corresponding division, by adding angle as parameter [9]. Similar to cross products, we are sure that we can solve the problem that dot product has no corresponding division by adding parameters. We naturally think again that, when computing dot product, the something important we unconsciously ignore includes angles, since there are many books in which the definition of dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is directly defined by their coordinates such as

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2+\cdots+a_nb_n \tag{1}$$

where $\boldsymbol{a}=\{a_1,a_2,\cdots,a_n\}$ and $\boldsymbol{b}=\{b_1,b_2,\cdots,b_n\}$ [4] [10]. Though the definition is simple and useful when computing dot products, it somehow hinders us to find the truth of inverse operation of dot product. If we want to inversely find the exact vector $\boldsymbol{a}$ from scalar c and vector $\boldsymbol{b}$ such that $\boldsymbol{a}\cdot\boldsymbol{b}=c$, we will face the fact that there are many many a's such that $\boldsymbol{a}\cdot\boldsymbol{b}=c$, which may be the main reason that makes us obtain that dot product does not have corresponding division. Fortunately some books stress the angle by presenting the following definition:

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors, and $\theta(0\le\theta\le\pi)$ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The dot product (also called scalar product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a}\cdot\boldsymbol{b}$, is a scalar, defined as

$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta \tag{2}$$

where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ indicate the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

The above definition can be easily found in the internet and in the most books related to vector analysis [1] [2] [3]. It is seen that if the angle is 0 or π then we can inversely get the exact $\boldsymbol{a}$ by $\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}$ such that $\boldsymbol{a}\cdot\boldsymbol{b}=c$ where $\boldsymbol{b}\ne\boldsymbol{0}$. But for other angles, we can not. We always think the dot product should have corresponding division like cross product. Thus, a problem naturally arises: Is there another hidden factor that we unconsciously ignore? We notice the fact: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are known, not only the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is specified, but also the cross product $\boldsymbol{a}\times\boldsymbol{b}$ is specified in three dimensional space. We find that, the direction of $\boldsymbol{a}\times\boldsymbol{b}$ is the second thing which is unconsciously ignored. However, when we inversely want to obtain $\boldsymbol{a}$ from c and $\boldsymbol{b}$ such that $\boldsymbol{a}\cdot\boldsymbol{b}=c$, we do not know the angle and the direction. If we grasp them and put them as parameters, we then establish the theory of indefinite dot quotients in three dimensional space. In fact, it is enough for us to inversely obtain the exact $\boldsymbol{a}$ from a constant c and a nonzero vector $\boldsymbol{b}$ such that $\boldsymbol{a}\cdot\boldsymbol{b}=c$ when we know an angle $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ and a direction $\boldsymbol{n}$.

We claim that there exist inverse operations on dot products if enough conditions are known. If we put unknown enough conditions as parameters we obtain indefinite dot quotients which can solve the problem that dot product has no corresponding division in three dimensional space.

In order to realize our purpose, this paper is arranged as 4 main sections. In Section 2, the definitions of indefinite dot quotients are introduced, and some basic properties are presented. In Section 3, some useful operations of indefinite dot quotients are discussed. In Section 4, the structures of indefinite dot quotients are studied in different angles. In Section 5, some useful coordinate formulas of indefinite dot quotients are simply obtained by their structures. And two simple examples are presented not only to support coordinate formulas but also to show that if we know sufficient information, we really can back to find the exact $\boldsymbol{a}$ such that $\boldsymbol{a} \cdot \boldsymbol{b} = c$ from c and $\boldsymbol{b}$.

2. Indefinite Dot Quotients

In this section, we build the foundation of this paper: indefinite dot quotients and their basic properties. Before introduction of them, we stress that: $c\cos\theta > 0$ on computation of dot product when $c \neq 0$ since $c = 0$ if and only if $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$ or $\theta = \frac{\pi}{2}$.

Definition 2.1. Let c be a scalar and $\boldsymbol{b}$ be a nonzero vector, and let $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ be an angle parameter such that $c\cos\theta \geq 0$, and $\boldsymbol{n}(\neq \boldsymbol{0})$ be a normal vector parameter such that $\boldsymbol{n} \perp \boldsymbol{b}$. The vector, denoted by $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ ($\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$), is called left (right) indefinite dot quotient (or division) of scalar c and vector $\boldsymbol{b}$, if its magnitude is defined as

$$\left|\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\right| = \left|\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}\right| = \frac{c}{|\boldsymbol{b}|\cos\theta}$$

and its direction, when $c \neq 0$, is determined by the following 3 steps:

Step 1. In Figure 1, let O be any point in 3D space, and set $\boldsymbol{ON} = \boldsymbol{n}$, $\boldsymbol{OB} = \boldsymbol{b}$.

Step 2. Spread the left (right) hand, satisfying that five fingers are on the plane BON and the thumb is perpendicular to other 4 fingers; and pointing the thumb direction along $\boldsymbol{ON}$ and other four fingers along $\boldsymbol{OB}$.

Step 3. The left (right) open hand rotates around vector $\boldsymbol{ON}$ through the angle θ. Then, the direction that the four fingers point out is just the direction of $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ ($\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$).

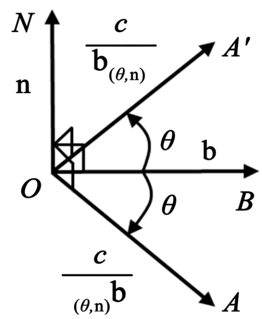

Figure 1. Dot quotients.

The left and right indefinite dot quotients are collectively called the indefinite dot quotients (or divisions), simply dot quotients (or divisions).

It is seen that $\frac{0}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{0}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \boldsymbol{0}$ when $c = 0$ and $\theta \neq \frac{\pi}{2}$. Notice that the above definition does not include the case of $\theta = \frac{\pi}{2}$, because it makes the denominator of $\frac{c}{|\boldsymbol{b}|\cos\theta}$ to be 0 so that the problem becomes complicated. We will present the supplementary definition on this special case in Section 4. Thus, in the rest of this paper, if no special statement, as we meet the notation $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ or $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$, we always suppose $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $\boldsymbol{b} \neq \boldsymbol{0}$, $\boldsymbol{n} \neq \boldsymbol{0}$, $\boldsymbol{n} \perp \boldsymbol{b}$ and $c\cos\theta \geq 0$.

From Definition 2.1, when $\theta = 0, \pi$, we have

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}. \tag{3}$$

Moreover, we have the following simple attributes:

(2.1) $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = c = \boldsymbol{b} \cdot \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} \cdot \boldsymbol{b}$.

(2.2) There is a real number $\mu > 0$ such that $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \times \boldsymbol{b} = \mu\boldsymbol{n}$, $\boldsymbol{b} \times \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \mu\boldsymbol{n}$.

(2.1.1) $\boldsymbol{n} \perp \frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\boldsymbol{n} \perp \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$.

(2.1.2) $\angle\left(\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}, \boldsymbol{b}\right) = \angle\left(\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}, \boldsymbol{b}\right) = \theta$.

(2.1.3) The ordered three vectors $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\boldsymbol{b}$, $\boldsymbol{n}$ obey the right hand rule, and $\boldsymbol{b}$, $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$, $\boldsymbol{n}$ also obey the right hand rule.

(2.3) If two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ have the same direction, then

$$\frac{c}{{}_{(\theta,\boldsymbol{n}_1)}\boldsymbol{b}}=\frac{c}{{}_{(\theta,\boldsymbol{n}_2)}\boldsymbol{b}},\quad \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n}_1)}}=\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n}_2)}}.$$

(2.4) $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{\boldsymbol{b}_{(\theta,-\boldsymbol{n})}}$, $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c}{{}_{(\theta,-\boldsymbol{n})}\boldsymbol{b}}$ (Conversion Formulas).

(2.5) $-\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{{}_{(\theta,\boldsymbol{n})}(-\boldsymbol{b})}$, $-\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c}{(-\boldsymbol{b})_{(\theta,\boldsymbol{n})}}$ (Inverse Formulas).

(2.6) $-\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{-c}{\boldsymbol{b}_{(\pi-\theta,\boldsymbol{n})}}$, $-\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{-c}{{}_{(\pi-\theta,\boldsymbol{n})}\boldsymbol{b}}$ (Angle Formulas).

The attributes of (2.4), (2.5) and (2.6) can be easily understood by **Figure 2**. There is a point, we should note, not to use $\frac{-c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$, since $(-c)\cos\theta\le 0$ in (2.6).

Note that, the definition of left indefinite dot quotient ensures that for any $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ is a vector such that $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=c$. Conversely, we have

Theorem 2.1. *Let* $\boldsymbol{b}\ne\boldsymbol{0}$ *be a vector and* $c\ne 0$ *be a scalar. If there is a vector* $\boldsymbol{a}$ *such that* $\boldsymbol{a}\cdot\boldsymbol{b}=c$ *and* $\boldsymbol{a}\times\boldsymbol{b}\ne\boldsymbol{0}$, *then there is the unique angle* $\theta\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$ *such that* $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\boldsymbol{a}$ *where* $\boldsymbol{n}$ *is any nonzero vector which has the same direction of* $\boldsymbol{a}\times\boldsymbol{b}$.

Proof. Since $c\ne 0$, the angle $\angle(\boldsymbol{a},\boldsymbol{b})$ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$. And since $\boldsymbol{a}\times\boldsymbol{b}\ne\boldsymbol{0}$, $\angle(\boldsymbol{a},\boldsymbol{b})$ is neither 0 nor π. Let $\theta=\angle(\boldsymbol{a},\boldsymbol{b})\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$. According to the definition of left indefinite dot quotient, we have the following three items:

1) $\boldsymbol{n}\perp\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\boldsymbol{n}\perp\boldsymbol{a}$, $\boldsymbol{n}\perp\boldsymbol{b}$.

2) The three ordered vectors $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\boldsymbol{b}$, $\boldsymbol{n}$ obey the right hand rule; and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{n}$ also obey the right hand rule.

3) $\angle\left(\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}},\boldsymbol{b}\right)=\theta$ and $\angle(\boldsymbol{a},\boldsymbol{b})=\theta$.

The above three items imply that $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{a}$ have the same direction, and θ is unique.

Since $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta=c$, they have the same magnitude $\frac{c}{|\boldsymbol{b}|\cos\theta}$. Thus $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\boldsymbol{a}$.

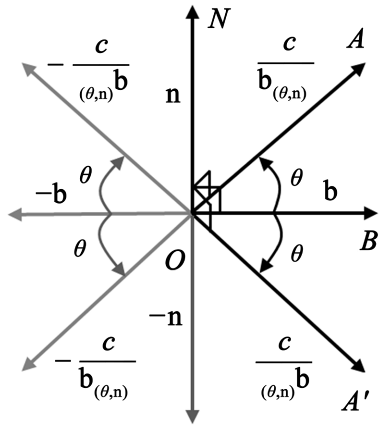

Figure 2. Conversion of left and right dot quotients.

Theorem 2.2. *Let* $\boldsymbol{b} \neq \boldsymbol{0}$ *be a vector and* $c \neq 0$ *be a scalar. If there is a vector* $\boldsymbol{a}$ *such that* $\boldsymbol{a} \cdot \boldsymbol{b} = c$ *and* $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, *then there is the unique angle* $\theta \in \{0, \pi\}$ *such that* $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \boldsymbol{a}$ *where* $\boldsymbol{n}$ *is any nonzero vector perpendicular to* $\boldsymbol{b}$.

Proof. Let $\theta = \angle(\boldsymbol{a}, \boldsymbol{b})$. Since $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, θ is one of 0 and π. If $c > 0$, then $\theta = 0$, that suggests $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction. If $c < 0$, then $\theta = \pi$, that suggests $\boldsymbol{a}$ and $\boldsymbol{b}$ have the opposite direction. For any nonzero vector $\boldsymbol{n}$, according to the definition of left indefinite dot quotient, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{b}$ are colinear; furthermore, when $\theta = 0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{b}$ have the same direction, and when $\theta = \pi$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{b}$ have the opposite direction. Thus, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{a}$ have the same direction, and θ is unique. Since $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = c$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{a}$ have the same magnitude $\frac{c}{|\boldsymbol{b}|\cos\theta}$. Thus $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \boldsymbol{a}$. □

Corollary 2.1. *Let* $\boldsymbol{b} \neq \boldsymbol{0}$ *be a vector and* $c \neq 0$ *be a scalar. If there is a vector* $\boldsymbol{a}$ *such that* $\boldsymbol{a} \cdot \boldsymbol{b} = c$, *then there are an angle* $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ *and a nonzero vector* $\boldsymbol{n}$ *such that* $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \boldsymbol{a}$.

Proof. It can be seen from Theorem 2.1 and Theorem 2.2. □

Symmetrically, for right indefinite dot quotients, we have the following three results:

Theorem 2.3. *Let* $\boldsymbol{a} \neq \boldsymbol{0}$ *be a vector and* $c \neq 0$ *be a scalar. If there is a vector* $\boldsymbol{b}$ *such that* $\boldsymbol{a} \cdot \boldsymbol{b} = c$ *and* $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{0}$, *then there is the unique angle* $\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ *such that* $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}} = \boldsymbol{b}$ *where* $\boldsymbol{n}$ *is any nonzero vector* *which has the same direction of* $\boldsymbol{a}\times\boldsymbol{b}$.

Theorem 2.4. *Let* $\boldsymbol{a}\neq\boldsymbol{0}$ *be a vector and* $c\neq 0$ *be a scalar. If there is a vector* $\boldsymbol{b}$ *such that* $\boldsymbol{a}\cdot\boldsymbol{b}=c$ *and* $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$, *then there is the unique angle* $\theta\in\{0,\pi\}$ *such that* $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}}=\boldsymbol{b}$ *where* $\boldsymbol{n}$ *is any nonzero vector perpendicular to* $\boldsymbol{a}$.

Corollary 2.2. *Let* $\boldsymbol{a}\neq\boldsymbol{0}$ *be a vector and* $c\neq 0$ *be a scalar. If there is a vector* $\boldsymbol{b}$ *such that* $\boldsymbol{a}\cdot\boldsymbol{b}=c$, *then there are an angle* $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ *and a nonzero vector* $\boldsymbol{n}$ *such that* $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}}=\boldsymbol{b}$.

3. Operations

Since a dot quotient involves four factors: a scalar and two vectors and an angle parameter, the rules of multiplications between scalars and dot quotients become very complicated. Thus, in this section, it is fully necessary to further study them. For the symmetry of left and right dot quotients, we only prove the properties with respect to left dot quotients. We always suppose that all factors involving dot quotients make the expressions valid.

For a real number $\lambda(\neq 0)$, in algebra, we have $\lambda\times\frac{a}{b}=\frac{\lambda a}{b}=\frac{a}{\frac{b}{\lambda}}$, $\frac{a}{b}=\frac{\lambda a}{\lambda b}$, etc. Can these properties be extended to dot quotients? In this section, if no special statement, we always assume $c\neq 0$. We have the following properties:

Theorem 3.1. *For any* $\lambda\neq 0$, $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}$, $\lambda\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_{(\theta,\boldsymbol{n})}}$.

Proof. When $\lambda>0$, since $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}$ have the same magnitude and direction, $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}$. When $\lambda<0$, based on the previous result and attribute (2.5),

$$\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=-|\lambda|\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=-\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(\frac{1}{|\lambda|}\boldsymbol{b}\right)}=\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(-\frac{1}{|\lambda|}\boldsymbol{b}\right)}=\frac{c}{{}_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}.\qquad\square$$

Theorem 3.2. 1) For any $\lambda>0$, $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\lambda\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{\lambda c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$.

2) For any $\lambda<0$, $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{-\lambda c}{{}_{(\theta,\boldsymbol{n})}(-\boldsymbol{b})}=\frac{\lambda c}{\boldsymbol{b}_{(\pi-\theta,\boldsymbol{n})}}$, $\lambda\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{-\lambda c}{(-\boldsymbol{b})_{(\theta,\boldsymbol{n})}}=\frac{\lambda c}{{}_{(\pi-\theta,\boldsymbol{n})}\boldsymbol{b}}$.

Proof. (The proof of left equation of (1))

For any $\lambda>0$, the magnitudes of $\lambda\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\frac{\lambda c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ are equal, because

$$\left|\lambda\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\right|=|\lambda|\left|\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\right|=\lambda\frac{c}{|\boldsymbol{b}|\cos\theta}=\frac{\lambda c}{|\boldsymbol{b}|\cos\theta}=\left|\frac{\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\right|.$$

Moreover, when $c\neq 0$, since $\lambda c\cos\theta>0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\frac{\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ have the same direction by the definition of left indefinite dot quotient. Thus $\lambda\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$.

(The proof of left equation of (2)) For any $\lambda<0$, we have

$$\lambda\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=-(-\lambda)\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=-\frac{-\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}.$$

By the attribute (2.5), $-\frac{-\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{-\lambda c}{_{(\theta,\boldsymbol{n})}(-\boldsymbol{b})}$;

by the attribute (2.6), $-\frac{-\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{\boldsymbol{b}_{(\pi-\theta,\boldsymbol{n})}}$. □

Corollary 3.1.

1) For any $\lambda>0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}=\frac{\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$, $\frac{c}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_{(\theta,\boldsymbol{n})}}=\frac{\lambda c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$.

2) For any $\lambda<0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\left(\frac{1}{\lambda}\boldsymbol{b}\right)}=\frac{-\lambda c}{_{(\theta,\boldsymbol{n})}(-\boldsymbol{b})}$, $\frac{c}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_{(\theta,\boldsymbol{n})}}=\frac{-\lambda c}{(-\boldsymbol{b})_{(\theta,\boldsymbol{n})}}$.

Proof. Obvious from Theorem 3.1 and Theorem 3.2. □

Corollary 3.2.

1) For any $\lambda>0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{_{(\theta,\boldsymbol{n})}(\lambda\boldsymbol{b})}$, $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{\lambda c}{(\lambda\boldsymbol{b})_{(\theta,\boldsymbol{n})}}$.

2) For any $\lambda<0$, $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{(\lambda\boldsymbol{b})_{(\pi-\theta,\boldsymbol{n})}}$, $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{\lambda c}{_{(\pi-\theta,\boldsymbol{n})}(\lambda\boldsymbol{b})}$.

Proof. 1) If $\lambda>0$, then $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\left(\frac{1}{\lambda}\times\lambda\right)\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{1}{\lambda}\frac{\lambda c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{\lambda c}{_{(\theta,\boldsymbol{n})}(\lambda\boldsymbol{b})}$.

2) If $\lambda<0$, then $\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\left(\frac{1}{\lambda}\times\lambda\right)\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{1}{\lambda}\frac{\lambda c}{\boldsymbol{b}_{(\pi-\theta,\boldsymbol{n})}}=\frac{\lambda c}{(\lambda\boldsymbol{b})_{(\pi-\theta,\boldsymbol{n})}}$. □

Theorem 3.3. *If* $c_i\cos\theta>0$ *for* $i=1,2$, *then*

1) $\frac{c_1+c_2}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}+\frac{c_2}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$; 2) $\frac{c_1+c_2}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c_1}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}+\frac{c_2}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$.

Proof. (1) Since $c_i\cos\theta>0$ for $i=1,2$, we have $c_1c_2>0$. Hence there is a real number $\lambda>0$, satisfying $c_2=\lambda c_1$. Thus,

$$\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}+\frac{c_2}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}+\frac{\lambda c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}+\lambda\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$$

$$=(1+\lambda)\frac{c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{(1+\lambda)c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c_1+\lambda c_1}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c_1+c_2}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}.$$

Similar to (2). □

Note that, if $c_1 c_2 < 0$, then the above results are not valid, since $c_1 \cos\theta > 0$ and $c_2 \cos\theta > 0$ can not hold at the same time.

4. Structures of Dot Quotients

In this section, we discuss the structures of dot quotients in two kinds of normal vector parameters: one is fixed, and another is not. We always assume that, $\boldsymbol{b} \neq \boldsymbol{0}$ is given, and c is a scalar, and $\boldsymbol{n} \neq \boldsymbol{0}$ always indicates a normal vector parameter such that $\boldsymbol{n} \perp \boldsymbol{b}$. Then we have the following geometric properties:

Theorem 4.1. *Let $c > 0$ and $\boldsymbol{n}$ be fixed. In Figure 3, let O be a point in 3-dimensional Space, and make $\boldsymbol{OB} = \boldsymbol{b}$ and $\boldsymbol{ON} = \boldsymbol{n}$. Take $\boldsymbol{OQ}_1 = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} = \frac{c}{{}_{(0,\boldsymbol{n})}\boldsymbol{b}}$ and $\boldsymbol{OQ}_2 = \frac{-c\boldsymbol{b}}{|\boldsymbol{b}|^2} = \frac{-c}{{}_{(\pi,\boldsymbol{n})}\boldsymbol{b}}$. For $i = 1,2$, through Q_i, draw a straight line l_i parallel to vector $\boldsymbol{b} \times \boldsymbol{n}$. Then*

1) Point P_1 is on the straight line l_1 if and only if there exists a $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ such that $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}}$ or $\boldsymbol{OP}_1 = \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}$;

2) Point P_2 is on the straight line l_2 if and only if there exists a $\theta_2 \in \left(\frac{\pi}{2}, \pi\right]$ such that $\boldsymbol{OP}_2 = \frac{-c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}}$ or $\boldsymbol{OP}_2 = \frac{-c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}$.

Proof. (Proof of (1)) In fact, when P_1 is on the right of line l_1 in Figure 3,

$$\boldsymbol{OP}_1 \cdot \boldsymbol{b} = \left(\boldsymbol{OQ}_1 + \boldsymbol{Q}_1\boldsymbol{P}_1\right) \cdot \boldsymbol{b} = \left[\frac{c}{{}_{(0,\boldsymbol{n})}\boldsymbol{b}} + \lambda\left(\boldsymbol{b} \times \boldsymbol{n}\right)\right] \cdot \boldsymbol{b} = \frac{c}{{}_{(0,\boldsymbol{n})}\boldsymbol{b}} \cdot \boldsymbol{b} + \lambda\left(\boldsymbol{b} \times \boldsymbol{n}\right) \cdot \boldsymbol{b} = c + 0 = c$$

, where $\lambda \geq 0$. According to Corollary 2.1, there exists a $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ such that $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}}$. Similarly, when P_1 is on the left of line l_1 (that is, $P_1 = A_1$), there exists a $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ such that $\boldsymbol{OP}_1 = \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}$.

Conversely, if there exists a $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ such that $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}}$ or $\frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}$. This tells that $\boldsymbol{n} \perp \boldsymbol{OP}_1$. And since $\boldsymbol{n} \perp \boldsymbol{OQ}_1$, we have $\boldsymbol{n} \perp \boldsymbol{Q}_1\boldsymbol{P}_1$. Moreover $\boldsymbol{Q}_1\boldsymbol{P}_1 \cdot \boldsymbol{b} = \left(\boldsymbol{OP}_1 - \boldsymbol{OQ}_1\right) \cdot \boldsymbol{b} = \boldsymbol{OP}_1 \cdot \boldsymbol{b} - \boldsymbol{OQ}_1 \cdot \boldsymbol{b} = c - c = 0$. This implies that $\boldsymbol{b} \perp \boldsymbol{Q}_1\boldsymbol{P}_1$. Thus, $\boldsymbol{Q}_1\boldsymbol{P}_1$ is parallel to $\boldsymbol{b} \times \boldsymbol{n}$. Therefore P_1 is on the line l_1.

Similarly, we can prove (2). □

Corollary 4.1. *Let $c > 0$ and $\boldsymbol{n}$ be fixed. In Figure 3, the point sets*

$$\left\{P \mid \boldsymbol{OP} = \frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \text{ or } \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}, \theta \in \left[0, \frac{\pi}{2}\right)\right\}$$

and $$\left\{P \mid \boldsymbol{OP} = \frac{-c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \text{ or } \frac{-c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}, \theta \in \left(\frac{\pi}{2}, \pi\right]\right\}$$

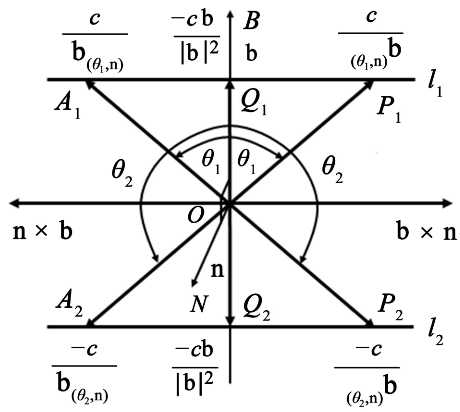

Figure 3. Structure for a fixed $\boldsymbol{n}$.

form two parallel lines, whose distance is $\frac{2c}{|\boldsymbol{b}|}$.

Proof. Obvious. □

Corollary 4.2. *Let* $c>0$ *and* $\boldsymbol{n}$ *be fixed. In* Figure 3, *the point sets* $\left\{P \mid \boldsymbol{OP}=\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}, \theta\in\left[0,\frac{\pi}{2}\right)\right\}$ *and* $\left\{P \mid \boldsymbol{OP}=\frac{-c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}, \theta\in\left(\frac{\pi}{2},\pi\right]\right\}$ *form two parallel rays on the right parts of* l_1 *and* l_2, *whose distance is* $\frac{2c}{|\boldsymbol{b}|}$. *Symmetrically, the point sets* $\left\{P \mid \boldsymbol{OP}=\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}, \theta\in\left[0,\frac{\pi}{2}\right)\right\}$ *and* $\left\{P \mid \boldsymbol{OP}=\frac{-c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}, \theta\in\left(\frac{\pi}{2},\pi\right]\right\}$ *form two parallel rays on the left parts of* l_1 *and* l_2, *whose distance is also* $\frac{2c}{|\boldsymbol{b}|}$.

Proof. Obvious. □

Corollary 4.3. *Let* $c\neq 0$ *and* $\boldsymbol{n}$ *be fixed. Given* $\theta_0\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, *for any* $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ *such that* $c\cos\theta\geq 0$ *and* $c\cos\theta_0\geq 0$, *there is a real number* $\lambda\geq 0$ *such that* $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{{}_{(\theta_0,\boldsymbol{n})}\boldsymbol{b}}+\lambda\boldsymbol{b}\times\boldsymbol{n}$ *and* $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c}{\boldsymbol{b}_{(\theta_0,\boldsymbol{n})}}+\lambda\boldsymbol{n}\times\boldsymbol{b}$.

Proof. They can be found from Figure 3. □

It is readily seen that, if $\theta_0\in\left[0,\frac{\pi}{2}\right)$, then $\theta\in\left[0,\frac{\pi}{2}\right)$ is required; and if $\theta_0\in\left(\frac{\pi}{2},\pi\right]$, then $\theta\in\left(\frac{\pi}{2},\pi\right]$.

Theorem 4.2. *Let* $c\neq 0$ *and* $\boldsymbol{n}$ *be fixed. Then, for any* $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ *such that* $c\cos\theta\geq 0$, *there is a real number* $\lambda\geq 0$ *satisfying*

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}=\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}+\lambda\boldsymbol{b}\times\boldsymbol{n} \text{ and } \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}=\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}+\lambda\boldsymbol{n}\times\boldsymbol{b} \tag{4}$$

where $\lambda = \frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}$.

Proof. When $\theta \in \left[0, \frac{\pi}{2}\right)$, we have $c > 0$. According to **Figure 3**, we can take $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$. Then there is a real number $\lambda \geq 0$ such that $\boldsymbol{Q}_1\boldsymbol{P}_1 = \lambda \boldsymbol{b} \times \boldsymbol{n}$. Hence, on the one hand, $|\boldsymbol{Q}_1\boldsymbol{P}_1| = |\lambda||\boldsymbol{b}||\boldsymbol{n}|\sin\frac{\pi}{2} = \lambda|\boldsymbol{b}||\boldsymbol{n}|$. On the other hand, $|\boldsymbol{Q}_1\boldsymbol{P}_1| = |\boldsymbol{OP}_1|\sin\theta = \left|\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\right|\sin\theta = \frac{c\sin\theta}{|\boldsymbol{b}|\cos\theta}$. Thus, $\frac{c\sin\theta}{|\boldsymbol{b}|\cos\theta} = \lambda|\boldsymbol{b}||\boldsymbol{n}| \Rightarrow$ $\lambda = \frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}$. Since $\boldsymbol{OP}_1 = \boldsymbol{OQ}_1 + \boldsymbol{Q}_1\boldsymbol{P}_1$, by Equation (3), we have

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{b} \times \boldsymbol{n} \quad \text{with} \quad \lambda = \frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}.$$

When $\theta \in \left(\frac{\pi}{2}, \pi\right]$, we have $c < 0$. In the previous proof, substituting Q_2 for Q_1 and P_2 for P_1, we also have $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{b} \times \boldsymbol{n}$ and $\lambda = \frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}$.

Symmetrically, we have $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{n} \times \boldsymbol{b}$. □

Corollary 4.4. *Let* $c > 0$ *and* $\boldsymbol{n}$ *be fixed. For any* $\theta_1 < \theta_2$ *with* $\theta_1, \theta_2 \in \left[0, \frac{\pi}{2}\right)$, *there is a real number* $\lambda > 0$ *such that*

$$\frac{c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}} - \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}} = \lambda \boldsymbol{b} \times \boldsymbol{n} = \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}} - \frac{c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}.$$

Proof. Obvious. □

Corollary 4.5. *Let* $c < 0$ *and* $\boldsymbol{n}$ *be fixed. For any* $\theta_1 < \theta_2$ *with* $\theta_1, \theta_2 \in \left(\frac{\pi}{2}, \pi\right]$, *there is a real number* $\lambda > 0$ *such that*

$$\frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}} - \frac{c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}} = \lambda \boldsymbol{b} \times \boldsymbol{n} = \frac{c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}} - \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}.$$

Proof. Obvious. □

Because Definition 2.1 does not include the case of $\theta = \frac{\pi}{2}$, it leads to something imperfect. How to define dot quotients on this special case?

In **Figure 3**, let $c > 0$ and $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ and $\theta_2 = \pi - \theta_1$, and let $\boldsymbol{Q}_1\boldsymbol{P}_1 = \lambda \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{Q}_2\boldsymbol{P}_2$ and $\boldsymbol{Q}_1\boldsymbol{A}_1 = \lambda \boldsymbol{n} \times \boldsymbol{b} = \boldsymbol{Q}_2\boldsymbol{A}_2$, $\lambda \geq 0$. We can find that, if we do not change the directions and magnitudes of four vectors $\boldsymbol{Q}_1\boldsymbol{P}_1$, $\boldsymbol{Q}_2\boldsymbol{P}_2$, $\boldsymbol{Q}_1\boldsymbol{A}_1$ and $\boldsymbol{Q}_2\boldsymbol{A}_2$, when c goes to 0, not only the vectors $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}}$ and

$OP_2 = \frac{-c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}}$ ($OA_1 = \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}$ and $OA_2 = \frac{-c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}$) are all closing to the vector $\boldsymbol{Q}_1\boldsymbol{P}_1$ ($\boldsymbol{Q}_1\boldsymbol{A}_1$), but also θ_1 and θ_2 are all closing to $\frac{\pi}{2}$. In other words, we have the following two facts:

1) $\lim_{c\to 0}\frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}} = \lim_{c\to 0}\frac{-c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}} = \frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$, and $\lim_{c\to 0}\frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}} = \lim_{c\to 0}\frac{-c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}} = \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$.

2) $\lim_{c\to 0}\frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}} = \boldsymbol{Q}_1\boldsymbol{P}_1 = \lambda\boldsymbol{b}\times\boldsymbol{n}$ and $\lim_{c\to 0}\frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}} = \boldsymbol{Q}_1\boldsymbol{A}_1 = \lambda\boldsymbol{n}\times\boldsymbol{b}$.

Thus, according to the above discussion, we can present a supplementary definition of indefinite dot quotients for the case of $\theta = \frac{\pi}{2}$ to complete our theory.

Definition 4.1. Let λ be an arbitrary nonnegative scalar. $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$ is defined as $\lambda\boldsymbol{b}\times\boldsymbol{n}$,

that is,

$$\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \lambda\boldsymbol{b}\times\boldsymbol{n};$$

and $\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$ is defined as $\lambda\boldsymbol{n}\times\boldsymbol{b}$, that is,

$$\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \lambda\boldsymbol{n}\times\boldsymbol{b};$$

where λ is called a scalar parameter (see Figure 4).

For $\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$ and $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$, Attribute (2.1)-(2.6) hold. In fact, they can be observed by Figure 4. For instance, Attribute (2.6) becomes

$$-\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}},\quad -\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}},$$

that is, $-\lambda\boldsymbol{b}\times\boldsymbol{n} = \lambda\boldsymbol{n}\times\boldsymbol{b}$ and $-\lambda\boldsymbol{n}\times\boldsymbol{b} = \lambda\boldsymbol{b}\times\boldsymbol{n}$. However, the most of results, involving the multiplications of scalars and numerators, in Section 3 do not hold. In other words, the properties except Theorem 3.1 do not hold. Since, at this time, the new numerator as the multiplication of a scalar and an old numerator is 0 which leads to invalid properties. For instance, the result: for $\lambda > 0$, $\lambda\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \frac{\lambda 0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$, does not hold. In fact, let $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = 2\boldsymbol{b}\times\boldsymbol{n}$. Then

$2\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = 4\boldsymbol{b}\times\boldsymbol{n} \neq \frac{2\times 0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = 2\boldsymbol{b}\times\boldsymbol{n}$.

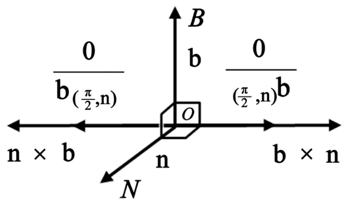

Figure 4. Structure for $\theta = \frac{\pi}{2}$.

If we know more valid information about $\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$ and $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$, then we can imply the exact them. We have

Theorem 4.3. *If there is a real number* $\mu \ge 0$ *such that* $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} \times \boldsymbol{b} = \mu \boldsymbol{n}$ *or* $\boldsymbol{b} \times \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \mu \boldsymbol{n}$, *then*

$$\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \frac{\mu \boldsymbol{b} \times \boldsymbol{n}}{|\boldsymbol{b}|^2} \text{ and } \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \frac{\mu \boldsymbol{n} \times \boldsymbol{b}}{|\boldsymbol{b}|^2}.$$

Proof. (using condition $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} \times \boldsymbol{b} = \mu \boldsymbol{n}$) According to the above definition, there is a real number $\lambda \ge 0$ such that $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} = \lambda \boldsymbol{b} \times \boldsymbol{n}$. We then have,

$$\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} \times \boldsymbol{b} = \mu \boldsymbol{n} \quad \Leftrightarrow \quad (\lambda \boldsymbol{b} \times \boldsymbol{n}) \times \boldsymbol{b} = \mu \boldsymbol{n} \quad \Leftrightarrow \quad \lambda\left[(\boldsymbol{b} \cdot \boldsymbol{b})\boldsymbol{n} - (\boldsymbol{b} \cdot \boldsymbol{n})\boldsymbol{b}\right] = \mu \boldsymbol{n} \quad \Leftrightarrow$$
$$\lambda |\boldsymbol{b}|^2 \boldsymbol{n} = \mu \boldsymbol{n} \quad \Leftrightarrow \quad \lambda = \frac{\mu}{|\boldsymbol{b}|^2}.$$

Similarly, we can get the other equation.

If we use condition $\boldsymbol{b} \times \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \mu \boldsymbol{n}$, we can obtain the same results. In fact, condition $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}} \times \boldsymbol{b} = \mu \boldsymbol{n}$ is equivalent to condition $\boldsymbol{b} \times \frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \mu \boldsymbol{n}$. □

Corollary 4.6. *For any* $\theta \in [0, \pi]$, *there is a real number* $\lambda \ge 0$ *such that*

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{b} \times \boldsymbol{n} \text{ and } \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{n} \times \boldsymbol{b} \tag{5}$$

where $\lambda = \begin{cases} \frac{c \tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}, & \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ \frac{\mu}{|\boldsymbol{b}|^2}, & \theta = \frac{\pi}{2} \end{cases}$ if there is a real number $\mu \ge 0$ such that $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)\boldsymbol{b}}\times\boldsymbol{b}=\mu\boldsymbol{n}$ or $\boldsymbol{b}\times\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}=\mu\boldsymbol{n}$ when $c=0$.

Proof. By Theorem 4.2 and Theorem 4.3. □

Theorem 4.4. *Let* $\lambda_1,\lambda_2\ge 0$ *such that* $\lambda_1+\lambda_2=1$. *If* $\theta_1,\theta_2\in\left[0,\frac{\pi}{2}\right)$ ($\left(\frac{\pi}{2},\pi\right]$), *then there is a* $\theta\in\left[0,\frac{\pi}{2}\right)$ ($\left(\frac{\pi}{2},\pi\right]$) *satisfying*

$$\lambda_1\frac{c}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}+\lambda_2\frac{c}{(\theta_2,\boldsymbol{n})\boldsymbol{b}}=\frac{c}{(\theta,\boldsymbol{n})\boldsymbol{b}} \text{ and } \lambda_1\frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}+\lambda_2\frac{c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}=\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$$

where $c\in\mathbb{R}$ and $c\cos\theta_1\ge 0$ and $c\cos\theta_2\ge 0$ and $c\cos\theta\ge 0$.

Proof. (We only prove the case of $\theta_1,\theta_2\in\left[0,\frac{\pi}{2}\right)$) If one of λ_1 and λ_2 and c is 0, or $\theta_1=\theta_2$, the results hold obviously.

Let us simply assume $\theta_1<\theta_2$. And let $\lambda_1,\lambda_2,c\ne 0$. Since $c\cos\theta_1>0$, $c>0$. In Figure 5, let O be a point in 3-dimensional Space, and make $\boldsymbol{OB}=\boldsymbol{b}$ and $\boldsymbol{ON}=\boldsymbol{n}$. Take $\boldsymbol{OQ}_1=\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}$ and $\boldsymbol{OQ}_2=-\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}$. And take $\boldsymbol{OA}_1=\frac{c}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}$ and $\boldsymbol{OA}_2=\frac{c}{(\theta_2,\boldsymbol{n})\boldsymbol{b}}$. And more, take $\boldsymbol{OP}=\lambda_2\boldsymbol{OA}_2$ and $\boldsymbol{PE}=\lambda_1\boldsymbol{OA}_1=\lambda_1\frac{c}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}$. From Theorem 4.1, the three points Q_1 and A_1 and A_2 are on the same line, which is the exact line l_1 in Figure 3. Since $\boldsymbol{n}\perp\boldsymbol{OA}_1,\perp\boldsymbol{OA}_2$, we have $\boldsymbol{n}\perp\boldsymbol{OE}$, that leads to $\boldsymbol{n}\perp\boldsymbol{Q_1E}$. Since $\boldsymbol{OE}\cdot\boldsymbol{b}=(\boldsymbol{OP}+\boldsymbol{PE})\cdot\boldsymbol{b}=\left(\lambda_2\frac{c}{(\theta_2,\boldsymbol{n})\boldsymbol{b}}+\lambda_1\frac{c}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}\right)\cdot\boldsymbol{b}=c$, $\boldsymbol{Q_1E}\cdot\boldsymbol{b}=(\boldsymbol{OE}-\boldsymbol{OQ}_1)\cdot\boldsymbol{b}=\boldsymbol{OE}\cdot\boldsymbol{b}-\boldsymbol{OQ}_1\cdot\boldsymbol{b}=c-c=0$, which shows $\boldsymbol{b}\perp\boldsymbol{Q_1E}$.

Thus, E is on the line Q_1A_2. From Theorem 4.1, there is a $\theta\in\left[0,\frac{\pi}{2}\right)$ satisfying $\boldsymbol{OE}=\frac{c}{(\theta,\boldsymbol{n})\boldsymbol{b}}$, that is, $\lambda_1\frac{c}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}+\lambda_2\frac{c}{(\theta_2,\boldsymbol{n})\boldsymbol{b}}=\frac{c}{(\theta,\boldsymbol{n})\boldsymbol{b}}$.

By symmetry, we have $\lambda_1\frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}+\lambda_2\frac{c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}=\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$. □

In the above theorem, interval $\left[0,\frac{\pi}{2}\right)$ can not be extended to $\left[0,\frac{\pi}{2}\right]$. In fact, if $\theta_1<\theta_2=\frac{\pi}{2}$, we have $c=0$. Let $\lambda_1=\lambda_2=\frac{1}{2}$, and let $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)\boldsymbol{b}}=\lambda\boldsymbol{b}\times\boldsymbol{n}$ with $\lambda\ne 0$. Then, there is no $\theta\in\left[0,\frac{\pi}{2}\right]$ satisfying $\frac{1}{2}\frac{0}{(\theta_1,\boldsymbol{n})\boldsymbol{b}}+\frac{1}{2}\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)\boldsymbol{b}}=\frac{0}{(\theta,\boldsymbol{n})\boldsymbol{b}}$, since $\frac{\lambda}{2}\boldsymbol{b}\times\boldsymbol{n}\ne\boldsymbol{0}$ and $\frac{\lambda}{2}\boldsymbol{b}\times\boldsymbol{n}\ne\lambda\boldsymbol{b}\times\boldsymbol{n}$ for any $\lambda\ne 0$. Interval $\left(\frac{\pi}{2},\pi\right]$, of course, can not be extended to $\left[\frac{\pi}{2},\pi\right]$.

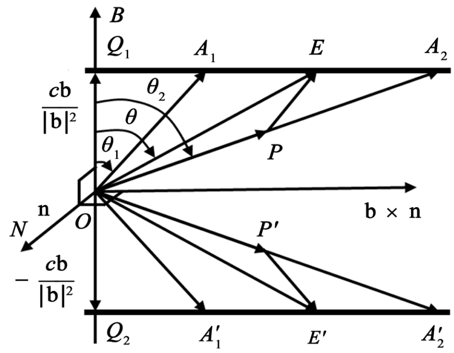

Figure 5. Linear operations.

Corollary 4.7. *Let* $\lambda_1,\lambda_2,\cdots,\lambda_s$ *be* s *nonnegative real numbers such that* $\sum_{i=1}^{n}\lambda_i=1$. *For* $i=1,2,\cdots,s$, *if* $\theta_i\in\left[0,\frac{\pi}{2}\right)$ $\left(\left(\frac{\pi}{2},\pi\right]\right)$, *then there is a* $\theta\in\left[0,\frac{\pi}{2}\right)$ $\left(\left(\frac{\pi}{2},\pi\right]\right)$ *satisfying*

$$\sum_{i=1}^{s}\lambda_i\frac{c}{_{(\theta_i,\boldsymbol{n})}\boldsymbol{b}}=\frac{c}{_{(\theta,\boldsymbol{n})}\boldsymbol{b}}\quad\text{and}\quad\sum_{i=1}^{s}\lambda_i\frac{c}{\boldsymbol{b}_{(\theta_i,\boldsymbol{n})}}=\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$$

where $c\in\mathbb{R}$ and $c\cos\theta_i\ge 0$ and $c\cos\theta\ge 0$.

Proof. It can be proved by mathematical induction based on Theorem 4.4. □

In the application, in order to fit new situation, we need to adjust direction parameter to find good new indefinite dot quotients. Considering that the direction parameter $\boldsymbol{n}$ is just perpendicular to $\boldsymbol{b}$, we have the following geometric properties:

Theorem 4.5. *Let* $c>0$. *In* Figure 6, *let O be a point in* 3*-dimensional Space, and* $\boldsymbol{OB}=\boldsymbol{b}$. *Take* $\boldsymbol{OQ}_1=\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}$, *and* $\boldsymbol{OQ}_2=-\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}$. *For* $i=1,2$, *through* Q_i, *draw a plane* Π_i *perpendicular to vector* $\boldsymbol{b}$. *Then*

1) *point* P_1 *is on the plane* Π_1 *if and only if* $\boldsymbol{OP}_1\cdot\boldsymbol{b}=c$ *and* $\angle(\boldsymbol{OP}_1,\boldsymbol{b})\in\left[0,\frac{\pi}{2}\right)$;

2) *point* P_2 *is on the plane* Π_2 *if and only if* $\boldsymbol{OP}_2\cdot\boldsymbol{b}=c$ *and* $\angle(\boldsymbol{OP}_2,\boldsymbol{b})\in\left(\frac{\pi}{2},\pi\right]$.

Proof. 1) Let P_1 be an arbitrary point on the plane Π_1. If P_1 is Q_1, the result, of course, holds. If P_1 is not Q_1, then $\boldsymbol{b}\perp\boldsymbol{Q}_1\boldsymbol{P}_1$. Thus
$\boldsymbol{OP}_1\cdot\boldsymbol{b}=(\boldsymbol{OQ}_1+\boldsymbol{Q}_1\boldsymbol{P}_1)\cdot\boldsymbol{b}=\boldsymbol{OQ}_1\cdot\boldsymbol{b}+\boldsymbol{Q}_1\boldsymbol{P}_1\cdot\boldsymbol{b}=c+0=c$. At this case, $\angle(\boldsymbol{OP}_1,\boldsymbol{b})\in\left[0,\frac{\pi}{2}\right)$ is obvious.

Conversely, if $\boldsymbol{OP}_1\cdot\boldsymbol{b}=c$ and $\angle(\boldsymbol{OP}_1,\boldsymbol{b})\in\left[0,\frac{\pi}{2}\right)$, Then,
$\boldsymbol{Q}_1\boldsymbol{P}_1\cdot\boldsymbol{b}=(\boldsymbol{OP}_1-\boldsymbol{OQ}_1)\cdot\boldsymbol{b}=\boldsymbol{OP}_1\cdot\boldsymbol{b}-\boldsymbol{OQ}_1\cdot\boldsymbol{b}=c-c=0$. This means P_1 is on the plane Π_1.

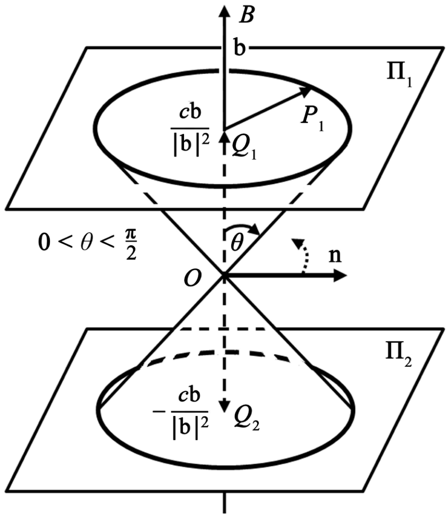

Figure 6. Structure for free $\boldsymbol{n}$.

Similarly, 2) can be proved. □

Corollary 4.8. *In* **Figure 6**, 1) *point* P_1 *is on the plane* Π_1 *if and only if there are a* $\theta_1 \in \left[0, \frac{\pi}{2}\right)$ *and an* $\boldsymbol{n}$ *such that* $\boldsymbol{OP}_1 = \frac{c}{{}_{(\theta_1,\boldsymbol{n})}\boldsymbol{b}}$ *or* $\boldsymbol{OP}_1 = \frac{c}{\boldsymbol{b}_{(\theta_1,\boldsymbol{n})}}$;

2) *point* P_2 *is on the plane* Π_2 *if and only if there are a* $\theta_2 \in \left(\frac{\pi}{2}, \pi\right]$ *and an* $\boldsymbol{n}$ *such that* $\boldsymbol{OP}_2 = \frac{c}{{}_{(\theta_2,\boldsymbol{n})}\boldsymbol{b}}$ or $\boldsymbol{OP}_2 = \frac{c}{\boldsymbol{b}_{(\theta_2,\boldsymbol{n})}}$.

Proof. Easy. □

In **Figure 6**, let $\theta \in \left[0, \frac{\pi}{2}\right)$ be a fixed angle. Then the point sets $\{P \mid \boldsymbol{OP}\cdot\boldsymbol{b} = c, \angle(\boldsymbol{OP},\boldsymbol{b}) = \theta\}$ and $\{P \mid \boldsymbol{OP}\cdot\boldsymbol{b} = -c, \angle(\boldsymbol{OP},\boldsymbol{b}) = \pi-\theta\}$ form two cycles. One's center is Q_1, and another's is Q_2.

In **Figure 6**, we can find that, when $c \to 0$, two planes Π_1 and Π_2 are all closed to the same plane through O and $\theta \to \frac{\pi}{2}$. Thus, we have

Theorem 4.6. *Let* $c = 0$. *In* **Figure 7**, *let O be a point in* 3*-dimensional Space, and* $\boldsymbol{OB} = \boldsymbol{b}$. *Through O, draw a plane* Π *perpendicular to vector* $\boldsymbol{b}$. *Then point P is on the plane* Π *if and only if* $\boldsymbol{OP}\cdot\boldsymbol{b} = 0$.

Proof. Obvious. □

In **Figure 7**, given $r > 0$, the point set $\{P \mid \boldsymbol{OP}\cdot\boldsymbol{b} = 0, |\boldsymbol{OP}| = r\}$ forms a cycle in the plane Π, whose center is O and radius is *r*.

5. Coordinates of Dot Quotients

Since our theory on dot quotients is built earlier in 3-dimensional Space, which does not use any coordinate system, it holds widely. For simplicity, in this section, we just consider the coordinate formulas of dot quotients in some given rectangular coordinate system $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Then, from the coordinates of $\boldsymbol{b}$

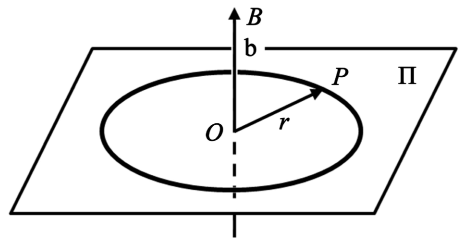

Figure 7. Structure for $c=0$.

and $\boldsymbol{n}$, by the results of Section 4, we can easily find the coordinates of $\frac{c}{(\theta,\boldsymbol{n})}\boldsymbol{b}$.

We have

Theorem 5.1. Let $c\in\mathbb{R}$, $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, $\boldsymbol{b}=\{X_2,Y_2,Z_2\}\neq\boldsymbol{0}$, and $\boldsymbol{n}=\{X_3,Y_3,Z_3\}\neq\boldsymbol{0}$ such that $\boldsymbol{n}\perp\boldsymbol{b}$. Suppose $\frac{c}{(\theta,\boldsymbol{n})}\boldsymbol{b}=\{X_1,Y_1,Z_1\}$. Then

$$\begin{cases} X_1=\dfrac{c\left[X_2+\dfrac{\tan\theta\left(Y_2Z_3-Y_3Z_2\right)}{\sqrt{X_3^2+Y_3^2+Z_3^2}}\right]}{X_2^2+Y_2^2+Z_2^2} \\ Y_1=\dfrac{c\left[Y_2+\dfrac{\tan\theta\left(X_3Z_2-X_2Z_3\right)}{\sqrt{X_3^2+Y_3^2+Z_3^2}}\right]}{X_2^2+Y_2^2+Z_2^2} \\ Z_1=\dfrac{c\left[Z_2+\dfrac{\tan\theta\left(X_2Y_3-X_3Y_2\right)}{\sqrt{X_3^2+Y_3^2+Z_3^2}}\right]}{X_2^2+Y_2^2+Z_2^2} \end{cases} \tag{6}$$

Proof. From Theorem 4.2, we have

$$\frac{c}{(\theta,\boldsymbol{n})}\boldsymbol{b}=\frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2}+\frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{b}|^2}\boldsymbol{b}\times\boldsymbol{n}=\frac{c}{|\boldsymbol{b}|^2}\left(\boldsymbol{b}+\frac{\tan\theta\boldsymbol{b}\times\boldsymbol{n}}{|\boldsymbol{n}|}\right).$$

We derive the formula (6) by substituting the coordinates of $\boldsymbol{b}$, $\boldsymbol{n}$ and $\boldsymbol{b}\times\boldsymbol{n}$. □

Corollary 5.1. *Let* $c\in R$, $\boldsymbol{b}=\{X_2,Y_2,Z_2\}\neq\boldsymbol{0}$ *and* $\boldsymbol{n}=\{X_3,Y_3,Z_3\}\neq\boldsymbol{0}$ *such that* $\boldsymbol{n}\perp\boldsymbol{b}$. *Suppose* $\frac{c}{(\theta,\boldsymbol{n})}\boldsymbol{b}=\{X_1,Y_1,Z_1\}$. *If* $\theta=0$ *or* π, *then*

$$\begin{cases} X_1=\dfrac{cX_2}{X_2^2+Y_2^2+Z_2^2} \\ Y_1=\dfrac{cY_2}{X_2^2+Y_2^2+Z_2^2} \\ Z_1=\dfrac{cZ_2}{X_2^2+Y_2^2+Z_2^2} \end{cases} \tag{7}$$

Proof. If $\theta=0$ or π, then $\tan\theta=0$. The result can be obtained by the coordinate formula (6). □

It is readily seen that the above result deduced from the coordinate formula (6) is quite equal to that deduced from Equation (3). If $\theta = \frac{\pi}{2}$, we have

Theorem 5.2. *Let* $\boldsymbol{b} = \{X_2, Y_2, Z_2\} \neq \boldsymbol{0}$ *and* $\boldsymbol{n} = \{X_3, Y_3, Z_3\} \neq \boldsymbol{0}$ *such that* $\boldsymbol{n} \perp \boldsymbol{b}$. *Suppose* $\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \{X_1, Y_1, Z_1\}$. *If* $\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} \times \boldsymbol{b} = \mu \boldsymbol{n}$, *then*

$$\begin{cases} X_1 = \dfrac{\mu(Y_2Z_3 - Y_3Z_2)}{X_2^2 + Y_2^2 + Z_2^2} \\ Y_1 = \dfrac{\mu(X_3Z_2 - X_2Z_3)}{X_2^2 + Y_2^2 + Z_2^2} \\ Z_1 = \dfrac{\mu(X_2Y_3 - X_3Y_2)}{X_2^2 + Y_2^2 + Z_2^2} \end{cases} \tag{8}$$

Proof. According to Theorem 4.3, we have

$$\frac{0}{\boldsymbol{b}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \frac{\mu \boldsymbol{b} \times \boldsymbol{n}}{|\boldsymbol{b}|^2} = \frac{\mu\{Y_2Z_3 - Y_3Z_2, X_3Z_2 - X_2Z_3, X_2Y_3 - X_3Y_2\}}{X_2^2 + Y_2^2 + Z_2^2}.$$

□

For right indefinite dot quotients, we have similar results:

Theorem 5.3. *Let* $c \in R$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $\boldsymbol{a} = \{X_1, Y_1, Z_1\} \neq \boldsymbol{0}$, *and* $\boldsymbol{n} = \{X_3, Y_3, Z_3\} \neq \boldsymbol{0}$ *such that* $\boldsymbol{n} \perp \boldsymbol{a}$. *Suppose* $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}} = \{X_2, Y_2, Z_2\}$. *Then*

$$\begin{cases} X_2 = \dfrac{c\left[X_1 + \dfrac{\tan\theta(Y_3Z_1 - Y_1Z_3)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}}\right]}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{c\left[Y_1 + \dfrac{\tan\theta(X_1Z_3 - X_3Z_1)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}}\right]}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{c\left[Z_1 + \dfrac{\tan\theta(X_3Y_1 - X_1Y_3)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}}\right]}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \tag{9}$$

Proof. From Theorem 4.2, we have

$$\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}} = \frac{c\boldsymbol{a}}{|\boldsymbol{a}|^2} - \frac{c\tan\theta}{|\boldsymbol{n}||\boldsymbol{a}|^2}\boldsymbol{a} \times \boldsymbol{n} = \frac{c}{|\boldsymbol{a}|^2}\left(\boldsymbol{a} + \frac{\tan\theta \boldsymbol{n} \times \boldsymbol{a}}{|\boldsymbol{n}|}\right).$$

Then, we derive Formula (9) by the coordinates of $\boldsymbol{a}$, $\boldsymbol{n}$ and $\boldsymbol{n} \times \boldsymbol{a}$. □

Corollary 5.2. *Let* $c \in R$, $\boldsymbol{a} = \{X_1, Y_1, Z_1\} \neq \boldsymbol{0}$, *and* $\boldsymbol{n} = \{X_3, Y_3, Z_3\} \neq \boldsymbol{0}$ *such that* $\boldsymbol{n} \perp \boldsymbol{a}$. *Suppose* $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}} = \{X_2, Y_2, Z_2\}$. *If* $\theta = 0$ *or* π, *then*

$$\begin{cases} X_2 = \dfrac{cX_1}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{cY_1}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{cZ_1}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \quad (10)$$

Proof. If $\theta = 0$ or π, then $\tan\theta = 0$. The result holds from the coordinate formula (9). □

Theorem 5.4. *Let* $\boldsymbol{a} = \{X_1, Y_1, Z_1\} \neq \boldsymbol{0}$ *and* $\boldsymbol{n} = \{X_3, Y_3, Z_3\} \neq \boldsymbol{0}$ *such that* $\boldsymbol{n} \perp \boldsymbol{a}$. *Suppose* $\dfrac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \{X_2, Y_2, Z_2\}$. *If* $\boldsymbol{a} \times \dfrac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \mu\boldsymbol{n}$, *then*

$$\begin{cases} X_2 = \dfrac{\mu(Y_3 Z_1 - Y_1 Z_3)}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{\mu(X_1 Z_3 - X_3 Z_1)}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{\mu(X_3 Y_1 - X_1 Y_3)}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \quad (11)$$

Proof. According to Theorem 4.3, we have

$$\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \frac{\mu\boldsymbol{n} \times \boldsymbol{a}}{|\boldsymbol{a}|^2} = \frac{\mu\{Y_3 Z_1 - Y_1 Z_3, X_1 Z_3 - X_3 Z_1, X_3 Y_1 - X_1 Y_3\}}{X_1^2 + Y_1^2 + Z_1^2}.$$

□

Next, we will discuss the applications of coordinate formulas. Although we can give some application examples of dot quotient in differential manifolds, in physics, in force, etc., here we just give a very simple examples to show how to use our formulas, and verify the correctness of our theory, by the way.

Example 1 It is given that two vectors $\boldsymbol{a} = \{X_1, Y_1, Z_1\} = \{1, 2, 2\}$ and $\boldsymbol{b} = \{X_2, Y_2, Z_2\} = \{2, 1, -3\}$. Then their dot product $c = \boldsymbol{a} \cdot \boldsymbol{b} = -2$, and their cross product $\boldsymbol{a} \times \boldsymbol{b} = \{Y_1 Z_2 - Y_2 Z_1, X_2 Z_1 - X_1 Z_2, X_1 Z_2 - X_2 Z_1\} = \{-8, 7, -3\}$. And let $\boldsymbol{n} = \{X_3, Y_3, Z_3\} = \{-16, 14, -6\} = 2\{-8, 7, -3\} = 2\boldsymbol{a} \times \boldsymbol{b}$. Of course, $\boldsymbol{n} \perp \boldsymbol{a}$ and $\boldsymbol{n} \perp \boldsymbol{b}$. Thus we have $|\boldsymbol{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2} = 3$, $|\boldsymbol{b}| = \sqrt{X_2^2 + Y_2^2 + Z_2^2} = \sqrt{14}$, $|\boldsymbol{n}| = \sqrt{X_3^2 + Y_3^2 + Z_3^2} = 2\sqrt{122}$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are known, the angle θ between them is determined by $\cos\theta = \dfrac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{-2}{3\sqrt{14}}$. By the way, we have

$\sin\theta = \sqrt{1 - \cos^2\theta} = \dfrac{\sqrt{122}}{3\sqrt{14}}$, $\cot\theta = \dfrac{-2}{\sqrt{122}}$, and

$\cot\theta\sqrt{X_3^2 + Y_3^2 + Z_3^2} = \dfrac{-2}{\sqrt{122}} \times 2\sqrt{122} = -4$.

It is no doubt that $\boldsymbol{n} \perp \boldsymbol{b}$ and $\boldsymbol{b} \neq \boldsymbol{0}$ and $c\cos\theta > 0$. From Formula (6), we, of course, have the coordinates of $\dfrac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$:

$$X_1=\frac{c\left(X_2+\dfrac{Y_2Z_3-Y_3Z_2}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_2^2+Y_2^2+Z_2^2}=\frac{-2}{14}\left[2+\frac{1\times(-6)-14\times(-3)}{-4}\right]=1;$$

$$Y_1=\frac{c\left(Y_2+\dfrac{X_3Z_2-X_2Z_3}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_2^2+Y_2^2+Z_2^2}=\frac{-2}{14}\left[1+\frac{(-16)\times(-3)-2\times(-6)}{-4}\right]=2;$$

$$Z_1=\frac{c\left(Z_2+\dfrac{X_2Y_3-X_3Y_2}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_2^2+Y_2^2+Z_2^2}=\frac{-2}{14}\left[-3+\frac{2\times14-(-16)\times1}{-4}\right]=2.$$

It is readily seen that $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ is exactly equal to $\boldsymbol{a}$.

Similarly, by Formula (9), we can obtain the coordinates of $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}}$:

$$X_2=\frac{c\left(X_1+\dfrac{Y_3Z_1-Y_1Z_3}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_1^2+Y_1^2+Z_1^2}=\frac{-2}{9}\left[1+\frac{14\times2-2\times(-6)}{-4}\right]=2;$$

$$Y_2=\frac{c\left(Y_1+\dfrac{X_1Z_3-X_3Z_1}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_1^2+Y_1^2+Z_1^2}=\frac{-2}{9}\left[2+\frac{1\times(-6)-(-16)\times2}{-4}\right]=1;$$

$$Z_2=\frac{c\left(Z_1+\dfrac{X_3Y_1-X_1Y_3}{\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}\right)}{X_1^2+Y_1^2+Z_1^2}=\frac{-2}{9}\left[2+\frac{(-16)\times2-1\times14}{-4}\right]=-3.$$

It is also seen that $\frac{c}{\boldsymbol{a}_{(\theta,\boldsymbol{n})}}$ is really equal to $\boldsymbol{b}$.

Example 2 Given two vectors $\boldsymbol{a}=\{X_1,Y_1,Z_1\}=\{1,2,1\}$ and $\boldsymbol{b}=\{X_2,Y_2,Z_2\}=\{2,1,-4\}$, their dot product $c=\boldsymbol{a}\cdot\boldsymbol{b}=0$, which means $\boldsymbol{a}\perp\boldsymbol{b}$, and $|\boldsymbol{b}|^2=X_2^2+Y_2^2+Z_2^2=21$, and their cross product $\boldsymbol{a}\times\boldsymbol{b}\triangleq\{X_3,Y_3,Z_3\}=\{Y_1Z_2-Y_2Z_1,X_2Z_1-X_1Z_2,X_1Y_2-X_2Y_1\}$ $=\{2\times(-4)-1\times1,2\times1-1\times(-4),1\times1-2\times2\}=\{-9,6,-3\}=3\{-3,2,-1\}$. If we regard $\boldsymbol{a}\times\boldsymbol{b}=\{-9,6,-3\}$ as known and let $\boldsymbol{n}=\{-3,2,-1\}$, then $\mu=3$.

Thus, from Formula (8), we have the coordinates of $\frac{0}{{}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}\boldsymbol{b}}$:

$$X_1=\frac{\mu(Y_2Z_3-Y_3Z_2)}{X_2^2+Y_2^2+Z_2^2}=\frac{\mu[1\times(-1)-2\times(-4)]}{21}=\frac{7\mu}{21}=\frac{\mu}{3}=1;$$

$$Y_1=\frac{\mu(X_3Z_2-X_2Z_3)}{X_2^2+Y_2^2+Z_2^2}=\frac{\mu[(-3)\times(-4)-2\times(-1)]}{21}=\frac{14\mu}{21}=\frac{2\mu}{3}=2;$$

$$Z_1 = \frac{\mu\left(X_2Y_3 - X_3Y_2\right)}{X_2^2 + Y_2^2 + Z_2^2} = \frac{\mu\left[2\times 2 - (-3)\times 1\right]}{21} = \frac{7\mu}{21} = \frac{\mu}{3} = 1.$$

It is readily seen that $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)^{\boldsymbol{b}}}$ is exactly equal to $\boldsymbol{a}$.

Similarly, we have the coordinates of $\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$:

$$X_2 = \frac{\mu\left(Y_3Z_1 - Y_1Z_3\right)}{X_1^2 + Y_1^2 + Z_1^2} = \frac{\mu\left[2\times 1 - 2\times(-1)\right]}{6} = \frac{4\mu}{6} = \frac{2\mu}{3} = 2;$$

$$Y_2 = \frac{\mu\left(X_1Z_3 - X_3Z_1\right)}{X_1^2 + Y_1^2 + Z_1^2} = \frac{\mu\left[1\times(-1) - (-3)\times 1\right]}{6} = \frac{2\mu}{6} = \frac{\mu}{3} = 1;$$

$$Z_2 = \frac{\mu\left(X_3Y_1 - X_1Y_3\right)}{X_1^2 + Y_1^2 + Z_1^2} = \frac{\mu\left[(-3)\times 2 - 1\times 2\right]}{6} = \frac{-8\mu}{6} = \frac{-4\mu}{3} = -4.$$

It is also seen that $\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$ is fully equal to $\boldsymbol{b}$.

We should note that, if we adjust the value of μ, we can get different $\boldsymbol{a}$ or $\boldsymbol{b}$ such that $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. For instance, if $\mu = 1$ instead of 3, we have $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)^{\boldsymbol{b}}} = \left\{\frac{1}{3},\frac{2}{3},\frac{1}{3}\right\}$ and $\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = \left\{\frac{2}{3},\frac{1}{3},-\frac{4}{3}\right\}$ such that $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)^{\boldsymbol{b}}}\cdot\boldsymbol{b} = 0$ and $\boldsymbol{a}\cdot\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}} = 0$. Based on our need, by adjusting the value of μ, we can get better or best $\frac{0}{\left(\frac{\pi}{2},\boldsymbol{n}\right)^{\boldsymbol{b}}}$ or $\frac{0}{\boldsymbol{a}_{\left(\frac{\pi}{2},\boldsymbol{n}\right)}}$.

Generally speaking, if we know sufficient supporting information, we really can back to find $\boldsymbol{a}$ ($\boldsymbol{b}$) such that $\boldsymbol{a}\cdot\boldsymbol{b} = c$ from the main information c and $\boldsymbol{b}$ ($\boldsymbol{a}$). Sometimes, we are not interested in finding original $\boldsymbol{a}$ ($\boldsymbol{b}$), but in finding a needed $\boldsymbol{a}'$ ($\boldsymbol{b}'$) such that $\boldsymbol{a}'\cdot\boldsymbol{b} = c$ ($\boldsymbol{a}\cdot\boldsymbol{b}' = c$). When $\theta \neq \frac{\pi}{2}$, the dot quotient theory tells that it is enough by adjusting angle parameter and normal vector parameter, otherwise we need additional conditions such as cross products.

6. Conclusions

This paper successfully set up the theory of indefinite dot quotients by adding angle and normal direction parameters.

First of all, we successfully define indefinite dot quotients by Definition 2.1. By the definition, as we know a real number c and a nonzero vector $\boldsymbol{b}$, we inversely obtain two vectors $\frac{c}{(\theta,\boldsymbol{n})^{\boldsymbol{b}}}$ and $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$ satisfying

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = c$$

where θ is an angle parameter and $\boldsymbol{n}$ direction parameter such that $\boldsymbol{n} \perp \boldsymbol{b}$, and both $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} \times \boldsymbol{b}$ and $\boldsymbol{n}$ have the same direction, and both $\boldsymbol{b} \times \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$ and $\boldsymbol{n}$ have the same direction. We further obtain the basic properties (2.1) to (2.6) and some expected operation properties.

Secondly, we obtained two geometric expressions of indefinite dot quotients (by Theorem 4.1 and Theorem 4.5) where one of them exposes that, when normal direction $\boldsymbol{n}$ is fixed, the end points of indefinite dot quotients form two parallel lines with the change of angle parameters, and another reveals that when θ is fixed, the end points form two circles with the change of normal directions. By the geometric expressions, Corollary 4.6 not only puts angle parameter into real parameter but also presents two unified expressions:

$$\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{b} \times \boldsymbol{n} \text{ and } \frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}} = \frac{c\boldsymbol{b}}{|\boldsymbol{b}|^2} + \lambda \boldsymbol{n} \times \boldsymbol{b}$$

which successfully avoid concerning the angle parameter is $\frac{\pi}{2}$ or not. We say that the structures of indefinite dot quotients are exposed completely in geometry.

Finally, we obtained the coordinate formulas (6)-(11), which help us to get the coordinates of $\frac{c}{{}_{(\theta,\boldsymbol{n})}\boldsymbol{b}}$ and $\frac{c}{\boldsymbol{b}_{(\theta,\boldsymbol{n})}}$ easily. With determined angle and direction parameters, we inversely find the exact vector $\boldsymbol{a}$ ($\boldsymbol{b}$) from a scalar c and a vector $\boldsymbol{b}$ ($\boldsymbol{a}$) such that $\boldsymbol{a} \cdot \boldsymbol{b} = c$ by the coordinations.

We want to say, we can design new indefinite dot quotients by adjusting angle parameters and direction parameters to fit new situation in the applications.

The relation between dot products and indefinite dot quotients likes that between derivatives and indefinite integrals, the only difference is that an indefinite integral has only one parameter (formed by an arbitrary constant), but an indefinite dot quotient has a parameter pair (an angle, a direction).

It is seen that this paper has successfully built the theory of indefinite dot quotients which solve the long time problem that dot product has no corresponding division in three dimensional space. Our theory of indefinite dot quotients makes the theory of vector analysis more perfect.

Conflicts of Interest

The author declares no conflicts of interest.

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