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Pseudo-Index Theory for a Schrödinger Equation with Competing Potentials

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Abstract

In this paper, we study a nonlinear Schrödinger equation with competing potentials $-\varepsilon^2\Delta v+V(x)v=W_1(x)|v|^{p-2}v+W_2(x)|v|^{q-2}v,\quad v\in H^1(\mathbb{R}^N)$, where $\varepsilon>0$, $p,q\in(2,2^*)$, $p>q$, $2^*:=\frac{2N}{N-2}(N>2)$, $V(x)$, $W_1(x)$ and $W_2(x)$ are continuous bounded positive functions. Under suitable assumptions on the potentials, we consider the existence, concentration, convergence and decay estimates of the ground state solution for this equation. Furthermore, the multiplicity of semi-classical solutions is established by using Benci pseudo-index theory, and the existence of sign-changing solutions is obtained via Nehari method.

Subject Areas

Partial Differential Equation

Keywords

Pseudo-Index, Multiplicity, Concentration, Sign-Changing Solution

1. Introduction

In this paper, we are interested in the nonlinear Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t}=-\varepsilon^2\Delta\psi+(V(x)+1)\psi-W_1(x)|\psi|^{p-2}\psi-W_2(x)|\psi|^{q-2}\psi, \tag{1.1}$$

where $(t,x)\in\mathbb{R}_+\times\mathbb{R}^N$, i is the imaginary unit, $\varepsilon>0$ is the Planck constant, $p,q\in(2,2^*)$, $p>q$, $2^*:=\frac{2N}{N-2}(N>2)$, $V(x)$, $W_1(x)$ and $W_2(x)$ are continuous bounded positive functions. An important issue concerning the above nonlinear evolution equation is to study its standing wave solutions of the form $\psi(x,t) = \mathrm{e}^{-it/\varepsilon} v(x)$. For small $\varepsilon > 0$, these standing wave solutions are referred to as semi-classical states. Byeon and Wang [1] are concerned with the existence and qualitative property of standing waves $\psi(x,t) = \mathrm{e}^{-iEt/\varepsilon} v(x)$ for the following Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\frac{\varepsilon^2}{2} \Delta \psi + V(x)\psi - |\psi|^{p-1} \psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where $\inf_{x \in \mathbb{R}^N} V(x) = E$ with E being a critical frequency. It is easy to see that $\psi(x,t) = \mathrm{e}^{-it/\varepsilon} v(x)$ solves Equation (1.1) if and only if $v(x)$ solves

$$-\varepsilon^2 \Delta v + V(x) v = W_1(x)|v|^{p-2} v + W_2(x)|v|^{q-2} v, \quad x \in \mathbb{R}^N. \tag{1.2}$$

Research on concentration phenomenon began many years ago, Ambrosetti, Badiale and Cingolani [2] considered

$$-\Delta v + (V(x) + \lambda) v = |v|^{p-2} v, \quad x \in \mathbb{R}^N,$$

where $\lambda \in \mathbb{R}$ and v is a real-valued function, $\lim_{|x| \to \infty} v(x) = 0$, V has a possibly degenerate local minimum or maximum at x_0. Up to translations, they assumed that $x_0 = 0$ and $V(0) = 0$, then obtained the solution v_ε concentrates near $x_0 = 0$ as $\varepsilon \to 0$. Wang and Zeng [3] studied the nonlinear elliptic equation with competing poentials V, K, Q

$$-\varepsilon^2 \Delta v + V(x) v = K(x)|v|^{p-2} v + Q(x)|v|^{q-2} v, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $2 < q < p < 2^*$, and they proved the ground state concentrates at a global minimum point of ground energy function by the concentration-compactness lemma. Ding and Liu [4] considered the existence, convergence and concentration phenomena of the ground state solution by using Mountain pass technique for

$$(-i\varepsilon\nabla + A(x))^2 v + V(x) v = W(x)|v|^{p-2} v, \quad x \in \mathbb{R}^N,$$

where $p \in (2, 2^*)$, V and W are bounded positive functions. For other convengence and concentration results on nonlinear elliptic equation, we can refer to [5] [6] [7].

In the past few decades, the research on the multiplicity of solutions has been widely concerned. For example, Cingolani and Lazzo [8] improved the existence result for Equation (1.3) in [3], and they studied the multiple positive solutions by the topology of the global minima set for energy function. Sun [9] studied the existence and multiplicity for a class of the quasilinear elliptic equations by Morse theory and the minimax method. Bartolo and Bisci [10] proved the existence and multiplicity of solutions to a fractional equation whose nonlinearity is subcritical and asymptotically linear at infinity by using a pseudo-index theory related to the genus. Papageorgiou, Rădulescu and Repovš [11] studied the existence and multiplicity to a class of double-phase Robin problems by the Morse theory, and using the notion of homological local linking. Wu, Tahar, Rafik, Rahmoune and Yang [12] established the existence of infinitely many solutions for the sublinear Schrödinger equations by using the linking theorem and the variant fountain theorem. Wang, Cheng and Wang [13] proved the multiplicity of positive solutions for the fractional Kirchhoff-Choquard equation with magnetic fields by using the penalization method and the Ljusternik-Schnirelmann theory. In [14], Guo and Li considered the multiplicity of nontrivial solutions by using a global compactness result and Krasnoselskii's genus theory for the following fractional Schrödinger equation in an open bounded domain of $\mathbb{R}^N$,

$$(-\Delta)^s v + V(x)v = |v|^{\frac{2N}{N-2s}-2} v,$$

where $s \in (0,1)$, $N > 2s$, V is a sign-changing function. For the multiplicity of solutions to the nonlinear Schrödinger equation, we can refer to [15] [16] [17].

Recently, Ding and Wei [18] considered the nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta v + V(x)v = W(x)|v|^{p-2} v, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $p \in (2, 2^*)$, $V(x)$, $W(x)$ are bounded positive functions, and studied the existence, concentration phenomena of the positive ground state and multiplicity of semi-classical solutions by Benci pseudo-index theory and Nehari method. Liu and Tang [19] studied the following Choquard equation

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-\theta} W(x)\left(I_\theta * \left(W|v|^p\right)\right)|v|^{p-2} v, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $N > 2$, I_θ is the Riesz potential with order $\theta \in (0, N)$, $p \in \left[2, \frac{N+\theta}{N-2}\right)$, $\min V > 0$ and $\inf W > 0$, they established the multiplicity of semi-classical solutions by Benci pseudo-index theory and the existence of sign-changing solutions by minimizing the energy on Nehari nodal set, they also studied the concentration phenomenon, convergence, decay estimate of ground state solutions. Similar studies appear in [20] [21].

Motivated by the above works, in this paper, we consider the multiplicity of solutions and the existence, concentration, convergence and decay estimates of the ground state solution for Equation (1.2). There appear the combined nonlinearities in our equation, which make more difficulties in our arguments. Finally, we use the Benci pesudo-index theory to obtain the multiplicity of the semi-classical solutions for Equation (1.2), and we get the sign-changing solutions by resorting to the method. We extend the research in [18] and develop the method in [4] [19] [20].

Our basic assumptions and the main results are the following.

(P1): $V, W_j \in C(\mathbb{R}^N, \mathbb{R})$ are bounded, $V(x)$ attains a global minimum on $\mathbb{R}^N$ with $\min_{\mathbb{R}^N} V(x) > 0$, and $W_j(x)$ attains a global maximum on $\mathbb{R}^N$ with $\inf_{\mathbb{R}^N} W_j(x) > 0$, $j = 1, 2$.

To describe our results, for $j = 1, 2$, we denote by

$$\tau := \min_{\mathbb{R}^N} V, \ \mathscr{V} := \left\{x \in \mathbb{R}^N : V(x) = \tau\right\}, \ \tau_\infty := \liminf_{|x|\to\infty} V(x);$$

$$\varsigma_j := \max_{\mathbb{R}^N} W_j, \ \mathscr{W}_j := \left\{x \in \mathbb{R}^N : W_j(x) = \varsigma_j\right\}, \ \varsigma_{j\infty} := \limsup_{|x|\to\infty} W_j(x).$$

(P2): $\mathscr{W}_1 \cap \mathscr{W}_2 \neq \varnothing$.

We continue to denote by

$$x_{jv} \in \mathscr{V}, \ \varsigma_{jv} := \max_{\mathscr{V}} W_j(x) = W_j(x_{jv}), \ j = 1,2;$$

$$x_w \in \mathscr{W}_1 \cap \mathscr{W}_2, \ \tau_w := \min_{\mathscr{W}_1 \cap \mathscr{W}_2} V(x) = V(x_w).$$

For vector $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$, we set

$$m(a,\vec{b}) = \begin{cases} \left(\frac{\tau_\infty}{a}\right)^{\frac{p}{p-2}-\frac{N}{2}} \left(\frac{b_1}{\varsigma_{1\infty}}\right)^{\frac{2}{p-2}} & \text{if } \left(\frac{\varsigma_{2\infty}}{b_2}\right) \le \left(\frac{a}{\tau_\infty}\right)^{\frac{q-p}{p-2}} \left(\frac{\varsigma_{1\infty}}{b_1}\right)^{\frac{q-2}{p-2}} \\ \left(\frac{\tau_\infty}{a}\right)^{\frac{q}{q-2}-\frac{N}{2}} \left(\frac{b_2}{\varsigma_{2\infty}}\right)^{\frac{2}{q-2}} & \text{otherwise} \end{cases},$$

and let $\vec{\varsigma} = (\varsigma_1, \varsigma_2)$, $\vec{\varsigma}_\infty = (\varsigma_{1\infty}, \varsigma_{2\infty})$, $\vec{\varsigma}_v = (\varsigma_{1v}, \varsigma_{2v})$. For $\vec{b}^i = (b_1^i, b_2^i) \in \mathbb{R}^2 \ (i = 1,2)$, we use $\vec{b}^1 \le \vec{b}^2$ to signify $\min\{b_1^2 - b_1^1, b_2^2 - b_2^1\} \ge 0$, and use $\vec{b}^1 < \vec{b}^2$ to signify $\min\{b_1^2 - b_1^1, b_2^2 - b_2^1\} \ge 0$ and $\max\{b_1^2 - b_1^1, b_2^2 - b_2^1\} > 0$.

(P3): 1) $\tau < \tau_\infty$, and there is $R_v > 0$ such that $W_j(x) \le \varsigma_{jv}$, $j = 1,2$ for $|x| \ge R_v$;

2) $\vec{\varsigma} > \vec{\varsigma}_\infty$, and there is $R_w > 0$ such that $V(x) \ge \tau_w$ for $|x| \ge R_w$.

If $(P3)$ - (1) holds, we set $\mathcal{A}_v := \{x \in \mathscr{V} : W_j(x) = \varsigma_{jv}, j = 1,2\} \cup \{x \notin \mathscr{V} : W_1(x) > \varsigma_{1v} \text{ or } W_2(x) > \varsigma_{2v}\}$. If $(P3)$ - (2) holds, we set $\mathcal{A}_w := \{x \in \mathscr{W}_1 \cap \mathscr{W}_2 : V(x) = \tau_w\} \cup \{x \notin \mathscr{W}_1 \cap \mathscr{W}_2 : V(x) < \tau_w\}$. In the following, $\mathcal{A}$ stands for $\mathcal{A}_v$ in the case $(P3)$ - (1), and $\mathcal{A}_w$ in the case $(P3)$ - (2). Clearly, $\mathcal{A}$ is bounded. Furthermore, $\mathcal{A} = \mathscr{V} \cap (\mathscr{W}_1 \cap \mathscr{W}_2)$, if $\mathscr{V} \cap (\mathscr{W}_1 \cap \mathscr{W}_2)$ is not empty.

Theorem 1.1. Assume that $(P1)$ holds and $\tau < \tau_\infty$, $\vec{\varsigma}_v \ge \vec{\varsigma}_\infty$. Then there exists $m_v \ge m(\tau, \vec{\varsigma}_v)$ such that for the maximal integer $m \in \mathbb{Z}_+$ with $m < m_v$, Equation (1.2) has at least m pairs of solutions for small $\varepsilon > 0$. Moreover, among the solutions, at least one is positive, one is negative and two change sign if $m \ge 2$.

Theorem 1.2. Assume that $(P1)$ - $(P2)$ hold and $\tau_w \le \tau_\infty$, $\vec{\varsigma} > \vec{\varsigma}_\infty$. Then there exists $m_w \ge m(\tau_w, \vec{\varsigma})$ such that for the maximal integer $m \in \mathbb{Z}_+$ with $m < m_w$, Equation (1.2) has at least m pairs of solutions for small $\varepsilon > 0$. Moreover, among the solutions, at least one is positive, one is negative and two change sign if $m \ge 2$.

Theorem 1.3. Assume that $(P1)$ - $(P3)$ hold. Then for $\varepsilon > 0$ large small, Equation (1.2) has a positive ground state solution v_ε. If $V, W_j \in C^1(\mathbb{R}^N, \mathbb{R})$ additionally and $\nabla V, \nabla W_j$ are bounded, $j = 1,2$, then v_ε satisfies that

1) There is a maximum point x_ε of v_ε such that $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathcal{A}) = 0$;

2) There are $C, c > 0$ such that $v_\varepsilon(x) \le Ce^{-\frac{c}{\varepsilon}|x - x_\varepsilon|}$ for all $x \in \mathbb{R}^N$;

3) Setting $\hat{u}_\varepsilon(x) := v_\varepsilon(\varepsilon x + x_\varepsilon)$, then for any sequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, there holds $\hat{u}_\varepsilon \to u$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, where u is a ground state solution of

$$-\Delta u+V(x_0)u=W_1(x_0)u^{p-1}+W_2(x_0)u^{q-1},\quad u>0. \tag{1.4}$$

If particularly $\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2)$ is not empty, then $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon,\mathscr{V}\cap(\mathscr{W}_1\cap\mathscr{W}_2))=0$, and up to a sequence, $\hat{u}_\varepsilon\to u$ in $H^1(\mathbb{R}^N)$ as $\varepsilon\to0$, where u is a ground state solution of

$$-\Delta u+\tau u=\varsigma_1u^{p-1}+\varsigma_2u^{q-1},\quad u>0. \tag{1.5}$$

Now we give some preliminary lemmas which will be useful for our arguments.

Lemma 1.4. ([22]) For every $v\in H^1(\mathbb{R}^N)$ and $v\ge0$, there are $v^*\in H_r^1(\mathbb{R}^N):=\{v\in H^1(\mathbb{R}^N):v(x)=v(|x|)\}$ and $v^*\ge0$ such that

$$\int_{\mathbb{R}^N}|\nabla v^*|^2\,\mathrm{d}x\le\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x,\quad \int_{\mathbb{R}^N}|v^*|^\mu\,\mathrm{d}x=\int_{\mathbb{R}^N}|v|^\mu\,\mathrm{d}x,\quad \forall\,\mu>1.$$

Lemma 1.5. ([22]) The embedding $H^1(\mathbb{R}^N)\hookrightarrow L^\mu(\mathbb{R}^N)$ is continuous for $\mu\in[2,2^*]$ and the embedding $H^1(\mathbb{R}^N)\hookrightarrow L_{loc}^\mu(\mathbb{R}^N)$ is compact for $\mu\in[2,2^*)$. Furthermore, $H_r^1(\mathbb{R}^N)\hookrightarrow L^\mu(\mathbb{R}^N)$ is compact for $\mu\in(2,2^*)$.

Lemma 1.6. ([23]) Let $R>0$ and $\mu\in[2,2^*)$. If $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|v_n(x)|^\mu\,\mathrm{d}x\to0$ as $n\to\infty$, then $v_n\to0$ in $L^t(\mathbb{R}^N)$ for $t\in(2,2^*)$ as $n\to\infty$.

For simplicity, we denote by

$$\|v\|:=\|v\|_{H^1(\mathbb{R}^N)},\quad |v|_\mu:=\|v\|_{L^\mu(\mathbb{R}^N)},\quad (u,v):=(u,v)_{H^1(\mathbb{R}^N)}.$$

$$v^+:=\max\{0,v\},\quad v^-:=\min\{0,v\},\quad \mathbb{R}_+:=(0,\infty),\quad \mathbb{Z}_+:=\mathbb{Z}\cap\mathbb{R}_+.$$

And we shall use different patterns of C to denote any positive constant, whose values may change from line to line, and $o(1)$ to denote the quantities that tend to 0 as $n\to\infty$ or $k\to\infty$.

This paper is organized as follows: In Section 2, we give some preliminary results which are proved by Nehari method and play a key role in the arguments of main theorems. In Section 3, we prove the multiplicity of semi-classical solutions by using Benci pseudo-index theory and show the existence of sign-changing solutions. In order to get more detailed and accurate characterization of the properties of solutions, we also study the convergence, concentration phenomenon, and exponential decay estimates of the positive ground state solution.

2. Preliminary Results

2.1. Constant Coefficient Equation

We first consider the following equation

$$-\Delta u+au=b_1|u|^{p-2}u+b_2|u|^{q-2}u,\quad u\in H^1(\mathbb{R}^N), \tag{2.1}$$

where $p,q\in(2,2^*)$, $p>q$, $a>0$, $b_j>0$, $j=1,2$.

For each $u\in H^1(\mathbb{R}^N)$, the energy functional associated to Equation (2.1) is

$$\mathcal{J}^{a\vec{b}}(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+au^2)\mathrm{d}x-\frac{b_1}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x-\frac{b_2}{q}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x.$$

The weak solutions of Equation (2.1) are critical points of $\mathcal{J}^{a\vec{b}} \in C^1\left(H^1\left(\mathbb{R}^N\right),\mathbb{R}\right)$. We denote the least energy by $\Theta^{a\vec{b}} = \inf_{\mathcal{N}^{a\vec{b}}} \mathcal{J}^{a\vec{b}}$, where $\mathcal{N}^{a\vec{b}} := \left\{u \in H^1\left(\mathbb{R}^N\right)\setminus\{0\} : \left\langle \left(\mathcal{J}^{a\vec{b}}\right)'(u), u\right\rangle = 0\right\}$ is the Nehari manifold. The set of least energy solutions can be denoted by $\mathcal{S}^{a\vec{b}} = \left\{u \in H^1\left(\mathbb{R}^N\right) : \mathcal{J}^{a\vec{b}}(u) = \Theta^{a\vec{b}}, \left(\mathcal{J}^{a\vec{b}}\right)'(u) = 0\right\}$. In particular, we set $\mathcal{J}^\infty := \mathcal{J}^{\tau_\infty\vec{\varsigma}_\infty}$, $\mathcal{N}^\infty := \mathcal{N}^{\tau_\infty\vec{\varsigma}_\infty}$ and $\Theta^\infty := \Theta^{\tau_\infty\vec{\varsigma}_\infty}$.

Lemma 2.1. The functional $\mathcal{J}^{a\vec{b}}$ satisfies that

1) There exist $\rho > 0$ and $\kappa > 0$ such that $\mathcal{J}^{a\vec{b}}(u) > \kappa$ for all $\|u\| = \rho$;

2) For $u \neq 0$, $\mathcal{J}^{a\vec{b}}(tu) \to -\infty$ as $t \to +\infty$.

Similar to the proof of Lemma 2.4 in [20], we have the following result.

Lemma 2.2. Let $\Upsilon^{a\vec{b}} := \left\{\gamma \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right) : \gamma(0) = 0, \mathcal{J}^{a\vec{b}}(\gamma(1)) < 0\right\}$, then

$$\Theta^{a\vec{b}} = \inf_{u \in H^1\left(\mathbb{R}^N\right)\setminus\{0\}} \max_{t \geq 0} \mathcal{J}^{a\vec{b}}(tu) = \inf_{\gamma \in \Upsilon^{a\vec{b}}} \max_{t \in [0,1]} \mathcal{J}^{a\vec{b}}(\gamma(t)) > 0.$$

Lemma 2.3. $\Theta^{a\vec{b}}$ is achieved and $\mathcal{S}^{a\vec{b}}$ is compact in $H^1\left(\mathbb{R}^N\right)$.

Proof. For any $u \in H^1\left(\mathbb{R}^N\right)$, we choose the equivalent norm $\|u\|_1^2 = \int_{\mathbb{R}^N}\left(|\nabla u|^2 + au^2\right)\mathrm{d}x$. Clearly, $\mathcal{N}^{a\vec{b}}$ is not empty. Let $u_n \in \mathcal{N}^{a\vec{b}}$ with $u_n \geq 0$ and $\mathcal{J}^{a\vec{b}}(u_n) \to \Theta^{a\vec{b}}$ as $n \to \infty$, By Lemma 1.4, there is $u_n^* \in H_r^1\left(\mathbb{R}^N\right)$ with $u_n^* \geq 0$ such that $\|u_n^*\|_1 \leq \|u_n\|_1$, $|u_n^*|_p = |u_n|_p$, $|u_n^*|_q = |u_n|_q$. Observe that $\|u_n^*\|_1^2 \leq b_1|u_n^*|_p^p + b_2|u_n^*|_q^q$. If $\|u_n^*\|_1^2 = b_1|u_n^*|_p^p + b_2|u_n^*|_q^q$, then $u_n^* \in \mathcal{N}^{a\vec{b}}$. If $\|u_n^*\|_1^2 < b_1|u_n^*|_p^p + b_2|u_n^*|_q^q$, then there exists $t_n \in (0,1)$ such that $t_n u_n^* \in \mathcal{N}^{a\vec{b}}$ and $\Theta^{a\vec{b}} \leq \mathcal{J}^{a\vec{b}}(t_n u_n^*) < \frac{q-2}{2q}\|u_n\|_1^2 + \frac{p-q}{pq}b_1|u_n|_p^p = \mathcal{J}^{a\vec{b}}(u_n) \to \Theta^{a\vec{b}}$ as $n \to \infty$. Hence $\mathcal{J}^{a\vec{b}}(t_n u_n^*) \to \Theta^{a\vec{b}}$ as $n \to \infty$. Define $w_n := t_n u_n^*$, then $w_n \in H_r^1\left(\mathbb{R}^N\right) \cap \mathcal{N}^{a\vec{b}}$, $w_n \geq 0$, and $\mathcal{J}^{a\vec{b}}(w_n) \to \Theta^{a\vec{b}}$ as $n \to \infty$.

Clearly, $\{w_n\} \subset H^1\left(\mathbb{R}^N\right)$ is bounded. We assume $w_n \rightharpoonup w$ in $H_r^1\left(\mathbb{R}^N\right)$ as $n \to \infty$ up to a subsequence if necessary. By Lemma 1.5 and $w_n \in \mathcal{N}^{a\vec{b}}$, we have $\|w_n\|_1^2 \leq C\left(\|w_n\|_1^p + \|w_n\|_1^q\right)$, which ensures that $w \neq 0$ by letting $n \to \infty$. Due to the weakly lower semi-continuity of norm, we obtain $\|w\|_1^2 \leq b_1|w|_p^p + b_2|w|_q^q$. Thus we have $w \in \mathcal{N}^{a\vec{b}}$.

In the end, we can obtain $\left(\mathcal{J}^{a\vec{b}}\right)'(w) = 0$, where $w \in \mathcal{S}^{a\vec{b}}$ is positive and radially symmetric. With similar arguments as aboves, $\mathcal{S}^{a\vec{b}}$ is compact in $H^1\left(\mathbb{R}^N\right)$.

Lemma 2.4. Let $a_i > 0$, $b_i^1, b_i^2 > 0$, $i = 1,2$.

1) If $\min\left\{a_2 - a_1, b_1^1 - b_2^1, b_1^2 - b_2^2\right\} \geq 0$, then $\Theta^{a_1\vec{b}_1} \leq \Theta^{a_2\vec{b}_2}$;

2) If $\min\left\{a_2 - a_1, b_1^1 - b_2^1, b_1^2 - b_2^2\right\} \geq 0$ and $\max\left\{a_2 - a_1, b_1^1 - b_2^1, b_1^2 - b_2^2\right\} > 0$, then $\Theta^{a_1\vec{b}_1} < \Theta^{a_2\vec{b}_2}$.

Lemma 2.5. If u is a ground state solution of

$$-\Delta u + \tau_\infty u = \varsigma_{1\infty}|u|^{p-2}u + \varsigma_{2\infty}|u|^{q-2}u, \quad u \in H^1\left(\mathbb{R}^N\right) \tag{2.2}$$

with the energy Θ^{∞}. Setting $z(x)=\lambda u\left(\sqrt{\frac{a}{\tau_{\infty}}}x\right)$, then z is a ground state solution of

$$-\Delta z+az=\left(\frac{a\varsigma_{1\infty}}{\tau_{\infty}b_1}\lambda^{2-p}\right)b_1|z|^{p-2}z+\left(\frac{a\varsigma_{2\infty}}{\tau_{\infty}b_2}\lambda^{2-q}\right)b_2|z|^{q-2}z,\quad z\in H^1\left(\mathbb{R}^N\right) \tag{2.3}$$

with the energy $\Theta(\lambda)=\lambda^2\left(\frac{a}{\tau_{\infty}}\right)^{1-\frac{N}{2}}\Theta^{\infty}$.

Proof. Observe that u is a ground state solution of Equation (2.2) if and only if z is a ground state solution of Equation (2.3). Indeed,

$$\begin{aligned}-\Delta z+az&=\frac{\lambda a}{\tau_{\infty}}\left(-\Delta u\left(\sqrt{\frac{a}{\tau_{\infty}}}x\right)+\tau_{\infty}u\left(\sqrt{\frac{a}{\tau_{\infty}}}x\right)\right)\\&=\left(\frac{a\varsigma_{1\infty}}{\tau_{\infty}}\lambda^{2-p}\right)|z|^{p-2}z+\left(\frac{a\varsigma_{2\infty}}{\tau_{\infty}}\lambda^{2-q}\right)|z|^{q-2}z.\end{aligned}$$

Furthermore, $u\in\mathcal{N}^{\infty}$ if and only if $z\in\mathcal{N}(\lambda)$. Hence

$\Theta(\lambda)=\lambda^2\left(\frac{a}{\tau_{\infty}}\right)^{1-\frac{N}{2}}\Theta^{\infty}$.

Lemma 2.6. Assume that $a\le\tau_{\infty}$, $\vec{b}\ge\vec{\varsigma}_{\infty}$. Then $m(a,\vec{b})\Theta^{a\vec{b}}\le\Theta^{\infty}$.

Proof. Note that if $\lambda>0$ satisfies $\max\left\{\frac{a\varsigma_{1\infty}}{\tau_{\infty}b_1}\lambda^{2-p},\frac{a\varsigma_{2\infty}}{\tau_{\infty}b_2}\lambda^{2-q}\right\}\le 1$, we have $\Theta^{a\vec{b}}\le\Theta(\lambda)$. By the definition of $m(a,\vec{b})$, we can find two cases:

$$\left(\frac{\varsigma_{2\infty}}{b_2}\right)\le\left(\frac{a}{\tau_{\infty}}\right)^{\frac{q-p}{p-2}}\left(\frac{\varsigma_{1\infty}}{b_1}\right)^{\frac{q-2}{p-2}} \tag{2.4}$$

or

$$\left(\frac{\varsigma_{1\infty}}{b_1}\right)<\left(\frac{a}{\tau_{\infty}}\right)^{\frac{p-q}{q-2}}\left(\frac{\varsigma_{2\infty}}{b_2}\right)^{\frac{p-2}{q-2}}. \tag{2.5}$$

If (2.4) holds, we choose $\lambda=\left(\frac{a\varsigma_{1\infty}}{\tau_{\infty}b_1}\right)^{\frac{1}{p-2}}$, then

$\Theta(\lambda)=\left(\frac{a}{\tau_{\infty}}\right)^{\frac{p}{p-2}-\frac{N}{2}}\cdot\left(\frac{\varsigma_{1\infty}}{b_1}\right)^{\frac{2}{p-2}}\Theta^{\infty}$. Thus we get $m(a,\vec{b})\Theta^{a\vec{b}}\le\Theta^{\infty}$.

If (2.5) holds, we choose $\lambda=\left(\frac{a\varsigma_{2\infty}}{\tau_{\infty}b_2}\right)^{\frac{1}{q-2}}$, then

$\Theta(\lambda)=\left(\frac{a}{\tau_{\infty}}\right)^{\frac{q}{q-2}-\frac{N}{2}}\cdot\left(\frac{\varsigma_{2\infty}}{b_2}\right)^{\frac{2}{q-2}}\Theta^{\infty}$. Thus we get $m(a,\vec{b})\Theta^{a\vec{b}}\le\Theta^{\infty}$.

Lemma 2.7. If $\tau<\tau_{\infty},\vec{\varsigma}_v\ge\vec{\varsigma}_{\infty}$, then $m(\tau,\vec{\varsigma}_v)>1$ and $\Theta^{\tau\vec{\varsigma}_v}<\Theta^{\infty}$. If $\tau_w\le\tau_{\infty},\vec{\varsigma}>\vec{\varsigma}_{\infty}$, then $m(\tau_w,\vec{\varsigma})\ge 1$ and $\Theta^{\tau_w\vec{\varsigma}}<\Theta^{\infty}$.

Proof. Choose $a=\tau, b_j=\varsigma_{jv}, j=1,2$ in Equation (2.1), Equations (2.3), (2.4) and (2.5), respectively. By the definition of $m(\tau,\vec{\varsigma}_v)$, we have $m(\tau,\vec{\varsigma}_v)>1$. By Lemma 2.6, we have $\Theta^{\tau\vec{\varsigma}_v}<\Theta^\infty$.

Similarly, we choose $a=\tau_w, b_j=\varsigma_j, j=1,2$ in Equation (2.1), Equations (2.3), (2.4) and (2.5), respectively. By the definition of $m(\tau_w,\vec{\varsigma})$, we have $m(\tau_w,\vec{\varsigma})\ge 1$. If (2.4) holds, we choose $\lambda=\left(\frac{\tau_w\varsigma_{1\infty}}{\tau_\infty\varsigma_1}\right)^{\frac{1}{p-2}}$, then $\Theta^{\tau_w\vec{\varsigma}}\le\Theta(\lambda)\le\Theta^\infty$ by Lemma 2.4 and Lemma 2.5. If $\varsigma_1>\varsigma_{1\infty}$, then $\Theta^{\tau_w\vec{\varsigma}}\le\Theta(\lambda)<\Theta^\infty$ by Lemma 2.5. If $\varsigma_2>\varsigma_{2\infty}$, then $\Theta^{\tau_w\vec{\varsigma}}<\Theta(\lambda)\le\Theta^\infty$ by Lemma 2.4. Hence $\Theta^{\tau_w\vec{\varsigma}}<\Theta^\infty$. If (2.5) holds, we choose $\lambda=\left(\frac{\tau_w\varsigma_{2\infty}}{\tau_\infty\varsigma_2}\right)^{\frac{1}{q-2}}$, then $\Theta^{\tau_w\vec{\varsigma}}\le\Theta(\lambda)\le\Theta^\infty$. If $\varsigma_1>\varsigma_{1\infty}$, then $\Theta^{\tau_w\vec{\varsigma}}<\Theta(\lambda)\le\Theta^\infty$. If $\varsigma_2>\varsigma_{2\infty}$, then $\Theta^{\tau_w\vec{\varsigma}}\le\Theta(\lambda)<\Theta^\infty$. Thus $\Theta^{\tau_w\vec{\varsigma}}<\Theta^\infty$.

Lemma 2.8. There exist constants $C,c>0$ such that for every $u\in\mathcal{S}^{a\vec{b}}$, $u(x)\le C\mathrm{e}^{-c|x|}$ for all $x\in\mathbb{R}^N$.

Proof. Let $a':=a-b_1u^{p-2}-b_2u^{q-2}$, we obtain $-\Delta u+a'u=0$. For R large enough, we get $2a'\ge a$ for $|x|\ge R$. Define $\phi(x)=C_1\mathrm{e}^{-c_2|x|}$ or $\phi(s)=C_1\mathrm{e}^{-c_2 s}$, where $C_1>0$, $s=|x|$. Choose C_1 large enough such that $\phi(x)\ge u(x)$ for $|x|=R$. Since $-\Delta\phi(x)+a'\phi(x)\ge\left(\frac{a}{2}-c_2^2\right)\phi(s)$, we choose $0<c_2\le\sqrt{\frac{a}{2}}$ such that $-\Delta\phi(x)+a'\phi(x)\ge 0$ for $|x|\ge R$. Therefore,

$$\begin{cases}-\Delta\phi(x)+a'\phi(x)\ge-\Delta u+a'u & \forall|x|\ge R\\ \phi(x)\ge u(x) & \forall|x|=R\end{cases}$$

By comparison principle, $\phi(x)\ge u(x)$ for all $|x|\ge R$, then $u(x)\le C_1\mathrm{e}^{-c_2|x|}$ for all $|x|\ge R$. For C_1 large enough, we get that $u(x)\le C_1\mathrm{e}^{-c_2|x|}$ for all $|x|<R$. Thus $u(x)\le C_1\mathrm{e}^{-c_2|x|}$ for all $x\in\mathbb{R}^N$.

2.2. Auxiliary Equation

In this subsection, we consider the following equation for $p,q\in(2,2^*)$ and $p>q$,

$$-\Delta u+V_\varepsilon^a(x)u=W_{1\varepsilon}^{b_1}(x)|u|^{p-2}u+W_{2\varepsilon}^{b_2}(x)|u|^{q-2}u,\quad u\in H^1(\mathbb{R}^N),\tag{2.6}$$

where $\tau\le a\le\tau_\infty$, $\vec{\varsigma}\ge\vec{b}\ge\vec{\varsigma}_\infty$, $V_\varepsilon^a(x):=V^a(\varepsilon x):=\max\{a,V(x)\}$ and $W_{j\varepsilon}^{b_j}(x):=W_j^{b_j}(\varepsilon x):=\min\{b_j,W_j(x)\}$, $j=1,2$.

For each $u\in H^1(\mathbb{R}^N)$, the energy functional associated to Equation (2.6) is

$$\mathcal{J}_\varepsilon^{a\vec{b}}(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V_\varepsilon^a(x)u^2\right)\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^N}W_{1\varepsilon}^{b_1}(x)|u|^p\,\mathrm{d}x$$
$$-\frac{1}{q}\int_{\mathbb{R}^N}W_{2\varepsilon}^{b_2}(x)|u|^q\,\mathrm{d}x.$$

The weak solutions of Equation (2.6) are critical points of $\mathcal{J}_\varepsilon^{a\vec{b}}\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$. We denote the least energy by $\Theta_\varepsilon^{a\vec{b}}=\inf_{\mathcal{N}_\varepsilon^{a\vec{b}}}\mathcal{J}_\varepsilon^{a\vec{b}}$, where

$\mathcal{N}_{\varepsilon}^{a\vec{b}} := \left\{ u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} : \left\langle \left(\mathcal{J}_{\varepsilon}^{a\vec{b}}\right)'(u), u \right\rangle = 0 \right\}$ is the Nehari manifold. The set of least energy solutions was denoted by

$\mathcal{S}_{\varepsilon}^{a\vec{b}} = \left\{ u_{\varepsilon} \in H^1\left(\mathbb{R}^N\right) : \mathcal{J}_{\varepsilon}^{a\vec{b}}(u_{\varepsilon}) = \Theta_{\varepsilon}^{a\vec{b}}, \left(\mathcal{J}_{\varepsilon}^{a\vec{b}}\right)'(u_{\varepsilon}) = 0 \right\}$. In particular, we set

$$\mathcal{J}_{\varepsilon}^{\infty} := \mathcal{J}_{\varepsilon}^{\tau_\infty \vec{\varsigma}_\infty},\ \mathcal{N}_{\varepsilon}^{\infty} := \mathcal{N}_{\varepsilon}^{\tau_\infty \vec{\varsigma}_\infty},\ \Theta_{\varepsilon}^{\infty} := \Theta_{\varepsilon}^{\tau_\infty \vec{\varsigma}_\infty};$$

$$V_{\varepsilon}^{\infty} := V_{\varepsilon}^{\tau_\infty},\ W_{j\varepsilon}^{\infty} := W_{j\varepsilon}^{\varsigma_{j\infty}},\ j = 1,2.$$

Lemma 2.9. The functional $\mathcal{J}_{\varepsilon}^{a\vec{b}}$ satisfies that

1) There exist $\rho > 0$ and $\kappa > 0$ both dependent on $N, p, q, \tau, \vec{\varsigma}$ and independent of $a, \vec{b}$ such that $\mathcal{J}_{\varepsilon}^{a\vec{b}}(u) > \kappa$ for all $\|u\| = \rho$;

2) For $u \neq 0$, $\mathcal{J}_{\varepsilon}^{a\vec{b}}(tu) \to -\infty$ as $t \to +\infty$.

Lemma 2.10. Let $\Upsilon_{\varepsilon}^{a\vec{b}} := \left\{ \gamma \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right) : \gamma(0) = 0, \mathcal{J}_{\varepsilon}^{a\vec{b}}(\gamma(1)) < 0 \right\}$, then

$$\Theta_{\varepsilon}^{a\vec{b}} = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} \mathcal{J}_{\varepsilon}^{a\vec{b}}(tu) = \inf_{\gamma \in \Upsilon_{\varepsilon}^{a\vec{b}}} \max_{t \in [0,1]} \mathcal{J}_{\varepsilon}^{a\vec{b}}(\gamma(t)) > 0.$$

Lemma 2.11. If $\mathcal{J}_{\varepsilon}^{\infty}$ has a $(PS)_c$ sequence, then either $c = 0$ or $c \geq \Theta_{\varepsilon}^{\infty}$. Furthermore, $\Theta_{\varepsilon}^{\infty} \geq \Theta^{\infty}$.

Proof. Let $\{u_n\} \subset H^1\left(\mathbb{R}^N\right)$ is a $(PS)_c$ sequence of $\mathcal{J}_{\varepsilon}^{\infty}$, then $\mathcal{J}_{\varepsilon}^{\infty}(u_n) \to c$ and $\left(\mathcal{J}_{\varepsilon}^{\infty}\right)'(u_n) \to 0$ in $H^{-1}\left(\mathbb{R}^N\right)$ as $n \to \infty$. We will show that $c \geq \Theta_{\varepsilon}^{\infty}$ when $c \neq 0$. Since $\{u_n\} \subset H^1\left(\mathbb{R}^N\right)$ is bounded, we may assume $u_n \rightharpoonup u$ in $H^1\left(\mathbb{R}^N\right)$ as $n \to \infty$. Hence $\left(\mathcal{J}_{\varepsilon}^{\infty}\right)'(u) = 0$. Set $y_n := u_n - u$. By Lemma 1.32 in [23], we obtain

$$\mathcal{J}_{\varepsilon}^{\infty}(y_n) \to c - \mathcal{J}_{\varepsilon}^{\infty}(u) \quad \text{as } n \to \infty. \tag{2.7}$$

For all $\varphi \in H^1\left(\mathbb{R}^N\right)$, we have

$$\begin{aligned} &\left\langle \left(\mathcal{J}_{\varepsilon}^{\infty}\right)'(y_n), \varphi \right\rangle - \left\langle \left(\mathcal{J}_{\varepsilon}^{\infty}\right)'(u_n), \varphi \right\rangle \\ &= \int_{\mathbb{R}^N} W_{1\varepsilon}^{\infty}(x)\left(|u_n|^{p-2}u_n - |y_n|^{p-2}y_n - |u|^{p-2}u\right)\varphi \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} W_{2\varepsilon}^{\infty}(x)\left(|u_n|^{q-2}u_n - |y_n|^{q-2}y_n - |u|^{q-2}u\right)\varphi \mathrm{d}x. \end{aligned} \tag{2.8}$$

For any $\sigma > 0$, there is $R > 0$ such that $\int_{|x|>R} |u|^p \mathrm{d}x < \sigma^p$ and $\int_{|x|>R} |u|^q \mathrm{d}x < \sigma^q$. By mean value theorem and Hölder inequality, we obtain $\int_{|x|>R} \left| \left(|u_n|^{p-2}u_n - |y_n|^{p-2}y_n\right)\varphi \right| \mathrm{d}x \leq C\sigma\|\varphi\|$. Furthermore, by Hölder inequality again, we get that $\int_{|x|>R} \left| |u|^{p-2}u\varphi \right| \mathrm{d}x \leq C\sigma\|\varphi\|$. Thus

$$\left| \int_{|x|>R} W_{1\varepsilon}^{\infty}(x)\left(|u_n|^{p-2}u_n - |y_n|^{p-2}y_n - |u|^{p-2}u\right)\varphi \mathrm{d}x \right| \leq C\sigma\|\varphi\|. \tag{2.9}$$

Similarly,

$$\left| \int_{|x|>R} W_{2\varepsilon}^{\infty}(x)\left(|u_n|^{q-2}u_n - |y_n|^{q-2}y_n - |u|^{q-2}u\right)\varphi \mathrm{d}x \right| \leq C\sigma\|\varphi\|. \tag{2.10}$$

By Lemma 1.5, we obtain $|u_n|^{\mu-2}u_n-|u|^{\mu-2}u\to 0$ in $L_{loc}^{\frac{\mu}{\mu-1}}(\mathbb{R}^N)$ as $n\to\infty$ with $\mu=p,q$, respectively. Hence

$$\left|\int_{|x|\le R}W_{1\varepsilon}^{\infty}(x)\left(|u_n|^{p-2}u_n-|y_n|^{p-2}y_n-|u|^{p-2}u\right)\varphi\mathrm{d}x\right|=o(1)\|\varphi\|. \tag{2.11}$$

Similarly,

$$\left|\int_{|x|\le R}W_{2\varepsilon}^{\infty}(x)\left(|u_n|^{q-2}u_n-|y_n|^{q-2}y_n-|u|^{q-2}u\right)\varphi\mathrm{d}x\right|=o(1)\|\varphi\|. \tag{2.12}$$

By (2.8) - (2.12), we get that $\left(J_\varepsilon^\infty\right)'(y_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n\to\infty$.

For all $n\in\mathbb{Z}_+$, if $y_n\neq 0$, there is $t_n>0$ such that $t_ny_n\in\mathcal{N}_\varepsilon^\infty$. Thus

$$\mathcal{J}_\varepsilon^\infty(t_ny_n)\ge\Theta_\varepsilon^\infty \tag{2.13}$$

and $0=\left\langle\left(\mathcal{J}_\varepsilon^\infty\right)'(t_ny_n),t_ny_n\right\rangle$. By $o(1)=\left\langle\left(\mathcal{J}_\varepsilon^\infty\right)'(y_n),y_n\right\rangle$, then we obtain

$$o(1)=\left(1-t_n^{p-2}\right)\int_{\mathbb{R}^N}W_{1\varepsilon}^{\infty}(x)|y_n|^p\mathrm{d}x+\left(1-t_n^{q-2}\right)\int_{\mathbb{R}^N}W_{2\varepsilon}^{\infty}(x)|y_n|^q\mathrm{d}x. \tag{2.14}$$

Moreover, $\|y_n\|^2\le C\int_{\mathbb{R}^N}\left(|\nabla y_n|^2+V_\varepsilon^\infty(x)y_n^2\right)\mathrm{d}x\le C\left(|y_n|_p^p+|y_n|_q^q\right)+o(1)$. If $|y_n|_p\to 0$ and $|y_n|_q\to 0$ as $n\to\infty$, we can get $u_n\to u$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$ and $c\ge\Theta_\varepsilon^\infty$. If $|y_n|_p\ge\sigma>0$ or $|y_n|_q\ge\sigma>0$ in (2.14), we obtain $t_n\to 1$ as $n\to\infty$. Hence by (2.7), we have $\mathcal{J}_\varepsilon^\infty(t_ny_n)\to c-\mathcal{J}_\varepsilon^\infty(u)$ as $n\to\infty$. By (2.13), we get that $c\ge\Theta_\varepsilon^\infty$. If there is $y_{n_k}\equiv 0$, then $\mathcal{J}_\varepsilon^\infty(u)=c\neq 0$ and $u\in\mathcal{N}_\varepsilon^\infty$. Hence $c\ge\Theta_\varepsilon^\infty$.

Observe that $\mathcal{J}_\varepsilon^\infty(u)\ge\mathcal{J}^\infty(u)$ for all $u\in H^1(\mathbb{R}^N)$. According to Lemma 2.2 and Lemma 2.10, we obtain $\Theta_\varepsilon^\infty\ge\Theta^\infty$.

Similar to the proof of Lemma 2.11, we also have the following result.

Lemma 2.12. If $\mathcal{J}_\varepsilon^{a\vec{b}}$ has a $(PS)_c$ sequence, then either $c=0$ or $c\ge\Theta_\varepsilon^{a\vec{b}}$.

Lemma 2.13. For all $c<\Theta_\varepsilon^\infty$, $\mathcal{J}_\varepsilon^{a\vec{b}}$ satisfies $(PS)_c$ condition.

Proof. Let $\{u_n\}\subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence of $\mathcal{J}_\varepsilon^{a\vec{b}}$, then $\mathcal{J}_\varepsilon^{a\vec{b}}(u_n)\to c$ and $\left(\mathcal{J}_\varepsilon^{a\vec{b}}\right)'(u_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n\to\infty$. We assume $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$. Hence $\left(\mathcal{J}_\varepsilon^{a\vec{b}}\right)'(u)=0$. Set $y_n:=u_n-u$. Due to the proof of Lemma 2.11, we obtain

$$\mathcal{J}_\varepsilon^{a\vec{b}}(y_n)\to c-\mathcal{J}_\varepsilon^{a\vec{b}}(u),\ \left(\mathcal{J}_\varepsilon^{a\vec{b}}\right)'(y_n)\to 0 \text{ in } H^{-1}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{2.15}$$

Next, we will show that $\mathcal{J}_\varepsilon^\infty(y_n)\to c-\mathcal{J}_\varepsilon^{a\vec{b}}(u)$ and $\left(\mathcal{J}_\varepsilon^\infty\right)'(y_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n\to\infty$. By definition, we get that for all $\sigma>0$, there is $\bar{R}>0$ such that for all $|x|>\bar{R}$,

$$\left|V_\varepsilon^\infty(x)-V_\varepsilon^a(x)\right|\le\sigma,\quad\left|W_{j\varepsilon}^\infty(x)-W_{j\varepsilon}^{b_j}(x)\right|\le\sigma,\ \ j=1,2. \tag{2.16}$$

Thus by (2.16), we have

$$\begin{aligned}&\left|\mathcal{J}_\varepsilon^\infty(y_n)-\mathcal{J}_\varepsilon^{a\vec{b}}(y_n)\right|\\&\le\sigma\left(\frac{1}{2}|y_n|_2^2+\frac{1}{p}|y_n|_p^p+\frac{1}{q}|y_n|_q^q\right)+C\left(|y_n|_{L^2(B_{\bar{R}})}^2+|y_n|_{L^p(B_{\bar{R}})}^p+|y_n|_{L^q(B_{\bar{R}})}^q\right),\end{aligned}$$

which together with Lemma 1.5 and (2.15) imply that

$$\mathcal{J}_\varepsilon^\infty(y_n) \to c - \mathcal{J}_\varepsilon^{a\vec{b}}(u) \quad \text{as } n \to \infty. \tag{2.17}$$

Similarly, by Lemma 1.5 and (2.15) again, we have

$$\left(\mathcal{J}_\varepsilon^\infty\right)'(y_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{2.18}$$

By (2.17) and (2.18), we obtain $\{y_n\}$ is a $(PS)_{c-\mathcal{J}_\varepsilon^{a\vec{b}}(u)}$ sequence of $\mathcal{J}_\varepsilon^\infty$. By Lemma 2.11, either $c = \mathcal{J}_\varepsilon^{a\vec{b}}(u)$ or $c \geq \mathcal{J}_\varepsilon^{a\vec{b}}(u) + \Theta_\varepsilon^\infty$. The latter contradicts our assumption $c < \Theta_\varepsilon^\infty$. Hence $c = \mathcal{J}_\varepsilon^{a\vec{b}}(u)$ and

$$\mathcal{J}_\varepsilon^{a\vec{b}}(u_n) \to \mathcal{J}_\varepsilon^{a\vec{b}}(u) \quad \text{as } n \to \infty. \tag{2.19}$$

Now we prove that $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Since $o(1) = \left\langle \left(\mathcal{J}_\varepsilon^{a\vec{b}}\right)'(u_n), u_n \right\rangle$, we get

$\mathcal{J}_\varepsilon^{a\vec{b}}(u_n) = \frac{q-2}{2q}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + V_\varepsilon^a(x)u_n^2\right)dx + \frac{p-q}{pq}\int_{\mathbb{R}^N} W_{1\varepsilon}^{b_1}(x)|u_n|^p dx + o(1)$. By $0 = \left\langle \left(\mathcal{J}_\varepsilon^{a\vec{b}}\right)'(u), u \right\rangle$, we obtain

$\mathcal{J}_\varepsilon^{a\vec{b}}(u) = \frac{q-2}{2q}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V_\varepsilon^a(x)u^2\right)dx + \frac{p-q}{pq}\int_{\mathbb{R}^N} W_{1\varepsilon}^{b_1}(x)|u|^p dx$. By Lemma 1.6, we assume there exist $R > 0$ and $\delta > 0$ such that $\int_{B_R(x_n)}|y_n|^p dx \geq \delta > 0$ for some $x_n \in \mathbb{R}^N$. Moreover,

$$\begin{aligned}\lim_{n\to\infty}\mathcal{J}_\varepsilon^{a\vec{b}}(y_n) &\geq \lim_{n\to\infty}\left(\frac{p-q}{pq}\int_{B_R(x_n)} W_{1\varepsilon}^{b_1}(x)|y_n|^p dx + o(1)\right) \\ &\geq \lim_{n\to\infty}\left(C\int_{B_R(x_n)}|y_n|^p dx + o(1)\right) \geq C\delta > 0,\end{aligned}$$

which is impossible. By (2.19), we conclude that $\|u_n\| \to \|u\|$ as $n \to \infty$. Thus $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 2.14. $\limsup_{\varepsilon\to 0}\Theta_\varepsilon^{a\vec{b}} \leq \Theta^{\tilde{a}\vec{\tilde{b}}}$, where $\tilde{a} = V^a(0)$, $\tilde{b}_j = W_j^{b_j}(0)$, $j = 1,2$, $\vec{\tilde{b}} = (\tilde{b}_1, \tilde{b}_2)$. Meanwhile, if $V(0) \leq a$, $W_j(0) \geq b_j$, $j = 1,2$, then $\lim_{\varepsilon\to 0}\Theta_\varepsilon^{a\vec{b}} = \Theta^{a\vec{b}}$.

Proof. Setting $\tilde{V}_\varepsilon(x) := V_\varepsilon^a(x) - \tilde{a}$ and $\tilde{W}_{j\varepsilon}(x) := \tilde{b}_j - W_{j\varepsilon}^{b_j}(x)$, $j = 1,2$, we have

$$\tilde{V}_\varepsilon(x) \to 0,\ \tilde{W}_{j\varepsilon}(x) \to 0,\ j = 1,2 \ \text{ a.e. on } \mathbb{R}^N \quad \text{as } \varepsilon \to 0. \tag{2.20}$$

Furthermore,

$$\mathcal{J}_\varepsilon^{a\vec{b}}(u) - \mathcal{J}^{\tilde{a}\vec{\tilde{b}}}(u) = \frac{\int_{\mathbb{R}^N}\tilde{V}_\varepsilon(x)u^2 dx}{2} + \frac{\int_{\mathbb{R}^N}\tilde{W}_{1\varepsilon}(x)|u|^p dx}{p} + \frac{\int_{\mathbb{R}^N}\tilde{W}_{2\varepsilon}(x)|u|^q dx}{q}. \tag{2.21}$$

Due to Lemma 2.3, there is $\alpha \in \mathcal{S}^{\tilde{a}\vec{\tilde{b}}}$ satisfying $\mathcal{J}^{\tilde{a}\vec{\tilde{b}}}(\alpha) = \Theta^{\tilde{a}\vec{\tilde{b}}}$ for $\alpha \in \mathcal{N}^{\tilde{a}\vec{\tilde{b}}}$. Let $t_\varepsilon > 0$ such that $t_\varepsilon\alpha \in \mathcal{N}_\varepsilon^{a\vec{b}}$, we obtain

$$\max_{t\geq 0}\mathcal{J}_{\varepsilon}^{a\vec{b}}(t\alpha)=\mathcal{J}_{\varepsilon}^{a\vec{b}}(t_{\varepsilon}\alpha)\geq\Theta_{\varepsilon}^{a\vec{b}}. \tag{2.22}$$

Observe that $\lim_{t\to+\infty}\mathcal{J}_{\varepsilon}^{a\vec{b}}(t\alpha)=-\infty$, there is $T>0$ such that

$$\mathcal{J}_{\varepsilon}^{a\vec{b}}(t\alpha)<0,\quad \forall t>T. \tag{2.23}$$

Combining (2.22) with (2.23), we have $t_{\varepsilon}\leq T$. Let $t_{\varepsilon}\to t_0$ as $\varepsilon\to 0$. By applying (2.20) - (2.22) and the Lebesgue dominated convergence theorem, we obtain $\Theta_{\varepsilon}^{a\vec{b}}\leq\mathcal{J}_{\varepsilon}^{a\vec{b}}(t_{\varepsilon}\alpha)\to\mathcal{J}^{\tilde{a}\vec{\tilde{b}}}(t_0\alpha)\leq\mathcal{J}^{\tilde{a}\vec{\tilde{b}}}(\alpha)=\Theta^{\tilde{a}\vec{\tilde{b}}}$ as $\varepsilon\to 0$. Hence $\limsup_{\varepsilon\to 0}\Theta_{\varepsilon}^{a\vec{b}}\leq\Theta^{\tilde{a}\vec{\tilde{b}}}$.

In the end, $\tilde{a}=a$ and $\tilde{b}_j=b_j$, $j=1,2$ when $V(0)\leq a$ and $W_j(0)\geq b_j$, $j=1,2$, namely, for all $x\in\mathbb{R}^N$, we obtain $\tilde{V}_{\varepsilon}(x)\geq 0$, $\tilde{W}_{j\varepsilon}(x)\geq 0$, $j=1,2$. By Lemma 2.2, Lemma 2.10 and (2.21), we have $\Theta_{\varepsilon}^{a\vec{b}}\geq\Theta^{\tilde{a}\vec{\tilde{b}}}$. According to $\Theta^{\tilde{a}\vec{\tilde{b}}}\leq\liminf_{\varepsilon\to 0}\Theta_{\varepsilon}^{a\vec{b}}\leq\limsup_{\varepsilon\to 0}\Theta_{\varepsilon}^{a\vec{b}}\leq\Theta^{\tilde{a}\vec{\tilde{b}}}$, we obtain $\lim_{\varepsilon\to 0}\Theta_{\varepsilon}^{a\vec{b}}=\Theta^{\tilde{a}\vec{\tilde{b}}}=\Theta^{a\vec{b}}$.

Lemma 2.15. If $\tau\leq a<\tau_{\infty},\vec{\varsigma}\geq\vec{b}\geq\vec{\varsigma}_{\infty}$ or $\tau\leq a\leq\tau_{\infty},\vec{\varsigma}>\vec{b}\geq\vec{\varsigma}_{\infty}$, then there is $\varepsilon^{a\vec{b}}>0$ such that $\Theta_{\varepsilon}^{a\vec{b}}$ is achieved at $u_{\varepsilon}^{a\vec{b}}>0$ for all $\varepsilon\leq\varepsilon^{a\vec{b}}$.

Proof. By Lemma 2.7, we have $\Theta^{\tilde{a}\vec{\tilde{b}}}<\Theta^{\infty}$, where $\tilde{a}=V^{a}(0)$, $\tilde{b}_j=W_j^{b_j}(0)$, $j=1,2$. By Lemma 2.11 and Lemma 2.14, there is $\varepsilon^{a\vec{b}}>0$ such that $\Theta_{\varepsilon}^{a\vec{b}}<\Theta^{\infty}\leq\Theta_{\varepsilon}^{\infty}$ for all $\varepsilon\leq\varepsilon^{a\vec{b}}$. By Lemma 2.13, $\mathcal{J}_{\varepsilon}^{a\vec{b}}$ satisfies the $(PS)_{\Theta_{\varepsilon}^{a\vec{b}}}$ condition for all $\varepsilon\leq\varepsilon^{a\vec{b}}$, which combined Lemma 2.9 with Lemma 2.10, we have $\Theta_{\varepsilon}^{a\vec{b}}$ is achieved at $u_{\varepsilon}^{a\vec{b}}\in H^1(\mathbb{R}^N)$. We set $u_{\varepsilon}^{a\vec{b}}$ is a ground state solution of Equation (2.6). If $\left(u_{\varepsilon}^{a\vec{b}}\right)^{\pm}\neq 0$, by

$0=\left\langle\left(\mathcal{J}_{\varepsilon}^{a\vec{b}}\right)'\left(u_{\varepsilon}^{a\vec{b}}\right),\left(u_{\varepsilon}^{a\vec{b}}\right)^{\pm}\right\rangle=\left\langle\left(\mathcal{J}_{\varepsilon}^{a\vec{b}}\right)'\left(\left(u_{\varepsilon}^{a\vec{b}}\right)^{\pm}\right),\left(u_{\varepsilon}^{a\vec{b}}\right)^{\pm}\right\rangle$ implies that

$\left(u_{\varepsilon}^{a\vec{b}}\right)^{\pm}\in\mathcal{N}_{\varepsilon}^{a\vec{b}}$. Thus $\Theta_{\varepsilon}^{a\vec{b}}=\mathcal{J}_{\varepsilon}^{a\vec{b}}\left(u_{\varepsilon}^{a\vec{b}}\right)=\mathcal{J}_{\varepsilon}^{a\vec{b}}\left(\left(u_{\varepsilon}^{a\vec{b}}\right)^{+}\right)+\mathcal{J}_{\varepsilon}^{a\vec{b}}\left(\left(u_{\varepsilon}^{a\vec{b}}\right)^{-}\right)\geq 2\Theta_{\varepsilon}^{a\vec{b}}$,

which is impossible. Hence $u_{\varepsilon}^{a\vec{b}}$ does not change the sign. Then we may assume $u_{\varepsilon}^{a\vec{b}}\geq 0$. By the elliptic regularity theory, $u_{\varepsilon}^{a\vec{b}}\in C^2(\mathbb{R}^N)$. By strong maximum principle, we have $u_{\varepsilon}^{a\vec{b}}>0$.

3. Proofs of the Main Results

Setting $u(x):=v(\varepsilon x)$, Equation (1.2) is a solution of

$$-\Delta u+V(\varepsilon x)u=W_1(\varepsilon x)|u|^{p-2}u+W_2(\varepsilon x)|u|^{q-2}u,\quad u\in H^1(\mathbb{R}^N). \tag{3.1}$$

If $u_{\varepsilon}(x)$ is a solution of Equation (3.1), then $v_{\varepsilon}(x)=u_{\varepsilon}\left(\frac{x}{\varepsilon}\right)$ is a solution of Equation (1.2).

Since $V(\varepsilon x)=V_{\varepsilon}^{\tau}(x)$, $W_j(\varepsilon x)=W_{j\varepsilon}^{\varsigma_j}(x)$, $j=1,2$, we denote by

$$\mathcal{J}_{\varepsilon}:=\mathcal{J}_{\varepsilon}^{\tau\vec{\varsigma}},\ \mathcal{N}_{\varepsilon}:=\mathcal{N}_{\varepsilon}^{\tau\vec{\varsigma}},\ \Theta_{\varepsilon}:=\Theta_{\varepsilon}^{\tau\vec{\varsigma}},\ \mathcal{S}_{\varepsilon}:=\mathcal{S}_{\varepsilon}^{\tau\vec{\varsigma}}.$$

3.1. Proof of Theorem 1.1

Without loss of generality, we assume $x_{jv}=0$. Then $V(0)=\tau$, $W_j(0)=\varsigma_{jv}$, $j=1,2$.

Lemma 3.1. Equation (3.1) has at least m pairs of solutions.

Proof. We choose $a=\tau, b_j=\varsigma_{jv}$, $j=1,2$ in Equation (2.1) and by Lemma 2.3 and Lemma 2.8, there are $u\in\mathcal{S}^{\tau\vec{\varsigma}_v}$ and $u>0$. Let $s>0$, $\zeta_s\in C_0^\infty(\mathbb{R}_+)$ satisfies $\zeta_s(t)=0$ if $t\geq s+1$ and $\zeta_s(t)=1$ if $t\leq s$ with $|\zeta_s'(t)|\leq 1$. Assume $u_s(x):=\zeta_s(|x|)u(x)$ for $x\in\mathbb{R}^N$. By $\|u_s-u\|\to 0$ as $s\to\infty$, we get that $u_s\to u$ in $H^1(\mathbb{R}^N)$ as $s\to\infty$ and $u_s\to u$ in $L^\mu(\mathbb{R}^N)$ for $\mu\in[2,2^*]$ as $s\to\infty$. There is a unique $t_s>0$ such that $t_su_s\in\mathcal{N}^{\tau\vec{\varsigma}}$. Therefore,

$$\begin{aligned}\max_{t\geq 0}\mathcal{J}^{\tau\vec{\varsigma}_v}(tu_s)&=\frac{p-2}{2p}t_s^p\int_{\mathbb{R}^N}\varsigma_{1v}|u_s|^p\,\mathrm{d}x+\frac{q-2}{2q}t_s^q\int_{\mathbb{R}^N}\varsigma_{2v}|u_s|^q\,\mathrm{d}x\\&\to\frac{p-2}{2p}\int_{\mathbb{R}^N}\varsigma_{1v}|u|^p\,\mathrm{d}x+\frac{q-2}{2q}\int_{\mathbb{R}^N}\varsigma_{2v}|u|^q\,\mathrm{d}x\,(s\to\infty)\\&=\max_{t\geq 0}\mathcal{J}^{\tau\vec{\varsigma}_v}(tu)=\mathcal{J}^{\tau\vec{\varsigma}_v}(u)=\Theta^{\tau\vec{\varsigma}_v}.\end{aligned}\tag{3.2}$$

Furthermore,

$$V(\varepsilon x)\to V(0)=\tau,\quad W_j(\varepsilon x)\to W_j(0)=\varsigma_{jv},\ j=1,2\quad\text{as }\varepsilon\to 0\tag{3.3}$$

uniformly of x on any bounded set. There is a unique $t_{s\varepsilon}>0$ such that $t_{s\varepsilon}u_s\in\mathcal{N}_\varepsilon$. Observe that $t_{s\varepsilon}\to t_s$ as $\varepsilon\to 0$. Hence (3.2) and (3.3) imply that

$$\begin{aligned}\max_{t\geq 0}\mathcal{J}_\varepsilon(tu_s)&=\frac{p-2}{2p}t_{s\varepsilon}^p\int_{|x|\leq s+1}W_1(\varepsilon x)|u_s|^p\,\mathrm{d}x+\frac{q-2}{2q}t_{s\varepsilon}^q\int_{|x|\leq s+1}W_2(\varepsilon x)|u_s|^q\,\mathrm{d}x\\&\to\frac{p-2}{2p}t_s^p\int_{|x|\leq s+1}\varsigma_{1v}|u_s|^p\,\mathrm{d}x+\frac{q-2}{2q}t_s^q\int_{|x|\leq s+1}\varsigma_{2v}|u_s|^q\,\mathrm{d}x\,(\varepsilon\to 0)\\&=\max_{t\geq 0}\mathcal{J}^{\tau\vec{\varsigma}_v}(tu_s)\to\Theta^{\tau\vec{\varsigma}_v}\ (s\to\infty).\end{aligned}\tag{3.4}$$

By Lemma 2.7, $m(\tau,\vec{\varsigma}_v)>1$. We choose $m_v=m(\tau,\vec{\varsigma}_v)$. For the maximal integer $m\in\mathbb{Z}_+$ with $m<m_v$, we have $m\geq 1$. Define $\xi_{sl}(x):=u_s(x_1-2l(s+1),x_2,\cdots,x_N)$ for $l=0,1,\cdots,m-1$, and set $E_{sm}:=\operatorname{span}\{\xi_{sl}(x):l=0,1,\cdots,m-1\}$. Clearly, $(\xi_{si},\xi_{sj})=0$ if $i\neq j$. Hence $\dim E_{sm}=m$. Combining (3.2) with (3.3) again, for all $l=1,2,\cdots,m-1$, we have

$$\begin{aligned}\max_{t\geq 0}\mathcal{J}_\varepsilon(t\xi_{sl})&=\frac{p-2}{2p}t_{l\varepsilon}^p\int_{|x|\leq s+1}W_1(\varepsilon x)|\xi_{sl}|^p\,\mathrm{d}x+\frac{q-2}{2q}t_{l\varepsilon}^q\int_{|x|\leq s+1}W_2(\varepsilon x)|\xi_{sl}|^q\,\mathrm{d}x\\&\to\frac{p-2}{2p}t_l^p\int_{|x|\leq s+1}\varsigma_{1v}|\xi_{sl}|^p\,\mathrm{d}x+\frac{q-2}{2q}t_l^q\int_{|x|\leq s+1}\varsigma_{2v}|\xi_{sl}|^q\,\mathrm{d}x\,(\varepsilon\to 0)\\&=\max_{t\geq 0}\mathcal{J}^{\tau\vec{\varsigma}_v}(tu_s)\to\Theta^{\tau\vec{\varsigma}_v}\ (s\to\infty),\end{aligned}$$

where $t_{l\varepsilon}$ and t_l are the unique constants satisfying $t_{l\varepsilon}\xi_{sl}\in\mathcal{N}_\varepsilon$ and $t_l\xi_{sl}\in\mathcal{N}^{\tau\vec{\varsigma}_v}$, respectively, and $t_{l\varepsilon}\to t_l$ as $\varepsilon\to 0$. Therefore, for all $\delta>0$, there are $s_\delta>0$ and $\varepsilon_\delta>0$ such that for all $l=0,1,\cdots,m-1$, we get

$$\max_{t\geq 0}\mathcal{J}_\varepsilon(t\xi_{sl})\leq\Theta^{\tau\vec{\varsigma}_v}+\delta,\quad\forall s\geq s_\delta,\forall\varepsilon\leq\varepsilon_\delta.\tag{3.5}$$

Let $u=t_0\xi_{s0}+t_1\xi_{s1}+\cdots+t_{m-1}\xi_{s(m-1)}$ for any $u\in E_{sm}$, where $t_0,\cdots,t_{m-1}\in\mathbb{R}$. According to (3.5), for all $s\geq s_\delta$ and $\varepsilon\leq\varepsilon_\delta$, we obtain $\mathcal{J}_\varepsilon(u)=\mathcal{J}_\varepsilon(t_0\xi_{s0})+\mathcal{J}_\varepsilon(t_1\xi_{s1})+\cdots+\mathcal{J}_\varepsilon(t_{m-1}\xi_{s(m-1)})\leq m(\Theta^{\tau\vec{\varsigma}_v}+\delta)$. Thus $\sup_{u\in E_{sm}}\mathcal{J}_\varepsilon(u)\leq m(\Theta^{\tau\vec{\varsigma}_v}+\delta)$ for all $s\geq s_\delta$ and $\varepsilon\leq\varepsilon_\delta$. By Lemma 2.6, $m\Theta^{\tau\vec{\varsigma}_v}<\Theta^\infty$. We choose $0<\delta<\frac{\Theta^\infty}{m}-\Theta^{\tau\vec{\varsigma}_v}$, then there exist $s_m>0$ and $\varepsilon_m>0$

such that

$$\sup_{u\in E_{sm}} \mathcal{J}_\varepsilon(u) < \Theta^\infty, \quad \forall s \ge s_m, \forall \varepsilon \le \varepsilon_m. \tag{3.6}$$

Next, we shall define constants $c_1, c_2, \cdots, c_m$ and prove that they are critical values of $\mathcal{J}_\varepsilon$. Consider the symmetric group $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ and we denote by $\Sigma := \{A \subset H^1(\mathbb{R}^N) : A \text{ is closed and } A = -A\}$ and

$$\mathcal{H} := \{h \in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) : h \text{ is an odd homeomorphism}\}.$$

For any $A \in \Sigma$, we define a version of Benci pseudo-index of A as follows, $i(A) := \min_{h\in\mathcal{H}} \mathrm{gen}(h(A) \cap \partial B_\rho)$, where $\mathrm{gen}(A) := \inf\{n : \exists g \in C(A, \mathbb{R}^n \setminus \{0\}) \text{ and } g \text{ is odd}\}$ is the Krasnoselskii genus of A, and $\rho > 0$ is a constant given in Lemma 2.9. Let $c_l := \inf_{i(A)\ge l} \sup_{u\in A} \mathcal{J}_\varepsilon(u)$, $l = 1, 2, \cdots, m$. Observe that $c_1 \le c_2 \le \cdots \le c_m$. For any $A \in \Sigma$ and $i(A) \ge 1$, we have $\mathrm{gen}(A \cap \partial B_\rho) \ge 1$, then $A \cap \partial B_\rho$ is not empty. By Lemma 2.9, it follows from $\sup_{u\in A} \mathcal{J}_\varepsilon(u) > \kappa$ that $c_1 \ge \kappa$, where κ is defined in Lemma 2.9.

Noticing that $\mathrm{gen}(A)$ satisfies dimension property in [24], for all $h \in \mathcal{H}$, we have $\mathrm{gen}(h(E_{sm}) \cap \partial B_\rho) = \dim E_{sm} = m$. Hence $i(E_{sm}) = m$, then we obtain $c_m \le \sup_{u\in E_{sm}} \mathcal{J}_\varepsilon(u)$. Combining (3.6) with Lemma 2.11, we have

$$\kappa \le c_1 \le c_2 \le \cdots \le c_m \le \sup_{u\in E_{sm}} \mathcal{J}_\varepsilon(u) < \Theta^\infty \le \Theta^\infty_\varepsilon. \tag{3.7}$$

Let $c_0 := \kappa$, $c_\infty := \sup_{u\in E_{sm}} \mathcal{J}_\varepsilon(u)$, $\mathcal{J}^c_\varepsilon := \{u \in H^1(\mathbb{R}^N) : \mathcal{J}_\varepsilon(u) \le c\}$, and $\Psi_c := \{u \in H^1(\mathbb{R}^N) : \mathcal{J}_\varepsilon(u) = c, \mathcal{J}'_\varepsilon(u) = 0\}$. Clearly, $\mathcal{J}_\varepsilon$ is an even functional. For all $c \in [c_0, c_\infty]$, we obtain

$$\mathcal{J}^c_\varepsilon \in \Sigma \quad \text{and} \quad \Psi_c \in \Sigma. \tag{3.8}$$

By using (3.7) and Lemma 2.13, for all $c \in [c_0, c_\infty]$, $\mathcal{J}_\varepsilon$ satisfies $(PS)_c$ condition and

$$\Psi_c \text{ is compact in } H^1(\mathbb{R}^N). \tag{3.9}$$

Set $(\Psi_c)_\iota := \{u \in H^1(\mathbb{R}^N) : \mathrm{dist}(u, \Psi_c) < \iota\}$, where $\iota > 0$ for any $c \in [c_0, c_\infty]$, then we choose $\delta = \frac{\iota}{4}$, we have there is $\tilde{\varepsilon} > 0$ such that

$$\|\mathcal{J}'_\varepsilon(u)\| \ge \frac{8\tilde{\varepsilon}}{\delta}, \quad \forall u \in \mathcal{J}^{-1}_\varepsilon([c - 2\tilde{\varepsilon}, c + 2\tilde{\varepsilon}]) \setminus \overline{(\Psi_c)_{\frac{\iota}{2}}}. \tag{3.10}$$

Let $P := H^1(\mathbb{R}^N) \setminus (\Psi_c)_\iota$, then $P_{2\delta} = H^1(\mathbb{R}^N) \setminus \overline{(\Psi_c)_{\frac{\iota}{2}}}$. By (3.10), we have $\|\mathcal{J}'_\varepsilon(u)\| \ge \frac{8\tilde{\varepsilon}}{\delta}$ for all $u \in \mathcal{J}^{-1}_\varepsilon([c - 2\tilde{\varepsilon}, c + 2\tilde{\varepsilon}]) \cap P_{2\delta}$. By Lemma 2.3 in [23], there is $\overline{\eta} \in C([0,1] \times H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$ such that for all $t \in [0,1]$, $\overline{\eta}(t, \cdot)$ is an odd homeomorphism of $H^1(\mathbb{R}^N)$ and $\overline{\eta}(1, \mathcal{J}^{c+\tilde{\varepsilon}}_\varepsilon \cap P) \subset \mathcal{J}^{c-\tilde{\varepsilon}}_\varepsilon$. Set $\eta := \overline{\eta}(1, \cdot)$, then η is an odd homeomorphism of $H^1(\mathbb{R}^N)$ and

$$\eta(\mathcal{J}^{c+\tilde{\varepsilon}}_\varepsilon \setminus (\Psi_c)_\iota) \subset \mathcal{J}^{c-\tilde{\varepsilon}}_\varepsilon. \tag{3.11}$$

For any $A \in \Sigma$ and $A \subset \mathcal{J}^{c_0}_\varepsilon$, it follows from $\mathcal{J}_\varepsilon(u) > \kappa$ for all $u \in \partial B_\rho$

that $A\cap\partial B_\rho=\varnothing$. Hence $\mathrm{gen}\left(A\cap\partial B_\rho\right)=0$ and

$$i\left(A\right)=\min_{h\in\mathcal{H}}\mathrm{gen}\left(h\left(A\right)\cap\partial B_\rho\right)=0. \tag{3.12}$$

Moreover,

$$E_{sm}\subset\mathcal{J}_\varepsilon^{c_\infty}\quad\text{and}\quad i\left(E_{sm}\right)=m\geq 1. \tag{3.13}$$

By applying the Theorem 1.4 in [24], (3.8), (3.9) and (3.11) - (3.13), we have $c_1,\cdots,c_m$ are critical values of $\mathcal{J}_\varepsilon$, and $\mathrm{gen}\left(\Psi_c\right)\geq s+1$, if $c:=c_k=c_{k+1}=\cdots=c_{k+s}$ with $k\geq 1$ and $k+s\leq m$. Since $\mathcal{J}_\varepsilon$ is even, then $\mathcal{J}_\varepsilon$ has at least m pairs of critical points being solutions of Equation (3.1).

Lemma 3.2. Equation (3.1) has at least one positive and one negative ground state solutions for $m\geq 1$ and has at least a pair of sign-changing solutions for $m\geq 2$.

Proof. If $a=\tau$, $b_j=\varsigma_j$, $j=1,2$ in Equation (2.1), then $\tilde{a}=V^\tau\left(0\right)=V\left(0\right)=\tau$, $\tilde{b}_j=W_j^{\varsigma_j}\left(0\right)=W_j\left(0\right)=\varsigma_{jv}$, $j=1,2$. By Lemma 2.9 and Lemma 2.13, $\mathcal{J}_\varepsilon^{\tau\vec{\varsigma}}$ has a $\left(PS\right)_{\Theta_\varepsilon}$ sequence and satisfies $\left(PS\right)_{\Theta_\varepsilon}$ condition. By Lemma 2.15, there exists $\varepsilon_0>0$ such that Θ_ε is achieved at $u_\varepsilon>0$ for all $\varepsilon\leq\varepsilon_0$. Thus u_ε and $-u_\varepsilon$ are positive and negative ground state solutions of Equation (3.1), respectively.

Let $\alpha^\pm\in\mathcal{S}^{\tau\vec{\varsigma}_v}$ with $\alpha^+>0$. Define $\alpha_s^\pm\left(x\right):=\zeta_s\left(\left|x\right|\right)\alpha^\pm\left(x\right)$ for $x\in\mathbb{R}^N$, where ζ_s is given in Lemma 3.1. Then $\alpha_s^\pm\to\alpha^\pm$ in $H^1\left(\mathbb{R}^N\right)$ as $s\to\infty$. Choose $s>0$, $x_s\in\mathbb{R}$ with $\left|x_s\right|$ large enough and $\mathrm{dist}\left(\overline{B_{s+1}\left(0\right)},\overline{B_{s+1}\left(x_s\right)}\right)>0$. Let $t_s^\pm\in\mathbb{R}$ such that $u_s^+:=t_s^+\alpha_s^+\in\mathcal{N}_\varepsilon$ and $u_s^-:=t_s^-\alpha_s^-\left(\cdot-x_s\right)\in\mathcal{N}_\varepsilon$. Then $u_s^+\geq 0$ and $u_s^-\leq 0$, $\mathrm{supp}u_s^+\cap\mathrm{supp}u_s^-$ is empty and $u_s:=u_s^++u_s^-\in\mathcal{N}_\varepsilon$. Define $L_\varepsilon:=\left\{u\in\mathcal{N}_\varepsilon:u^\pm\in\mathcal{N}_\varepsilon\right\}$, then we have $u_s\in L_\varepsilon$. Define $l_\varepsilon:=\inf_{u\in L_\varepsilon}\mathcal{J}_\varepsilon\left(u\right)$, then $l_\varepsilon\geq 2\Theta_\varepsilon>0$.

Next, we will prove $l_\varepsilon<\Theta_\varepsilon^\infty$ for ε small enough. Due to $l_\varepsilon\leq\mathcal{J}_\varepsilon\left(u_s\right)$, we get

$$\lim_{\varepsilon\to 0}l_\varepsilon\leq\lim_{s\to\infty}\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon\left(u_s\right). \tag{3.14}$$

Observe that $t_s^\pm\to 1$ as $s\to\infty$ and

$$\begin{aligned}\mathcal{J}_\varepsilon\left(u_s\right)&=\mathcal{J}_\varepsilon\left(u_s^+\right)+\mathcal{J}_\varepsilon\left(u_s^-\right)\to\mathcal{J}^{\tau\vec{\varsigma}_v}\left(u_s^+\right)+J^{\tau\vec{\varsigma}_v}\left(u_s^-\right)\left(\varepsilon\to 0\right)\\&=\mathcal{J}^{\tau\vec{\varsigma}_v}\left(t_s^+\alpha_s^+\right)+\mathcal{J}^{\tau\vec{\varsigma}_v}\left(t_s^-\alpha_s^-\left(x-x_s\right)\right)\to 2\mathcal{J}^{\tau\vec{\varsigma}_v}\left(\alpha\right)\left(s\to\infty\right)\\&=2\Theta^{\tau\vec{\varsigma}_v}.\end{aligned} \tag{3.15}$$

By $m\geq 2$ and combining Lemma 2.6 with Lemma 2.11, we have

$$2\Theta^{\tau\vec{\varsigma}_v}<\Theta^\infty\leq\Theta_\varepsilon^\infty. \tag{3.16}$$

By (3.14) - (3.16), we get that $l_\varepsilon<\Theta_\varepsilon^\infty$ for ε small enough, which implies $\mathcal{J}_\varepsilon$ satisfies $\left(PS\right)_{l_\varepsilon}$ condition for ε small enough.

Now we show that there is a $\left(PS\right)_{l_\varepsilon}$ sequence of $\mathcal{J}_\varepsilon$. Since $\left\{u_n\right\}\subset\mathcal{N}_\varepsilon$, then $u_n^\pm\in\mathcal{N}_\varepsilon$. We assume $u_n^\pm\rightharpoonup u^\pm$ in $H^1\left(\mathbb{R}^N\right)$ with $u^\pm\neq 0$. There exist $t^+>0$ and $t^-<0$ such that $t^\pm u^\pm\in\mathcal{N}_\varepsilon$, $u=t^+u^++t^-u^-\in L_\varepsilon$, we get $\mathcal{J}_\varepsilon\left(u\right)=l_\varepsilon$. Assume by contradiction that if $\tilde{u}$ is not a sign-changing solution of Equation (3.1), there exists $\varphi\in H^1\left(\mathbb{R}^N\right)$ such that $\left\langle\mathcal{J}_\varepsilon'\left(\tilde{u}\right),\varphi\right\rangle\leq -1/2$. We choose $\hat{\varepsilon}>0$ small enough, satisfying $\left\langle \mathcal{J}_{\varepsilon}'\left(tu^{+}+hu^{-}+\rho\varphi\right),\varphi\right\rangle \le -1/2$ for all $|t-1|+|h-1|+|\rho|\le\hat{\varepsilon}$. Let η be a cut off function such that

$$\eta(t,h)=\begin{cases}1 & |t-1|\le 1/2\,\hat{\varepsilon} \text{ and } |h-1|\le 1/2\,\hat{\varepsilon}\\ 0 & |t-1|\ge\hat{\varepsilon} \text{ or } |h-1|\ge\hat{\varepsilon}\end{cases}.$$

Then $\mathcal{J}_{\varepsilon}\left(tu^{+}+hu^{-}+\hat{\varepsilon}\eta(t,h)\varphi\right)\le\mathcal{J}_{\varepsilon}(\tilde{u})=l_{\varepsilon}$. Hence $\max_{0\le t,h\le 2}\mathcal{J}_{\varepsilon}\left(tu^{+}+hu^{-}+\hat{\varepsilon}\eta(t,h)\varphi\right)<l_{\varepsilon}$. By a degree theory argument in [25], we find $a,b\in(0,2)$ such that $\tilde{u}:=au^{+}+bu^{-}+\hat{\varepsilon}\eta(a,b)\varphi\in L_{\varepsilon}$ and $\mathcal{J}_{\varepsilon}(\tilde{u})<l_{\varepsilon}$, which contradits that the defination of l_{ε}.

In the end, we prove l_{ε} is achieved at some $u_{\varepsilon}\in L_{\varepsilon}$. Let $\{u_n\}\subset L_{\varepsilon}$ and $\mathcal{J}_{\varepsilon}(u_n)\to l_{\varepsilon}$ as $n\to\infty$. By Ekeland vainational principle there is $\{\bar{u}_n\}\subset L_{\varepsilon}$ such that $\mathcal{J}_{\varepsilon}(\bar{u}_n)\to l_{\varepsilon}$ and $\mathcal{J}_{\varepsilon}'(\bar{u}_n)\to 0$ as $n\to\infty$, then $\|\bar{u}_n-u_n\|\to 0$ as $n\to\infty$. Hence $\{\bar{u}_n\}$ is a $(PS)_{l_{\varepsilon}}$ sequence of $\mathcal{J}_{\varepsilon}$. Going of necessary to a subsequence, for ε small enough we may assume $\bar{u}_n\to u_{\varepsilon}$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$. Hence $\mathcal{J}_{\varepsilon}(u_{\varepsilon})=l_{\varepsilon}$ and $\mathcal{J}_{\varepsilon}'(u_{\varepsilon})=0$. Then $u_{\varepsilon}\in\mathcal{N}_{\varepsilon}$, we have $u_{\varepsilon}^{\pm}\ne 0$, $u_{\varepsilon}^{\pm}\in\mathcal{N}_{\varepsilon}$. Thus $u_{\varepsilon}\in L_{\varepsilon}$ and $\pm u_{\varepsilon}$ are a pair of sign-changing solutions of Equation (3.1). Let $v_{\varepsilon}(x)=u_{\varepsilon}\left(\frac{x}{\varepsilon}\right)$, then $\pm v_{\varepsilon}$ are a pair of sign-changing solutions of Equation (1.2).

This completes the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2

We can assume without loss of generality that $x_w=0$. Then $V(0)=\tau_w$, $W_j(0)=\varsigma_j$, $j=1,2$. Letting $a=\tau_w$, $b_j=\varsigma_j$, $j=1,2$ in Equation (2.1), there is $u\in\mathcal{S}^{\tau_w\vec{\varsigma}}$ by Lemma 2.3. Due to Lemma 2.7, $m(\tau_w,\vec{\varsigma})\ge 1$, we choose

$$m_w=\begin{cases}m(\tau_w,\vec{\varsigma}) & \text{if } m(\tau_w,\vec{\varsigma})>1\\ \dfrac{3}{2} & \text{if } m(\tau_w,\vec{\varsigma})=1\end{cases}.$$

For the maximal integer $m<m_w$, then $m\ge 1$. By Lemma 2.6 and Lemma 2.7, we have $m\Theta^{\tau_w\vec{\varsigma}}<\Theta^{\infty}$. The following proof of Theorem 1.2 is similar to that of Theorem 1.1 and so is omitted.

3.3. Proof of Theorem 1.3

In this subsection, we will consider the case $(P3)$ - (1), the other case can be handled similarly. Without loss of generality, we assume $x_{jv}=0$. Then $V(0)=\tau$, $W_j(0)=\varsigma_{jv}$, $j=1,2$.

Lemma 3.3. $u_{\varepsilon}\to u$ as $\varepsilon\to 0$ up to a sequence after translations.

Proof. Let $\varepsilon_k\to 0$ as $k\to\infty$, $u_k:=u_{\varepsilon_k}\in\mathcal{S}_{\varepsilon_k}$ with $u_k>0$. By Lemma 2.14, we obtain $\lim_{k\to\infty}\Theta_{\varepsilon_k}=\Theta^{\tau\vec{\varsigma}}$, which together with $\Theta_{\varepsilon_k}=\mathcal{J}_{\varepsilon_k}(u_k)\ge\frac{p-2}{2p}\int_{\mathbb{R}^N}\left(|\nabla u_k|^2+V(\varepsilon_k x)u_k^2\right)\mathrm{d}x\ge C\|u_k\|^2$, implies that $\{u_k\}\subset H^1(\mathbb{R}^N)$ is bounded. By Lemma 2.8, there exist $\sigma>0$, $R>0$ and

$z_k' \in \mathbb{R}^N$ such that

$$\int_{B_R(z_k')} u_k^2 \mathrm{d}x \ge \sigma. \tag{3.17}$$

Let $\hat{u}_k(x) := u_k(x+z_k')$, $\hat{V}_{\varepsilon_k}(x) := V(\varepsilon_k(x+z_k'))$, $\hat{W}_{j\varepsilon_k}(x) := W_j(\varepsilon_k(x+z_k'))$, $j=1,2$. Then $\hat{u}_k$ is a solution of

$$-\Delta \hat{u}_k + \hat{V}_{\varepsilon_k}(x)\hat{u}_k = \hat{W}_{1\varepsilon_k}(x)\hat{u}_k^{p-1} + \hat{W}_{2\varepsilon_k}(x)\hat{u}_k^{q-1}. \tag{3.18}$$

Furthermore,

$$\hat{\Theta}_{\varepsilon_k} = \hat{\mathcal{J}}_{\varepsilon_k}(\hat{u}_k) = \mathcal{J}_{\varepsilon_k}(u_k) = \Theta_{\varepsilon_k}. \tag{3.19}$$

Since $\{\hat{u}_k\}$ is bounded, we can assume that $\hat{u}_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Then $\hat{u}_k \to u$ in $L^\mu_{loc}(\mathbb{R}^N)$ for $\mu \in [2, 2^*)$ as $k \to \infty$. By (3.17), $u \ne 0$.

Since V and W_j, $j=1,2$ are bounded, up to a subsequence if necessary, we can assume

$$V(\varepsilon_k z_k') \to V_0, \quad W_j(\varepsilon_k z_k') \to W_{j0},\ j=1,2 \quad \text{as } k \to \infty, \tag{3.20}$$

and $\vec{W}_0 := (W_{10}, W_{20})$. For all $x \in \mathbb{R}^N$, by the boundedness of $\nabla V : |\nabla V(x)| \le C$, for given arbitrarily $R > 0$, we obtain $|V(\varepsilon_k x + \varepsilon_k z_k') - V(\varepsilon_k z_k')| \le \varepsilon_k CR$ for all $x \in B_R(0)$. Hence $\hat{V}_{\varepsilon_k}(x) \to V_0$ as $k \to \infty$ uniformly on any bounded set of *x*. Similarly, $\hat{W}_{j\varepsilon_k}(x) \to W_{j0}$, $j=1,2$ as $k \to \infty$ uniformly on any bounded set of *x*. Similar to the proof of Lemma 2.14, we have

$$\limsup_{k\to\infty} \hat{\Theta}_{\varepsilon_k} \le \Theta^{V_0\vec{W}_0}. \tag{3.21}$$

By (3.18), for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \int_{\mathbb{R}^N} \left(\nabla \hat{u}_k \nabla \varphi + \hat{V}_{\varepsilon_k}(x)\hat{u}_k\varphi - \hat{W}_{1\varepsilon_k}(x)\hat{u}_k^{p-1}\varphi - \hat{W}_{2\varepsilon_k}(x)\hat{u}_k^{q-1}\varphi \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(\nabla u \nabla \varphi + V_0 u\varphi - W_{10}u^{p-1}\varphi - W_{20}u^{q-1}\varphi \right) \mathrm{d}x, \end{aligned}$$

which implies that u is a ground state solution of

$$-\Delta u + V_0 u = W_{10}u^{p-1} + W_{20}u^{q-1} \tag{3.22}$$

with the energy functional

$$\mathcal{J}^{V_0\vec{W}_0}(u) = \frac{p-2}{2p}\int_{\mathbb{R}^N} W_{10}u^p \mathrm{d}x + \frac{q-2}{2q}\int_{\mathbb{R}^N} W_{20}u^q \mathrm{d}x \ge \Theta^{V_0\vec{W}_0}. \tag{3.23}$$

By Fatou's Lemma,

$$\begin{aligned} &\frac{p-2}{2p}\int_{\mathbb{R}^N} W_{10}u^p \mathrm{d}x + \frac{q-2}{2q}\int_{\mathbb{R}^N} W_{20}u^q \mathrm{d}x \\ &\le \liminf_{t\to 0} \int_{\mathbb{R}^N} \left(\frac{p-2}{2p}\hat{W}_{1\varepsilon_k}(x)\hat{u}_k^p + \frac{q-2}{2q}\hat{W}_{2\varepsilon_k}(x)\hat{u}_k^q \right) \mathrm{d}x, \end{aligned} \tag{3.24}$$

Combining (3.19) with (3.22) - (3.24), we have

$$\Theta^{V_0\vec{W}_0} \le \mathcal{J}^{V_0\vec{W}_0}(u) \le \liminf_{k\to\infty} \hat{\mathcal{J}}_{\varepsilon_k}(\hat{u}_k) \le \limsup_{k\to\infty} \hat{\Theta}_{\varepsilon_k} \le \Theta^{V_0\vec{W}_0}.$$

Hence

$$\lim_{k\to\infty} \hat{\Theta}_{\varepsilon_k} = \Theta^{V_0\vec{W}_0} = \mathcal{J}^{V_0\vec{W}_0}(u). \tag{3.25}$$

Set $\eta \in C_0^\infty(\mathbb{R})$ satisfies $\eta(t)=0$ if $t \ge 2$ and $\eta(t)=1$ if $t \le 1$. Define $\tilde{u}_k(x) := \eta\left(\frac{|2x|}{k}\right)u(x)$ and $z_k(x)=\hat{u}_k(x)-\tilde{u}_k(x)$ for $x \in \mathbb{R}^N$. Then $\tilde{u}_k \to u$ and $z_k \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$, $\tilde{u}_k \to u$ in $L^\mu(\mathbb{R}^N)$ for $\mu \in [2,2^*]$ and $z_k \to 0$ in $L^\mu_{loc}(\mathbb{R}^N)$ for $\mu \in [2,2^*)$ as $k \to \infty$, $\tilde{u}_k \to u$ and $z_k \to 0$ a.e. on $\mathbb{R}^N$ as $k \to \infty$. We define

$$\hat{\mathcal{J}}_{\varepsilon_k}(z_k) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla z_k|^2 \mathrm{d}x + \hat{V}_{\varepsilon_k}(x)z_k^2\right)\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\hat{W}_{1\varepsilon_k}(x)|z_k|^p \mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N}\hat{W}_{2\varepsilon_k}(x)|z_k|^q \mathrm{d}x$$

. Now we show that $\hat{\mathcal{J}}_{\varepsilon_k}(z_k) \to 0$ and $\left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(z_k), z_k \right\rangle \to 0$ as $k \to \infty$. By Remark 1.33 in [23], we have

$$\|z_k\|^2 = \|\hat{u}_k\|^2 - \|\tilde{u}_k\|^2 + o(1). \tag{3.26}$$

For any $\sigma > 0$, there exists $k_0 > 0$ such that

$$\int_{\mathbb{R}^N}\left||\hat{u}_k|^\mu - |z_k|^\mu - |\tilde{u}_k|^\mu\right|\mathrm{d}x \le C\sigma, \quad \forall k > k_0. \tag{3.27}$$

By choosing $\mu = 2, p, q$ in (3.27), respectively, we obtain

$$\int_{\mathbb{R}^N}\hat{V}_{\varepsilon_k}(x)\hat{u}_k^2\mathrm{d}x = \int_{\mathbb{R}^N}\hat{V}_{\varepsilon_k}(x)z_k^2\mathrm{d}x + \int_{\mathbb{R}^N}\hat{V}_{\varepsilon_k}(x)\tilde{u}_k^2\mathrm{d}x + o(1), \tag{3.28}$$

$$\int_{\mathbb{R}^N}\hat{W}_{1\varepsilon_k}(x)|\hat{u}_k|^p \mathrm{d}x = \int_{\mathbb{R}^N}\hat{W}_{1\varepsilon_k}(x)|z_k|^p \mathrm{d}x + \int_{\mathbb{R}^N}\hat{W}_{1\varepsilon_k}(x)|\tilde{u}_k|^p \mathrm{d}x + o(1), \tag{3.29}$$

$$\int_{\mathbb{R}^N}\hat{W}_{2\varepsilon_k}(x)|\hat{u}_k|^q \mathrm{d}x = \int_{\mathbb{R}^N}\hat{W}_{2\varepsilon_k}(x)|z_k|^q \mathrm{d}x + \int_{\mathbb{R}^N}\hat{W}_{2\varepsilon_k}(x)|\tilde{u}_k|^q \mathrm{d}x + o(1). \tag{3.30}$$

By using the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^N}\hat{V}_{\varepsilon_k}(x)\tilde{u}_k^2\mathrm{d}x = \int_{\mathbb{R}^N}V_0 u^2\mathrm{d}x + o(1), \tag{3.31}$$

$$\int_{\mathbb{R}^N}\hat{W}_{1\varepsilon_k}(x)|\tilde{u}_k|^p \mathrm{d}x = \int_{\mathbb{R}^N}W_{10}|u|^p \mathrm{d}x + o(1), \tag{3.32}$$

$$\int_{\mathbb{R}^N}\hat{W}_{2\varepsilon_k}(x)|\tilde{u}_k|^q \mathrm{d}x = \int_{\mathbb{R}^N}W_{20}|u|^q \mathrm{d}x + o(1). \tag{3.33}$$

Moreover,

$$|\nabla\tilde{u}_k|_2^2 = |\nabla u|_2^2 + o(1). \tag{3.34}$$

Combining (3.25) - (3.34) and (3.18) with (3.22), we have

$$\hat{\mathcal{J}}_{\varepsilon_k}(z_k) = \hat{\mathcal{J}}_{\varepsilon_k}(\hat{u}_k) - \hat{\mathcal{J}}_{\varepsilon_k}(\tilde{u}_k) + o(1) = \hat{\Theta}_{\varepsilon_k} - \mathcal{J}^{V_0\vec{W}_0}(u) + o(1) = o(1) \tag{3.35}$$

and

$$\begin{aligned}\left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(z_k), z_k \right\rangle &= \left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(\hat{u}_k), \hat{u}_k \right\rangle - \left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(\tilde{u}_k), \tilde{u}_k \right\rangle + o(1) \\ &= \left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(\hat{u}_k), \hat{u}_k \right\rangle - \left\langle \left(\mathcal{J}^{V_0\vec{W}_0}\right)'(u), u \right\rangle + o(1) = o(1).\end{aligned} \tag{3.36}$$

In the end, by (3.35) and (3.36), we have $o(1) = \hat{\mathcal{J}}_{\varepsilon_k}(z_k) - \frac{1}{p}\left\langle \hat{\mathcal{J}}'_{\varepsilon_k}(z_k), z_k \right\rangle \ge C\|z_k\|^2$, which implies that $z_k \to 0$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Thus $\hat{u}_k \to u$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$.

Lemma 3.4. $\hat{u}_k(x)\to 0$ as $|x|\to\infty$ uniformly in $k\in\mathbb{Z}_+$.

Proof. We use the contradiction method to obtain that there are $\sigma>0$ for $x_n\in\mathbb{R}^N$, $|x_n|\to\infty$ as $n\to\infty$ such that $\hat{u}_{k_n}(x_n)\ge\sigma$. Moreover, there exists $C>0$ (independent of *k*) such that $\hat{u}_{k_n}(x_n)\le C\left(\int_{B_1(x_n)}\hat{u}_{k_n}^2\mathrm{d}x\right)^{\frac{1}{2}}$. Thus by the Minkowski inequality, we have

$\hat{u}_{k_n}(x_n)\le C\left(\int_{\mathbb{R}^N}|\hat{u}_{k_n}-u|^2\mathrm{d}x\right)^{\frac{1}{2}}+C\left(\int_{B_1(x_n)}u^2\mathrm{d}x\right)^{\frac{1}{2}}\to 0$ as $n\to\infty$, which is impossible.

Lemma 3.5. $\{\varepsilon_k z_k'\}_k$ is bounded on $\mathbb{R}^N$.

Proof. Assume by contradiction that there is $|\varepsilon_k z_k'|\to\infty$ as $k\to\infty$ up to a subsequence. Hence $V_0\ge\tau_\infty>\tau$ and $W_{j0}\le\varsigma_{j\infty}<\varsigma_{jv}$, $j=1,2$. By Lemma 2.4, we have $\Theta^{V_0\vec{W}_0}>\Theta^{\tau\vec{\varsigma}_v}$. According to (3.19), (3.25) and Lemma 2.14,
$\Theta^{V_0\vec{W}_0}=\lim_{k\to\infty}\hat{\Theta}_{\varepsilon_k}=\lim_{k\to\infty}\Theta_{\varepsilon_k}\le\limsup_{k\to\infty}\Theta_{\varepsilon_k}\le\Theta^{\tau\vec{\varsigma}_v}$, which is impossible.

By Lemma 3.5, we may assume $\varepsilon_k z_k'\to x_0$ as $k\to\infty$. By (3.20), we obtain $V_0=V(x_0)$ and $W_{j0}=W_j(x_0)$, $j=1,2$. Applying (3.22), we get that u is a ground state solution of Equation (1.4).

Lemma 3.6. $\{\varepsilon z_\varepsilon\}_\varepsilon$ is bounded, where $z_\varepsilon\in\mathbb{R}^N$ is a maximum point of u_ε.

Proof. If the thesis were not true, there were $\varepsilon_k\to 0$ with $|\varepsilon_k z_k|\to\infty$, where $z_k:=z_{\varepsilon_k}$ is a maximum point of $u_k:=u_{\varepsilon_k}$. Repeating Lemma 3.3 - Lemma 3.5, we can get that there exists $z_k'\in\mathbb{R}^N$ such that $\hat{u}_k=u_k(\cdot+z_k')\to u\ne 0$ in $H^1(\mathbb{R}^N)$ as $k\to\infty$, $\hat{u}_k(x)\to 0$ as $|x|\to\infty$ uniformly in $k\in\mathbb{Z}_+$, $\{\varepsilon_k z_k'\}_k$ is bounded on $\mathbb{R}^N$. Thus $|\varepsilon_k z_k-\varepsilon_k z_k'|\to\infty$ as $k\to\infty$, then $|z_k-z_k'|\to\infty$ as $k\to\infty$. Since $\max_{\mathbb{R}^N}u_k=u_k(z_k)=\hat{u}_k(z_k-z_k')\to 0$ as $k\to\infty$, then $\hat{u}_k(x)\to 0$ as $k\to\infty$ uniformly in $x\in\mathbb{R}^N$, which contradicts with $u\ne 0$.

Lemma 3.7. $\lim_{\varepsilon\to 0}\mathrm{dist}(\varepsilon z_\varepsilon,\mathcal{A}_v)=0$.

Proof. By Lemma 3.5 and Lemma 3.6, there exists $\varepsilon_k\to 0$ as $k\to\infty$ with

$$\varepsilon_k z_k'\to x_0,\quad \varepsilon_k z_k\to z_0\quad \text{as } k\to\infty,\tag{3.37}$$

where $z_k=z_{\varepsilon_k}$ is a maximum point of $u_k=u_{\varepsilon_k}$. By Lemma 3.3 and Lemma 3.5, there exists $z_k'\in\mathbb{R}^N$ such that $\hat{u}_k(x)=u_k(x+z_k')$. By Lemma 3.4, we may assume $\hat{u}_k(x_k')=\max_{\mathbb{R}^N}\hat{u}_k$ and $\{x_k'\}_k$ is bounded on $\mathbb{R}^N$. Hence $z_k=x_k'+z_k'$ and $\varepsilon_k x_k'\to 0$ as $k\to\infty$. By (3.32) and (3.34), which imply that

$$z_0=x_0,\ V(z_0)=V_0,\ W_j(z_0)=W_{j0},\ j=1,2.\tag{3.38}$$

Assume indirectly that $z_0\notin\mathcal{A}_v$, then $V(z_0)>\tau$, $W_j(z_0)\le\varsigma_{jv}$, $j=1,2$ or $V(z_0)=\tau$, $W_1(z_0)<\varsigma_{1v}$, $W_2(z_0)=\varsigma_{2v}$ or $V(z_0)=\tau$, $W_1(z_0)=\varsigma_{1v}$, $W_2(z_0)<\varsigma_{2v}$. By Lemma 2.4,

$$\Theta^{V(z_0)\vec{W}(z_0)}>\Theta^{\tau\vec{\varsigma}_v}.\tag{3.39}$$

Combining (3.19), (3.25), (3.38) and (3.39) with Lemma 2.14, we have

$$\lim_{k\to\infty}\Theta_{\varepsilon_k}=\lim_{k\to\infty}\hat{\Theta}_{\varepsilon_k}=\Theta^{V_0\vec{W}_0}=\Theta^{V(z_0)\vec{W}(z_0)}>\Theta^{\tau\vec{\varsigma}_v}\ge\limsup_{k\to\infty}\Theta_{\varepsilon_k},$$

which is impossible. Hence $x_0 = z_0 \in \mathcal{A}_v$.

By Lemma 3.6, if $\mathcal{V} \cap (\mathcal{W}_1 \cap \mathcal{W}_2)$ is not empty, we assume $x_0 \in \mathcal{A}_v = \mathcal{V} \cap (\mathcal{W}_1 \cap \mathcal{W}_2)$, which implies that

$$\lim_{\varepsilon \to 0} \operatorname{dist}\left(\varepsilon z_\varepsilon, \mathcal{V} \cap (\mathcal{W}_1 \cap \mathcal{W}_2)\right) = 0 \text{ and } V(x_0) = \tau, W_j(x_0) = \varsigma_j,\ j = 1,2.$$

Hence u is a groundstate solution of Equation (1.5). This completes the proof of Theorem 1.3.

Similar to the proof of Step 6 in [18], we have the following result.

Lemma 3.8. There exists $C > 0$ such that for small $\varepsilon > 0$, $u_\varepsilon(x) \le Ce^{-\sqrt{\frac{\tau}{2}}|x - z_\varepsilon|}$ for all $x \in \mathbb{R}^N$.

Now we prove Theorem 1.3 by Lemma 3.3 - Lemma 3.8. Set $x_\varepsilon = \varepsilon z_\varepsilon$, then $v_\varepsilon(x_\varepsilon) = u_\varepsilon(z_\varepsilon)$. By Lemma 3.6, x_ε is a maximum point of v_ε and $\{x_\varepsilon\}_\varepsilon$ is bounded on $\mathbb{R}^N$. By Lemma 3.7, $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathcal{A}_v) = 0$. By Lemma 3.3 and Lemma 3.4, $\hat{u}_\varepsilon(x) = u_\varepsilon(x + z'_\varepsilon) = v_\varepsilon(\varepsilon x + x_\varepsilon - \varepsilon x'_\varepsilon)$, where $x'_\varepsilon = z_\varepsilon - z'_\varepsilon$ is a maximum point of $\hat{u}_\varepsilon$ with $\varepsilon x'_\varepsilon \to 0$ as $\varepsilon \to 0$. By Lemma 3.8, we obtain $v_\varepsilon(x) \le Ce^{-\frac{c}{\varepsilon}|x - x_\varepsilon|}$, where C depends on N, τ.

Consequently, we establish the multiplicity of the semi-classical solutions for Equation (1.2), and we obtain the existence, concentration, convergence, exponential decay estimates of the positive ground state solution. We also prove the existence of sign-changing solutions of Equation (1.2).

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Byeon, J. and Wang, Z.Q. (2002) Standing Waves with a Critical Frequency for Nonlinear Schrödinger Equations. *Archive for Rational Mechanics and Analysis*, **165**, 295-316. https://doi.org/10.1007/s00205-002-0225-6

[2] Ambrosetti, A., Badiale, M. and Cingolani, S. (1997) Semiclassical States of Nonlinear Schrödinger Equations. *Archive for Rational Mechanics and Analysis*, **140**, 285-300. https://doi.org/10.1007/s002050050067

[3] Wang, X. and Zeng, B. (1997) On Concentration of Positive Bound States of Nonlinear Schrödinger Equations with Competing Potential Functions. *SIAM Journal on Mathematical Analysis*, **28**, 633-655. https://doi.org/10.1137/S0036141095290240

[4] Ding, Y. and Liu, X. (2013) Semiclassical Solutions of Schrödinger Equations with Magnetic Fields and Critical Nonlinearities. *Manuscripta Mathematica*, **140**, 51-82. https://doi.org/10.1007/s00229-011-0530-1

[5] Ambrosetti, A. and Wang, Z.Q. (2005) Nonlinear Schrödinger Equations with Vanishing and Decaying Potentials. *Differential and Integral Equations*, **18**, 1321-1332. https://doi.org/10.57262/die/1356059712

[6] Bonheure, D., Cingolani, S. and Nys, M. (2016) Nonlinear Schrödinger Equation: Concentration on Circles Driven by an External Magnetic Field. *Calculus of Variations and Partial Differential Equations*, **55**, Article No. 82. https://doi.org/10.1007/s00526-016-1013-8

[7] Kurata, K. (2000) Existence and Semi-Classical Limit of the Least Energy Solution to a Nonlinear Schrödinger Equation with Electromagnetic Fields. *Nonlinear Analysis: Theory, Methods & Applications*, **41**, 763-778. https://doi.org/10.1016/S0362-546X(98)00308-3

[8] Cingolani, S. and Lazzo, M. (2000) Multiple Positive Solutions to Nonlinear Schrödinger Equations with Competing Potential Functions. *Journal of Differential Equations*, **160**, 118-138. https://doi.org/10.1006/jdeq.1999.3662

[9] Sun, M. (2012) Multiplicity of Solutions for a Class of the Quasilinear Elliptic Equations at Resonance. *Journal of Mathematical Analysis and Applications*, **386**, 661-668. https://doi.org/10.1016/j.jmaa.2011.08.030

[10] Bartolo, R. and Bisci, G.M. (2015) A Pseudo-Index Approach to Fractional Equations. *Expositiones Mathematicae*, **33**, 502-516. https://doi.org/10.1016/j.exmath.2014.12.001

[11] Papageorgiou, N.S., Rădulescu, V.D. and Repovš, D.D. (2020) Existence and Multiplicity of Solutions for Double-Phase Robin Problems. *Bulletin of the London Mathematical Society*, **52**, 546-560. https://doi.org/10.1112/blms.12347

[12] Wu, Y., Tahar, B., Rafik, G., Rahmoune, A. and Yang, L. (2022) The Existence and Multiplicity of Homoclinic Solutions for a Fractional Discrete p-Laplacian Equation. *Mathematics*, **10**, Article 1400. https://doi.org/10.3390/math10091400

[13] Wang, L., Cheng, K. and Wang, J. (2022) The Multiplicity and Concentration of Positive Solutions for the Kirchhoff-Choquard Equation with Magnetic Fields. *Acta Mathematica Scientia*, **42**, 1453-1484. https://doi.org/10.1007/s10473-022-0411-6

[14] Guo, L. and Li, Q. (2022) Existence and Multiplicity Results for Fractional Schrödinger Equation with Critical Growth. *The Journal of Geometric Analysis*, **32**, Article No. 277. https://doi.org/10.1007/s12220-022-01011-0

[15] Alves, C.O. and Figueiredo, G.M. (2014) Multiple Solutions for a Semilinear Elliptic Equation with Critical Growth and Magnetic Field. *Milan Journal of Mathematics*, **82**, 389-405. https://doi.org/10.1007/s00032-014-0225-7

[16] Bisci, G.M. and Rădulescu, V.D. (2015) Ground State Solutions of Scalar Field Fractional Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **54**, 2985-3008. https://doi.org/10.1007/s00526-015-0891-5

[17] Khoutir, S. (2023) Infinitely Many Solutions for a Fractional Schrödinger Equation in $\mathbb{R}^N$ with Combined Nonlinearities. *Bulletin of the Malaysian Mathematical Sciences Society*, **46**, Article No. 58. https://doi.org/10.1007/s40840-022-01457-z

[18] Ding, Y. and Wei, J. (2017) Multiplicity of Semiclassical Solutions to Nonlinear Schrödinger Equations. *Journal of Fixed Point Theory and Applications*, **19**, 987-1010. https://doi.org/10.1007/s11784-017-0410-8

[19] Liu, M. and Tang, Z. (2019) Multiplicity and Concentration of Solutions for Choquard Equation via Nehari Method and Pseudo-Index Theory. *Discrete and Continuous Dynamical Systems*, **39**, 3365-3398. https://doi.org/10.3934/dcds.2019139

[20] Liu, M. and Tang, Z. (2019) Multiplicity and Concentration of Solutions for a Fractional Schrödinger Equation via Nehari Method and Pseudo-Index Theory. *Journal of Mathematical Physics*, **60**, Article ID: 053502. https://doi.org/10.1063/1.5051462

[21] Liu, M. and Tang, Z. (2020) Pseudoindex Theory and Nehari Method for a Fractional Choquard Equation. *Pacific Journal of Mathematics*, **304**, 103-142. https://doi.org/10.2140/pjm.2020.304.103

[22] Badiale, M. and Serra, E. (2010) Semilinear Elliptic Equations for Beginners: Existence Results via the Variational Approach. Springer Science and Business Media,

London. https://doi.org/10.1007/978-0-85729-227-8

[23] Willem, M. (1996) Minimax Theorems. Birckhäuser, Boston. https://doi.org/10.1007/978-1-4612-4146-1

[24] Benci, V. (1982) On Critical Point Theory for Indefinite Functionals in the Presence of Symmetries. *Transactions of the American Mathematical Society*, **274**, 533-572. https://doi.org/10.1090/S0002-9947-1982-0675067-X

[25] Cerami, G., Solimini, S. and Struwe, M. (1986) Some Existence Results for Superlinear Elliptic Boundary Value Problems Involving Critical Exponents. *Journal of Functional Analysis*, **69**, 289-306. https://doi.org/10.1016/0022-1236(86)90094-7