Normalized Solutions of Mass-Subcritical Schrödinger-Maxwell Equations

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Abstract
In this paper, we investigate the existence of normalized solutions to the coupling of the nonlinear Schrödinger-Maxwell equations. In the mass-subcritical case, we by weak lower semicontinuity of norm prove that the equations satisfying normalization condition exist a normalized ground state solution.

Subject Areas
Mathematics

Keywords
Normalized Solutions, Schrödinger-Maxwell Equations

1. Introduction
In this paper, we study the existence of normalized ground state solution of the following Schrödinger-Maxwell equations
\begin{equation}
\begin{cases}
-\Delta u + u + \phi u + \lambda u = f(u) \text{ in } \mathbb{R}^N, \\
-\Delta \phi = u^2 \text{ in } \mathbb{R}^N,
\end{cases}
\end{equation}
where \( \phi : \mathbb{R}^N \to \mathbb{R} \) and \( 2 < N < 6 \), the parameter \( \lambda \in \mathbb{R} \) appears as a Lagrange multiplier. The unknowns of the equations are the field \( u \) associated to the particle and the electric potential \( \phi \), and satisfying the normalization condition
\begin{equation}
\int_{\mathbb{R}^N} |u|^2 \, dx = a,
\end{equation}
we prescribe \( a > 0 \). Hence, we have
\begin{equation}
\begin{cases}
-\Delta u + u + \phi u + \lambda u = f(u) \text{ in } \mathbb{R}^N, \\
-\Delta \phi = u^2 \text{ in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u|^2 \, dx = a.
\end{cases}
\end{equation}
where $u$ belongs to the Hilbert space
\[ \mathcal{H} = \{ u \in H^1_0(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx < \infty \}, \]
and
\[ H^1_0(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u(x) = u(\|x\|) \}. \]
The space $\mathcal{H}$ is endowed with the norm
\[ \| u \|^2_{\mathcal{H}} = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + u^2 \right) dx. \]

Let $D^{1,2} = D^{1,2}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \forall u \in L^2(\mathbb{R}^N) \}$ with respect to the norm
\[ \| u \|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2 dx. \]

For any $2 < s < 2^*$, $L^s(\mathbb{R}^N)$ is endowed with the norm
\[ \| u \|^s = \int_{\mathbb{R}^N} |u|^s dx. \]

Obviously, the embedding $\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N)$ is compact (see [1]).

By the variational nature, the weak solutions of (1.1) are critical points of the functional
\[ J : \mathcal{H} \times D^{1,2} \to \mathbb{R} \]
defined by
\[ J(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V(x)u^2 \right) dx - \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \phi u^2 dx - \int_{\mathbb{R}^N} F(u) dx, \]
where $F(t) = \int_0^t f(s) ds$ is a rather general nonlinearity. Then, it is clear that the function $J$ is $C^1$ on $\mathcal{H} \times D^{1,2}$ and has the strong indefiniteness. We can know that the weak solutions of (1.1) $(u, \phi) \in \mathcal{H} \times D^{1,2}$ are critical points of the functional $J$. By standard arguments, the function $J$ is $C^1$ on $\mathcal{H} \times D^{1,2}$.

In recent years, normalized solutions of Schrödinger equations have been widely studied. When searching for the existence of normalized solutions of Schrödinger equations in $\mathbb{R}^N$, appears a new mass-critical exponent
\[ l = 2 + \frac{4}{N}. \]
Now, let us review the involved works. In the mass-subcritical case, Zuo Yang and Shijie Qi [2] proved that for all $a > 0$, the following Schrödinger equations with potentials and non-autonomous nonlinearities
\[ \begin{cases} -\Delta u + V(x)u + \lambda u = f(x, u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a, u \in H^1(\mathbb{R}^N), \end{cases} \]
have a normalized solutions. Nicola Soave [3] in the mass-subcritical proved the nonlinear Schrödinger equation with combined power nonlinearities mass-critical and mass-supercritical cases studied of:
\[ \begin{cases} -\Delta u + \lambda u + \mu |u|^{p-2} u + |u|^{q-2} u & \text{in } \mathbb{R}^N, N \geq 3, \\ \int_{\mathbb{R}^N} |u|^2 dx = a, u \in H^1(\mathbb{R}^N). \end{cases} \]

have several stability/instability and existence/non-existence results of normalized ground state solutions. For $g(u)$ is a superlinear, subcritical, Thomas Bartsch [4]
studied the existence of infinitely many normalized solutions for the problem

\[-\Delta u - g(u) = \lambda u, u \in H^1(\mathbb{R}^N),\]

By establishing the compactness of the minimizing sequences, Tianxiang Gou and Louis Jeanjean [5] in the mass-subcritical studied the existence of multiple positive solutions to the nonlinear Schrödinger systems:

\[
\begin{cases}
-\Delta u = \lambda u + \mu_1 |u_1|^{p_1-2} u_1 + \beta_1 |u_2|^{q_1-2} u_2, \\
-\Delta u = \lambda u + \mu_2 |u_2|^{p_2-2} u_2 + \beta_2 |u_1|^{q_2-2} u_1.
\end{cases}
\]

In the mass-subcritical case, Masataka Shibata [6] studied for the nonlinear Schrödinger equations with the minimizing problem:

\[
E_a = \inf \left\{ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(|u|) \, dx \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 \, dx = a \right\}
\]

where \( F(t) = \int_0^t f(s) \, ds \) is a general nonlinear term. They proved \( E_a \) is attained. That is to say, the Schrödinger equations have normalized solutions. Moreover, for the \( I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) |u|^2 \, dx - \int_{\mathbb{R}^N} F(|u|) \, dx \) case, Norihisa Ikoma and Yasuhito Miyamoto [7] showed the existence of the minimizer of the minimization problem \( E_a \), where \( V(x) \to 0 \) as \( |x| \to \infty \). They also obtained the conclusions that the normalized solutions of Schrödinger equations exist. In the mass-subcritical condition, Zhen Chen and Wenming Zou [8] basing on the refined energy estimates proved the existence of normalized solutions to the Schrödinger equations.

Other related normalized solutions problems of Schrödinger can be seen in [9] [10] [11] [12] [13]. Thus, the main purpose of this paper is to study the solution of Schrödinger-Maxwell equations satisfying normalization condition by using above results. In particular, the situation we consider will involve the presence of potential \( \phi \). In addition, the nonlinear term \( f(u) \) is mass-subcritical and satisfies the following appropriate assumptions. In this case, the functional \( J \) is bounded from below and coercive on \( S(a) \), which will be proved in Lemma 2.5.

We assume the following conditions throughout the paper:

\[(f1)\] \( f : \mathbb{R}^N \to \mathbb{R} \) is continuous.

\[(f2)\] \( \lim_{s \to 0} \frac{f(s)}{s} = 0 \) and \( \lim_{|u| \to \infty} \frac{f(s)}{|u|^l} = 0 \) with \( l = 2 + \frac{4}{N} \).

Moreover, \( c \) and \( c_i \) are positive constants which may change from line to line.

Our main result is the following theorem:

**Theorem 1.1** Suppose (f1) and (f2) hold. Then, for any \( a > 0 \), problem (1.3) has a normalized ground state solution.

### 2. Proof of Main Results

Since the functional \( J \) exhibits a strong indefiniteness. To avoid the difficulty we use the reduction method. Thus, we shall introduce the method.
For any \( u \in \mathcal{H} \), we consider the linear operator \( T(u) : D^{1,2} \to \mathbb{R} \) defined as
\[
T(u) = \int_{\mathbb{R}^N} u^2 \, dx.
\]
(2.1)

Then, there exists a positive constant \( c_1 \) such that
\[
\int_{\mathbb{R}^N} u^2 \, dx \leq \| u \|_{L^p}^2 \leq c_1 \| u \|_{L^{p+2}}^2 \leq c_1 \| u \|_{L^{4/3}}^2 \leq c_1 \| u \|_{H^1}^2,
\]
because the following embeddings are continuous:
\[
\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N), \quad \forall s \in \left[ 2, 2^* \right] \quad \text{and} \quad D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N).
\]

We set
\[
g(\phi, v) = \int_{\mathbb{R}^N} \nabla \phi \cdot \nabla v \, dx, \quad \phi, v \in D^{1,2}.
\]

Obviously, \( g(\phi, v) \) is linear in \( \phi \) and \( v \) respectively.

Moreover, there exists a positive constant \( c_2 \) and \( c_3 \) such that for any \( \phi, v \in D^{1,2} \),
\[
|g(\phi, v)| \leq c_2 \| \phi \|_{H^1} \| v \|_{H^1},
\]
(2.2)
\[
g(\phi, v) \geq c_3 \| \phi \|_{H^1} \| v \|_{H^1}.
\]
(2.3)

Combining (2.2) and (2.3), we know that \( g(\phi, v) \) is bounded and coercive. Hence, by the Lax-Milgram theorem we have that for every \( u \in \mathcal{H} \), for any \( v \in D^{1,2} \), there exists a unique \( \phi_u \in D^{1,2} \) such that
\[
T(u) v = g(\phi_u, v).
\]

Then, for any \( v \in D^{1,2} \), we obtain
\[
\int_{\mathbb{R}^N} u^2 \, dx = \int_{\mathbb{R}^N} \nabla \phi_u \cdot \nabla v \, dx,
\]
(2.4)
and using integration by parts, we have
\[
\int_{\mathbb{R}^N} \nabla \phi_u \cdot \nabla v \, dx = -\int_{\mathbb{R}^N} v \Delta \phi_u \, dx.
\]
Therefore,
\[
-\Delta \phi_u = u^2
\]
(2.5)
in a weak sense, and \( \phi_u \) has the following integral expression:
\[
\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|} \, dy,
\]
(2.6)
The functions \( \phi_u \) possess the following properties:

**Lemma 2.1** For any \( u \in \mathcal{H} \), we have:
1) \( \| \phi_u \|_{L^p} \leq c_4 \| u \|_{L^{p+2}}^2 \), where \( c_4 > 0 \) is independent of \( u \). As a consequence there exists \( c_5 > 0 \) such that
\[
\int_{\mathbb{R}^N} \phi_u u^2 \, dx \leq c_5 \| u \|_{H^1}^2;
\]
2) \( \phi_u \geq 0 \).

**Proof.** 1) For any \( u \in \mathcal{H} \), using (2.5) we have
\[ \|\phi_n\|_{L^{2,\infty}}^2 = \int_{\mathbb{R}^N} |\nabla \phi_n|^2 \, dx = -\int_{\mathbb{R}^N} \phi_n \Delta \phi_n \, dx = \int_{\mathbb{R}^N} \phi_n^2 \, dx \leq \|\phi_n\|_{L^{2,\infty}}^2 \leq c_4 \|\phi_n\|_{L^{2,\infty}}^2 \|\phi_n\|_{L^4}^4, \]

where \( c_4 \) is a positive constant. Hence, we obtain that \[ \|\phi_n\|_{L^{2,\infty}}^2 \leq c_4 \|\phi_n\|_{L^4}^4, \]

therefore there exists a positive constant \( c_5 \) such that \[ \int_{\mathbb{R}^N} \phi_n^2 \, dx \leq c_4 \|\phi_n\|_{L^{2,\infty}}^2 \leq c_4 \|\phi_n\|_{L^4}^4 \leq c_4 \|\phi_n\|_{\mathcal{H}}^4, \quad (2.7) \]

because we know for any \( s \in \left[ 2, 2^* \right], \mathcal{H} \hookrightarrow \mathcal{H} \left( \mathbb{R}^N \right). \)

2) Obviously, by the expression (2.6) the conclusion holds. \( \square \)

Now let us consider the functional \( I : \mathcal{H} \rightarrow \mathbb{R}^N, \)

\[
I(u) := J(u, \phi_u). 
\]

Then \( I \) is \( C^1. \)

By the definition of \( J, \) we have

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \phi_u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \phi_u u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx. 
\]

Multiplying both members of (2.5) by \( \phi_u \) and integrating by parts, we obtain

\[
\int_{\mathbb{R}^N} |\nabla \phi_u|^2 \, dx = \int_{\mathbb{R}^N} \phi_u u^2 \, dx. 
\]

Therefore, the functional \( I \) may be written as

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi_u u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx. 
\]

The following lemma is Proposition 2.3 in [5].

**Lemma 2.2** The following statements are equivalent:

1) \( (u, \phi_u) \in \mathcal{H} \times D^{1,2} \left( \mathbb{R}^N \right) \) is a critical point of \( I. \)

2) \( u \) is a critical point of \( I \) and \( \phi = \phi_u. \)

Hence \( u \) is a solution to (1.3) if and only if \( u \) is the critical point of the functional (2.8). The critical point can be obtained as the minimizer under the constraint of \( \mathcal{L}^2 \)-sphere

\[
S(a) = \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^N} u^2 \, dx = a \right\}. 
\]

We shall study the constraint problem as follows:

\[
E_a = \inf_{u \in S(a)} I(u). 
\]

The solution of (13) \( u = \bar{u} \) is called a normalized ground state solution satisfying problem (3) if it has minimal energy among all solutions:

\[
dl_{I(u)}(\bar{u}) = 0 \text{ and } I(\bar{u}) = \inf \left\{ I(u) : dl_{I(u)}(\bar{u}) = 0, \bar{u} \in S(a) \right\}. 
\]

In this paper, we will be especially interested in the existence of normalized ground state solutions.

**Lemma 2.3** We define \( \Phi : \mathcal{H} \rightarrow D_{\infty}^{1,2}, \Phi(u) = \phi_u, \) which is also the solution
of the Equation (2.5) in $D^{1,2}$. Let $\{u_n\} \subset S(a)$ be a minimizing sequence of $I$ with satisfying $u_n \rightharpoonup u$ in $\mathcal{H}$. Then, $\Phi(u_n) \to \Phi(u)$ in $D^{1,2}$ and we obtain

$$\int_{\mathbb{R}^N} \Phi(u_n) u_n^2 dx \to \int_{\mathbb{R}^N} \Phi(u) u^2 dx \quad \text{as} \ n \to \infty.$$  \hspace{1cm} (2.11)

**Proof.** By (2.1), the following expressions hold

$$T(u_n) v = \int_{\mathbb{R}^N} u_n^2 v dx, \quad T(u) v = \int_{\mathbb{R}^N} u^2 v dx.$$  

Since $u \in \mathcal{H}$ and the embedding $H^1 \hookrightarrow L^s$ is compact for any $s \in (2, 2^*)$, clearly we have

$$u^2 \in L^1\left(\mathbb{R}^N\right) \cap L^{2^*}\left(\mathbb{R}^N\right),$$  \hspace{1cm} (2.12)

then, by interpolation we have

$$u^2 \in L^{\frac{2N}{2N+2}}\left(\mathbb{R}^N\right).$$

Using again (2.12), we get

$$u^2 \in L^{2}\left(\mathbb{R}^N\right).$$

Moreover, $\{u_n\}$ be a minimizing sequence and $u_n \rightharpoonup u$ in $\mathcal{H}$, we obtain

$$u_n^2 \to u^2 \quad \text{in} \quad L^{\frac{2N}{2N+2}}.$$  \hspace{1cm} (2.13)

Therefore, we get

$$\left| T(u_n) v - T(u) v \right| = \int_{\mathbb{R}^N} u_n^2 v dx - \int_{\mathbb{R}^N} u^2 v dx \leq |u_n^2 - u^2| \left\| v \right\|_{\mathcal{H}}^2,$$

which implies that $T(u_n)$ converges strongly to $T(u)$.

Hence, we obtain

$$\Phi(u_n) \to \Phi(u) \quad \text{in} \quad D^{1,2},$$

$$\Phi(u_n) \to \Phi(u) \quad \text{in} \quad L^{2^*}.$$  \hspace{1cm} (2.14)

By (2.13) and (2.14), we know that conclusion (2.11) holds.

**Lemma 2.4 (Gagliardo-Nirenberg inequality).** For all $u \in \mathcal{H}$, we have

$$\left\| u \right\|_{p'} \leq C(N) \left\| \nabla u \right\|_2 \left\| u \right\|^p_{p'}, \quad 2 < p < 2^*,$$

where $C(N)$ is a positive constant depending on $N$ and $p' = \frac{N(p-2)}{2p}$.

**Lemma 2.5** Suppose (A1) and (A2) hold, then for any $a > 0$, the functional $I$ is bounded from below and coercive on $S(a)$.

**Proof.** Assumptions (A1) and (A2) imply that for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that

$$F(s) \leq C_\varepsilon |s|^l + \varepsilon |s|^l, \quad \forall s \in \mathbb{R}.$$  

Hence, according to Lemma 2.4 with $p = l = 2 + \frac{4}{N}$, we obtain that

$$\int_{\mathbb{R}^N} F(u) ds \leq C_{\varepsilon} \left\| u \right\|^2_{l} + \varepsilon \left\| u \right\|^l_{l} \leq C_\varepsilon \left\| u \right\|^2_{l} + \varepsilon C(N) \left\| \nabla u \right\|_2 \left\| u \right\|^4_{l}.$$
Choose \( \varepsilon \) such that \( \varepsilon C(N) a^2 = \frac{1}{4} \), than

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + u^2 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi_0 u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx
\geq \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx
\geq \frac{1}{4} \|\nabla u\|_2^2 - Ca > -\infty
\]

Therefore, \( I \) is bounded from below and coercive on \( S(a) \). \( \square \)

The following lemma is Lemma 2.2 in [6].

**Lemma 2.6** Suppose \((A)\) and \((f_2)\) hold and \( \{u_n\}_{n\in\mathbb{N}} \) is a bounded sequence in \( \mathcal{H} \). If \( \lim_{n\to\infty} \|u_n\|_2^2 = 0 \) holds, then it is true that

\[
\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) \, dx = 0.
\]

Next, we collect a variant of Lemma 2.2 in [14]. The proof is similar, so we omit it.

**Lemma 2.7** Suppose \((A)\) and \((f_2)\) hold and \( \{u_n\}_{n\in\mathbb{N}} \) is a bounded sequence in \( \mathcal{H} \), then we have \( u_n \rightharpoonup u \) in \( \mathcal{H} \), thus

\[
\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[ F(u_n) - F(u) - F(u_n - u) \right] \, dx = 0.
\]

**Proof of Theorem 1.1.** Let \( \{u_n\} \subset S(a) \) be a minimizing sequence of \( I \) with concerning \( E_a \). Then, by (9) we obtain

\[
I(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u_n|^2 + u_n^2 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi_0 u_n^2 \, dx - \int_{\mathbb{R}^N} F(u_n) \, dx.
\]

According to Lemma 2.5, the sequence \( \{u_n\} \) is bounded in \( \mathcal{H} \). Letting \( u_0 \) be in \( \mathcal{H} \). Moreover, we know that the embedding \( \mathcal{H} \hookrightarrow L^s(\mathbb{R}^N) \) is compact. Hence, we conclude

\[
u_n \rightharpoonup u_0 \text{ in } \mathcal{H}, \quad u_n \rightharpoonup u_0 \text{ in } L^s(\mathbb{R}^N), \quad 2 < s < 2^*, \quad u_n \rightharpoonup u_0 \text{ a.e. in } \mathbb{R}^N.
\]

We also have

\[
I(u_0) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u_0|^2 + u_0^2 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi_0 u_0^2 \, dx - \int_{\mathbb{R}^N} F(u_0) \, dx.
\]

Since (19) holds, we have \( \lim_{n\to\infty} \|u_n - u_0\|_2^2 = 0 \). Then, by Lemma 2.6 we obtain

\[
\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n - u_0) \, dx = 0.
\]

Moreover, by Lemma 2.7 we have

\[
\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[ F(u_n) - F(u_0) \right] \, dx = 0.
\]

which implies

\[
\int_{\mathbb{R}^N} F(u_n) \, dx \to \int_{\mathbb{R}^N} F(u_0) \, dx \quad \text{as } n \to \infty.
\]
Hence, combining weak lower semicontinuity of the norm \( \| \cdot \|_E \), Lemma 2.3 and (2.17), we have
\[
E_a \leq I(u_0) \leq \liminf_{n \to \infty} I(u_n) = E_a,
\]
which implies \( I(u_0) = E_a \). Then, \( u_0 \) satisfies
\[
\begin{cases}
-\Delta u_0 + u_0 + \phi u_0 + \lambda u_0 = f(u_0) & \text{in } \mathbb{R}^N, \\
-\Delta \phi = u_0^2 & \text{in } \mathbb{R}^N,
\end{cases}
\]
and \( \int_{\mathbb{R}^N} |u_0|^2 \, dx = a \). Therefore, problem (1.3) has a normalized ground state solution. \( \square \)

**Conflicts of Interest**

The author declares no conflicts of interest.

**References**


