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Development of the Additive-Quadratic η-Function Inequality with $3k$-Variables Based on a General Quadratic Function Variables on a Complex Banach Spaces

Ly Van An

Faculty of Mathematics Teacher Education, Tay Ninh University, Tay Ninh, Vietnam
Email: lyvanan145@gmail.com, lyvananvietnam@gmail.com

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Abstract

In this article, I study the establishment of the quadratic-additive η-function inequality with $3k$-variables on the homogeneous complex Banach space and prove the quadratic-additive η-function equation related to the additive and quadratic η-functional inequalities in (α_1, α_2)-homogeneous Banach complex space.

Subject Areas

Mathematics

Keywords

Additive-Quadratic η-Functional Inequalities, (α_1, α_2)-Homogeneous Complex Banach Spaces, Hyers-Ulam-Rassias Stability

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f : \mathbf{X} \to \mathbf{Y}$. I use the notations $\|\cdot\|_{\mathbf{X}}$, $\|\cdot\|_{\mathbf{Y}}$ as the normals on $\mathbf{X}$ and the normals on $\mathbf{Y}$, respectively. In this paper, I investigate some additive-quadratic η-functional inequalities in (α_1, α_2)-homogeneous complex Banach spaces.

In fact, when $\mathbf{X}$ is a α_1-homogeneous real or complex normed spaces $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a α_2-homogeneous real or complex Banach spaces $\|\cdot\|_{\mathbf{Y}}$

I solve and prove the Hyers-Ulam-Rassias type stability of two following additive-quadratic η-functional inequalities.

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\| h(\eta)\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{1}$$

and when I change the role of the function inequality (1.1), I continue to prove the following function inequality.

$$\begin{aligned}&\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\| h(\eta)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{2}$$

based on following Generalized Quadratic functional equations with $2k$-variable.

$$f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)=2\sum_{i=1}^{k}f(x_i)+2\sum_{k=1}^{k}f(y_i) \tag{3}$$

The Hyers-Ulam stability was the first investigated for the functional equation of Ulam in [1] concerning the stability of group homomorphisms.

The Hyers [2] gave the first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbounded Cauchy difference. A ageneralization of the Rassias theorem was obtained by Găvruta [5] with replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The Hyers-Ulam stability for functional inequalities has been investigated such as Gilányi [6] showed that is if satisfies the functional inequality.

$$\|2f(x)+2f(y)-f(x-y)\|\le\|f(x+y)\| \tag{4}$$

Then f satisfies the Jordan-von Newman functional equation.

$$2f(x)+2f(y)=f(x+y)+f(x-y) \tag{5}$$

Gilányi [7] and Fechner [8] proved the Hyers-Ulam stability of the functional inequality (4).

Next Chookil [9] and [10] proved the of additive β-functional inequalities in non-Archimedean Banach spaces and in complex Banach spaces, and Harin Lee[a] [11] [12] [13] proved the Hyers-Ulam stability of additive β-functional inequalities in ρ-homogeneous F space.

Recently, the author has studied the additive-quadratic functional inequalities of mathematicians around the world, on spaces complex Banach spaces, non-Archimedan Banach spaces or additive β-functional inequalities in p-homogeneous F-space.... See [14]-[19].

So in this paper, I solve and prove the Hyers-Ulam stability for two additive-quadratic η-functional inequalities (1)-(2), *i.e.* the additive-quadratic η-functional inequalities with $3k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, I will prove that the mappings satisfy the additive-quadratic η-functional inequalities (1) or (2). Thus, the results in this paper are a generalization of those in [14]-[20] for additive-quadratic η-functional inequalities with $3k$-variables.

In this paper, I have constructed a general quadratic linear functional inequality to improve the classical linear linear inequality. This problem I think is one outstanding development for the mathematics industry modern studies in the field of functional equations in particular and mathematics in general. I would like to express my gratitude to the senior mathematicians [1]-[24] who have inspired today's mathematics researchers.

The paper is organized as follows: In section preliminaries, I remind a basic property such as I only redefine the solution definition of the equations of the additive function, the equations of the quadratic function and F^*-space.

Section 3: Constructing solution to the quadratic η-functional inequalities (1) in (α_1,α_2)-homogeneous complex Banach spaces.

Section 4: Constructing solution to the quadratic η-functional inequalities (2) in (α_1,α_2)-homogeneous complex Banach spaces.

Section 5: Constructing solution to the additive η-functional inequalities (1) in (α_1,α_2)-homogeneous complex Banach spaces.

Section 6: Constructing solution to the additive η-functional inequalities (2) in (α_1,α_2)-homogeneous complex Banach spaces.

2. Preliminaries

2.1. F^*-Spaces

Let $\mathbf{X}$ be a (complex) linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

1. $\|x\|=0$ if and only if $x=0$;
2. $\|\lambda x\|=\|x\|$ for all $x\in X$ and all λ with $|\lambda|=1$;

3. $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in X$;
4. $\|\lambda_n x\| \to 0$, $\lambda_n \to 0$;
5. $\|\lambda_n x\| \to 0$, $x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space. An F-norm is called β-homgeneous ($\beta > 0$) if $\|tx\| = |t|^{\beta} \|x\|$ for all $x \in \mathbf{X}$ and for all $t \in \mathbb{C}$ and $(X, \|\cdot\|)$ is called α-homogeneous F-space.

2.2. Solutions of the Inequalities

The functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called the qudratic equation. In particular, every solution of the quadratic equation is said to be a quadratic mapping.

The functional equation:

$$f(x+y) = f(x) + f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The functional equation:

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2} f(x) + \frac{1}{2} f(y)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be a Jensen mapping.

The functional equation:

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2} f(x) + \frac{1}{2} f(y)$$

is called the Jensen type qudratic equation. In particular, every solution of the quadratic equation is said to be a Jensen type quadratic mapping.

$$D = \left\{ \varphi : \mathbb{C} \to \mathbb{C} : g(\eta) = \eta, |g(\eta)| = |\eta| \le \frac{1}{2} \right\} \tag{6}$$

Note: With k is a positive integer and $h \in A$ $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1, \alpha_2 \le 1$.

3. Constructing Solution to the η-Functional Inequalities (2) in (α_1, α_2)-Homogeneous Complex Banach Spaces

Now, I first study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ is a α_1-homogeneous real or complex normed spaces $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a α_2-homogeneous real or complex Banach spaces $\|\cdot\|_{\mathbf{Y}}$. Under this setting, I can show that the mapping satisfying (1) is quadratic. These results are given in the following.

Lemma 1 Let $f : \mathbf{X} \to \mathbf{Y}$ be an even mapping satilies:

$$\begin{aligned}&\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|\eta\Bigg(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}\end{aligned}\tag{7}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (7).

I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (7), I have:

$$\|(4k-2)f(0)\|_{\mathbf{Y}}\le\|\eta f(0)\|_{\mathbf{Y}}\le 0$$

therefore,

$$\left(|4k-2|^{\alpha_2}-|\eta|^{\alpha_2}\right)\|f(0)\|_{\mathbf{Y}}\le 0$$

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (7), I have.

Thus

$$\|f(2kx)-4kf(x)\|_{\mathbf{Y}}\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{4k}f(x)\tag{8}$$

for all $x\in\mathbf{X}$.

From (7) and (8) I infer that:

$$\begin{aligned}&\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|\eta\Bigg(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}\\&=\frac{|\eta|^{\alpha_2}}{|2k|^{\alpha_2}}\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\end{aligned}\tag{9}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$ and so,

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)$$
$$=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+2\sum_{j=1}^{k}f\left(z_j\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$, as I expected. The couverse is obviously true. □

Corollary 1 *Let* $f:\mathbf{X}\to\mathbf{Y}$ *be an even mapping satilies:*

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\\&=\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(z_j\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-z_j\right)\right)\end{aligned}\tag{10}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

Note! The functional Equation (10) is called an quadratic η-functional equation.

Theorem 2 Assume for $r>\frac{2\alpha_2}{\alpha_1}$, θ be nonngative real number, and suppose $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping such that:

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(z_j\right)\\&\left.\left.-\frac{1}{2k}\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{11}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbf{X}$. Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\left\| f(x)-\phi(x) \right\|_{\mathbf{Y}} \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}-(4k)^{\alpha_2}} \theta \|x\|_{\mathbf{X}}^r \tag{12}$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (11).

I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (11), I have:

$$\left\|(4k-2) f(0)\right\|_{\mathbf{Y}} \le \left\|2\eta f(0)\right\|_{\mathbf{Y}} \le 0$$

therefore,

$$\left(|4k-2|^{\alpha_2}-|2\eta|^{\alpha_2}\right)\left\| f(0)\right\|_{\mathbf{Y}} \le 0$$

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (11) I have:

$$\left\| f(2x)-4kf(x)\right\|_{\mathbf{Y}} \le \left(2k^{\alpha_1 r+1}+1\right)\theta\|x\|_{\mathbf{X}}^r \tag{13}$$

for all $x \in \mathbf{X}$. Thus,

$$\left\| f(x)-4kf\left(\frac{x}{2k}\right)\right\|_{\mathbf{Y}} \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\|x\|_{\mathbf{X}}^r \tag{14}$$

for all $x \in \mathbf{X}$.

$$\begin{aligned} &\left\|(4k)^l f\left(\frac{x}{(2k)^l}\right)-(4k)^m f\left(\frac{x}{(2k)^m}\right)\right\|_{\mathbf{Y}} \\ &\le \sum_{j=1}^{m-1}\left\|(4k)^j f\left(\frac{x}{(2k)^j}\right)-(4k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\|_{\mathbf{Y}} \\ &\le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\sum_{j=1}^{m-1}\frac{(4k)^{\alpha_2 j}}{(2k)^{\alpha_1 r j}}\|x\|_{\mathbf{X}}^r \end{aligned} \tag{15}$$

for all nonnegative integers *p*, *l* with $p>l$ and all $x\in\mathbf{X}$. It follows from (15) that the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by,

$$\phi(x) := \lim_{n\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (15), I get (12).

Form $f:\mathbf{X}\to\mathbf{Y}$ is even, the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is even.

It follows from (11) that:

$$\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}}$$
$$=\lim_{n\to\infty}(4k)^{\alpha_2 n}\left\|f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.$$
$$+f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)$$
$$\left.-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|_{\mathbf{Y}}$$
$$\leq\lim_{n\to\infty}(4k)^{\alpha_2 n}|\eta|^{\alpha_2}\left\|2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.$$
$$+2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)$$
$$\left.+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)-\frac{1}{2}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\frac{1}{2}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|_{\mathbf{Y}}$$
$$+\lim_{n\to\infty}\frac{(4k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right)$$
$$=|\eta|^{\alpha_2}\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}} \tag{16}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for all $j=1\to n$.

$$\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.$$
$$\left.\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}} \tag{17}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, So by lemma 3.1, it follows that the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is additive. Now I need to prove uniqueness, Suppose $\phi':\mathbf{X}\to\mathbf{Y}$ is also a quadratic mapping that satisfies (12). Then I have:

$$\begin{aligned}\|\phi(x)-\phi'(x)\|_{\mathbf{Y}} &= (4k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}} \\ &\le (4k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\right) \\ &\le \frac{2\cdot(4k)^{\alpha_2 n}\cdot\left(2k^{\alpha_1 r+1}+1\right)}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(4k)^{\alpha_2}\right)}\theta\|x\|_{\mathbf{X}}^r\end{aligned} \tag{18}$$

which tends to zero as $n\to\infty$ for all $x\in X$. So I can conclude that $\phi(x)=\phi'(x)$ for all $x\in\mathbf{X}$. This proves thus the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is a unique mapping satisfying(12) as I expected.

Theorem 3 Assume for $r<\frac{2\alpha_2}{\alpha_1}$, θ be nonngative real number, and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satisfiying (11). Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\|f(x)-\phi(x)\|\le\frac{2k^{\alpha_1 r+1}+1}{(4k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|^r \tag{19}$$

for all $x\in\mathbf{X}$.

The proof is similar to the proof of theorem 3.3.

4. Constructing Solution to the η-Functional Inequalities (2) in (α_1,α_2)-Homogeneous Complex Banach Spaces

Now, I study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ is a α_1-homogeneous complex Banach spaces and that $\mathbf{Y}$ is a α_2-homogeneous complex Banach spaces.

Under this setting, I can show that the mapping satisfying (2) is quadratic. These results are given in the following.

Lemma 4 Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies $f(0)=0$ and:

$$\begin{aligned}&\left\|2f\left(\sum_{j=1}^k\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^k z_j\right)+2f\left(\sum_{j=1}^k\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^k z_j\right)\right. \\ &\left.-\frac{3}{2k}\sum_{j=1}^k f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^k f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^k f(z_j)-\frac{1}{2k}\sum_{j=1}^k f(-z_j)\right\|_{\mathbf{Y}} \\ &\le\left\|\eta\left(f\left(\sum_{j=1}^k\frac{x_j+y_j}{2k}+\sum_{j=1}^k z_j\right)+f\left(\sum_{j=1}^k\frac{x_j+y_j}{2k}-\sum_{j=1}^k z_j\right)\right.\right. \\ &\left.\left.-2\sum_{j=1}^k f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^k f(z_j)-\sum_{j=1}^k f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{20}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (20).

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (20), I have.

Thus

$$\left\|4f\left(\frac{x}{2k}\right)-\frac{1}{k}f(x)\right\|_{\mathbf{Y}}\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{4k}f(x) \tag{21}$$

for all $x\in\mathbf{X}$.

From (20) and (21) I infer that:

$$\begin{aligned}&\frac{1}{(2k)^{\alpha_2}}\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&=\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le|\eta|^{\alpha_2}\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\end{aligned} \tag{22}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ and so:

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+2\sum_{j=1}^{k}f(z_j)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, as I expected. The couverse is obviously true.

Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies,

$$\begin{aligned}&2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\\&=\eta\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\end{aligned} \tag{23}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic Note! The functional Equation (23) is called an quadratic λ-functional equation.

Theorem 5 Assume for $r>\frac{2\alpha_2}{\alpha_1}$, θ be nonngative real number, and suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that $f(0)=0$ and

$$\begin{aligned}&\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\|\eta\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right.\right.\\&\left.\left.-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right)\end{aligned}\tag{24}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$. Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\|f(x)-\phi(x)\|_{\mathbf{Y}}\le\frac{(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 r}-(4k)^{\alpha_2}}\theta\|x\|_{\mathbf{X}}^r\tag{25}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (24).

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (24) I have:

$$\left\|4f\left(\frac{x}{2k}\right)-\frac{1}{k}f(x)\right\|_{\mathbf{Y}}\le(2k)^{\alpha_1 r}\theta\|x\|_{\mathbf{X}}^r\tag{26}$$

for all $x\in\mathbf{X}$. Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-f(x)\right\|\le(2k)^{\alpha_1 r}k^{\alpha_2}\theta\|x\|_{\mathbf{X}}^r\tag{27}$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|(4k)^l f\left(\frac{x}{(2k)^l}\right)-(4k)^m f\left(\frac{x}{(2k)^m}\right)\right\|\\&\le\sum_{j=1}^{m-1}\left\|(4k)^j f\left(\frac{x}{(2k)^j}\right)-(4k)^{j+1}f\left(\frac{x}{(2k)^{j+1}}\right)\right\|_{\mathbf{Y}}\\&\le(2k)^{\alpha_1 r}k^{\alpha_2}\theta\sum_{j=1}^{m-1}\frac{(4k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|_{\mathbf{X}}^r\end{aligned}\tag{28}$$

for all nonnegative integers *p*, *l* with $p>l$ and all $x\in\mathbf{X}$. It follows from (28) that the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x):=\lim_{n\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (28), I get (25). Form $f : \mathbf{X} \to \mathbf{Y}$ is even, the mapping:

$$\phi : \mathbf{X} \to \mathbf{Y}$$

is even. It follows from (24) I have:

$$\begin{aligned}
&\Bigg\| 2\phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} + \frac{1}{2k}\sum_{j=1}^{k} z_j \right) + 2\phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} - \frac{1}{2k}\sum_{j=1}^{k} z_j \right) \\
&- \frac{3}{2k}\sum_{j=1}^{k} \phi\left(\frac{x_j + y_j}{2k} \right) + \frac{1}{2k}\sum_{j=1}^{k} \phi\left(-\frac{x_j + y_j}{2k} \right) - \frac{1}{2k}\sum_{j=1}^{k} \phi(z_j) - \frac{1}{2k}\sum_{j=1}^{k} \phi(-z_j) \Bigg\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty} (4k)^{\alpha_2 n} \Bigg\| 2f\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^{n+2}} + \frac{1}{(2k)^{n+1}}\sum_{j=1}^{k} z_j \right) \\
&+ 2f\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^{n+2}} - \frac{1}{(2k)^{n+1}}\sum_{j=1}^{k} z_j \right) - \frac{3}{2k}\sum_{j=1}^{k} f\left(\frac{x_j + y_j}{(2k)^{n+1}} \right) \\
&+ \frac{1}{2k}\sum_{j=1}^{k} f\left(-\frac{x_j + y_j}{(2k)^{n+1}} \right) - \frac{1}{2k}\sum_{j=1}^{k} f\left(\frac{z_j}{(2k)^n} \right) - \frac{1}{2k}\sum_{j=1}^{k} f\left(\frac{-z_j}{(2k)^n} \right) \Bigg\|_{\mathbf{Y}} \\
&\le \lim_{n\to\infty} (4k)^{\alpha_2 n} |\eta|^{\alpha_2} \Bigg\| 2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \frac{1}{(2k)^n}\sum_{j=1}^{k} z_j \right) \\
&+ 2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \frac{1}{(2k)^n}\sum_{j=1}^{k} z_j \right) - 2\sum_{j=1}^{k} f\left(\frac{1}{(2k)^n}\frac{x_j + y_j}{2k} \right) \\
&- \frac{1}{2k}\sum_{j=1}^{k} f\left(\frac{1}{(2k)^n} z_j \right) - \frac{1}{2k}\sum_{j=1}^{k} f\left(-\frac{1}{(2k)^n} z_j \right) \Bigg\|_{\mathbf{Y}} \\
&+ \lim_{n\to\infty} \frac{(4k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}} \theta\left(\sum_{j=1}^{k} \|x_j\|_{\mathbf{X}}^r + \sum_{j=1}^{k} \|y_j\|_{\mathbf{X}}^r + \sum_{j=1}^{k} \|z_j\|_{\mathbf{X}}^r \right) \\
&= \Bigg\| \phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) + \phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j \right) \\
&- 2\sum_{j=1}^{k} \phi\left(\frac{x_j + y_j}{2k} \right) - \sum_{j=1}^{k} \phi(z_j) - \sum_{j=1}^{k} \phi(-z_j) \Bigg\|_{\mathbf{Y}}
\end{aligned} \tag{29}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbf{X}$.

$$\begin{aligned}
&\Bigg\| 2\phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} + \frac{1}{2k}\sum_{j=1}^{k} z_j \right) + 2\phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} - \frac{1}{2k}\sum_{j=1}^{k} z_j \right) \\
&- \frac{3}{2k}\sum_{j=1}^{k} \phi\left(\frac{x_j + y_j}{2k} \right) + \frac{1}{2k}\sum_{j=1}^{k} \phi\left(-\frac{x_j + y_j}{2k} \right) - \frac{1}{2k}\sum_{j=1}^{k} \phi(z_j) - \frac{1}{2k}\sum_{j=1}^{k} \phi(-z_j) \Bigg\|_{\mathbf{Y}} \\
&\le \Bigg\| \eta\left(\phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) + \phi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j \right) \right. \\
&- 2\sum_{j=1}^{k} \phi\left(\frac{x_j + y_j}{2k} \right) - \sum_{j=1}^{k} \phi(z_j) - \sum_{j=1}^{k} \phi(-z_j) \Bigg\|_{\mathbf{Y}}
\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$, So by lemma 4.1 it follows that the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is quadratic. Now I need to prove uniqueness, Suppose $\phi':\mathbf{X}\to\mathbf{Y}$ is also a quadratic mapping that satisfies (25). Then I have:

$$\begin{aligned}\left\|\phi(x)-\phi'(x)\right\|_{\mathbf{Y}}&=(4k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\\&\le(4k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\right)\\&\le\frac{2\cdot(4k)^{\alpha_2 n}\cdot(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(4k)^{\alpha_2}\right)}\theta\|x\|_{\mathbf{X}}^r\end{aligned}\tag{30}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbf{X}$. So I can conclude that $\phi(x)=\phi'(x)$ for all $x\in\mathbf{X}$.This proves thus the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is a unique mapping satisfying(25) as I expected. □

Theorem 6 Assume for $r<\frac{2\alpha_2}{\alpha_1}$, θ be nonngative real number, $f(0)=0$ and suppose $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping (24). Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\left\|f(x)-\phi(x)\right\|_{\mathbf{Y}}\le\frac{(2k)^{\alpha_1 r}}{(4k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|_{\mathbf{X}}^r\tag{31}$$

for all $x\in\mathbf{X}$.

The proof is similar to the proof of theorem 4.3.

5. Constructing Solution to the Additive η-Functional Inequalities (1) in (α_1,α_2)-Homogeneous Complex Banach Spaces

Now, I first study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ is a α_1-homogeneous complex Banach spaces and that $\mathbf{Y}$ is a α_2-homogeneous complex Banach spaces. Under this setting, I can show that the mapping satisfying (1) is additive. These results are given in the following.

Lemma 7 Let $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping satilies:

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^k\frac{x_j+y_j}{2k}+\sum_{j=1}^k z_j\right)+f\left(\sum_{j=1}^k\frac{x_j+y_j}{2k}-\sum_{j=1}^k z_j\right)\right.\\&\left.-2\sum_{j=1}^k f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^k f(z_j)-\sum_{j=1}^k f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\|\eta\left(2f\left(\sum_{j=1}^k\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^k z_j\right)+2f\left(\sum_{j=1}^k\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^k z_j\right)\right.\right.\\&\left.\left.-\frac{3}{2k}\sum_{j=1}^k f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^k f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^k f(z_j)-\frac{1}{2k}\sum_{j=1}^k f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{32}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (32).

I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (32), I have:

$$\left\|(4k-2)f(0)\right\|_{\mathbf{Y}}\le|\eta|^{\alpha_2}\left\|5f(0)\right\|_{\mathbf{Y}}\le 0$$

therefore $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (32), I have.

Thus

$$\left\|f(2kx)-2kf(x)\right\|_{\mathbf{Y}}\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f(x) \tag{33}$$

for all $x\in\mathbf{X}$ From (32) and (33) I infer that:

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\|\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\\&=\frac{|\eta|^{\alpha_2}}{|k|^{\alpha_2}}\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\end{aligned} \tag{34}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ and so.

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right) \tag{35}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$.

Next I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,z,\cdots,z)$ in (35), I have

$$f(kx+kz)+f(kx-kz)=2kf(x) \tag{36}$$

for all $x,z\in\mathbf{X}$ Now letting $p=kx+kz,q=kx-kz$ when that in (36), I get

$$f(p)+f(q)=2kf\left(\frac{p+q}{2k}\right)=2k\cdot\frac{1}{2k}f(p+q)=f(p+q) \tag{37}$$

for all $p,q\in\mathbf{X}$. So f is an additive mapping. as I expected. The couverse is obviously true. □

Corollary 2 Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies:

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\\&=\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(z_j\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-z_j\right)\right)\end{aligned}\tag{38}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbf{X}$ if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

Note! The functional Equation (38) is called an additive η-functional equation.

Theorem 8 Assume for $r>\frac{\alpha_2}{\alpha_1}$, θ be nonngative real number, and suppose $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping such that:

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(z_j\right)\\&\left.\left.-\frac{1}{2k}\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^r\right)\end{aligned}\tag{39}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in X$. Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\left\|f(x)-\phi(x)\right\|_{\mathbf{Y}}\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}-(2k)^{\alpha_2}}\theta\|x\|_{\mathbf{X}}^r.\tag{40}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (39). I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (39), I have:

$$O\left\|(4k-2)f(0)\right\|_{\mathbf{Y}}\le\left\|5\lambda f(0)\right\|_{\mathbf{Y}}$$

therefore,

$$\left(|4k-2|^{\alpha_2}-|5\lambda|^{\alpha_2}\right)\left\|f(0)\right\|_{\mathbf{Y}}\le 0$$

So $f(0)=0$. Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (39) I have:

$$\left\|f(2kx)-2kf(x)\right\|_{\mathbf{Y}} \le \left(2k^{\alpha_1 r+1}+1\right)\theta\|x\|_{\mathbf{X}}^r \tag{41}$$

for all $x\in\mathbf{X}$. Thus

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\|_{\mathbf{Y}} \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\|x\|_{\mathbf{X}}^r \tag{42}$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|(2k)^l f\left(\frac{x}{(2k)^l}\right)-(2k)^m f\left(\frac{x}{(2k)^m}\right)\right\|_{\mathbf{Y}}\\ &\le \sum_{j=1}^{m-1}\left\|(2k)^j f\left(\frac{x}{(2k)^j}\right)-(2k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\|_{\mathbf{Y}}\\ &\le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\sum_{j=1}^{m-1}\frac{(2k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|_{\mathbf{X}}^r\end{aligned} \tag{43}$$

for all nonnegative integers *p*, *l* with $p>l$ and all $x\in\mathbf{X}$. It follows from (15) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x):=\lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (15), I get (40).

Form $f:\mathbf{X}\to\mathbf{Y}$ is even, the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is even.

It follows from (39) I have:

$$\begin{aligned}&\Bigg\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\ &-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\Bigg\|_{\mathbf{Y}}\\ &=\lim_{n\to\infty}(2k)^{\alpha_2 n}\Bigg\|f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\\ &+f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)\\ &-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\Bigg\|_{\mathbf{Y}}\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty}(2k)^{\alpha_2 n}|\lambda|^{\alpha_2}\Bigg\|2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)\\
&+2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)\\
&-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\Bigg\|_{\mathbf{Y}}\\
&+\lim_{n\to\infty}\frac{(2k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right)\\
&=|\lambda|^{\alpha_2}\Bigg\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\
&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}
\end{aligned}
\tag{44}
$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$.

$$
\begin{aligned}
&\Bigg\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\
&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\
&\le\Bigg\|\eta\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right)\Bigg\|_{\mathbf{Y}}
\end{aligned}
$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, So by lemma 5.1, it follows that the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is additive. Now I need to prove uniqueness, suppose $\phi' : \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (40). Then I have:

$$
\begin{aligned}
\|\phi(x)-\phi'(x)\|_{\mathbf{Y}} &= (2k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\\
&\le (2k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\right)\\
&\le \frac{2\cdot(2k)^{\alpha_2 n}\cdot\left(2k^{\alpha_1 r+1}+1\right)}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(2k)^{\alpha_2}\right)}\theta\|x\|_{\mathbf{X}}^r
\end{aligned}
\tag{45}
$$

which tends to zero as $n \to \infty$ for all $x \in X$. So I can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$. This proves thus the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying(40) as I expected. □

Theorem 9 Assume for $r < \frac{\alpha_2}{\alpha_1}$, θ be nonngative real number, and suppose

$f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping satisfying (1). Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\left\|f(x)-\phi(x)\right\|_{\mathbf{Y}}\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|_{\mathbf{X}}^{r} \tag{46}$$

for all $x\in\mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 5.3.

6. Constructing Solution to the Additive η-Functional Inequalities (2) in (α_1,α_2)-Homogeneous Complex Banach Spaces

Now, I first study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ is a α_1-homogeneous complex Banach spaces and that $\mathbf{Y}$ is a α_2-homogeneous complex Banach spaces. Under this setting, I can show that the mapping satisfying (2) is additive. These results are given in the following.

Lemma 10 Let $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping satilies:

$$\begin{aligned}&\Bigg\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|\eta\Bigg(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}\end{aligned} \tag{47}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (47).

I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (20), I have:

$$\left\|2kf(0)\right\|_{\mathbf{Y}}\le|\eta|^{\alpha_2}\left\|(4k-2)f(0)\right\|_{\mathbf{Y}}$$

So $f(0)=0$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (47), I have.

Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-2f(x)\right\|_{\mathbf{Y}}\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f(x) \tag{48}$$

for all $x\in\mathbf{X}$. From (47) and (48) I infer that:

$$\begin{aligned}
&\frac{1}{|k|^{\alpha_2}}\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\
&=\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\
&\le|\eta|^{\alpha_2}\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}
\end{aligned} \tag{49}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, and so:

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, as I expected. The couverse is obviously true. □

Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies.

Theorem 11 Assume for $r>\frac{\alpha_2}{\alpha_1}$, θ be nonngative real number, and suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that $f(0)=0$ and:

$$\begin{aligned}
&\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\
&\le\left\|\eta\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right.\right.\\
&\left.\left.-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^{r}\right)
\end{aligned} \tag{50}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$. Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that:

$$\|f(x)-\phi(x)\|_{\mathbf{Y}}\le\frac{(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 r}-(4k)^{\alpha_2}}\theta\|x\|_{\mathbf{X}}^{r} \tag{51}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (50).

I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (50), I have:

$$\|2f(0)\|_{\mathbf{Y}}\le|\eta|^{\alpha_2}\|(4k-2)f(0)\|_{\mathbf{Y}}$$

therefore,

$$\left(|4k-2|^{\alpha_2}-|2\eta|^{\alpha_2}\right)\|f(0)\|_{\mathbf{Y}}\le 0$$

So $f(0)=0$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (50) I have:

$$\left\|4f\left(\frac{x}{2k}\right)-\frac{1}{k}f(x)\right\|_{\mathbf{Y}}\le(2k)^{\alpha_1 r}\theta\|x\|_{\mathbf{X}}^r \tag{52}$$

for all $x\in\mathbf{X}$. Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-f(x)\right\|_{\mathbf{Y}}\le(2k)^{\alpha_1 r}k^{\alpha_2}\theta\|x\|_{\mathbf{X}}^r \tag{53}$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|(4k)^l f\left(\frac{x}{(2k)^l}\right)-(4k)^m f\left(\frac{x}{(2k)^m}\right)\right\|_{\mathbf{Y}}\\&\le\sum_{j=1}^{m-1}\left\|(4k)^j f\left(\frac{x}{(2k)^j}\right)-(4k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\|_{\mathbf{Y}}\\&\le(2k)^{\alpha_1 r}k^{\alpha_2}\theta\sum_{j=1}^{m-1}\frac{(4k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|_{\mathbf{X}}^r\end{aligned} \tag{54}$$

for all nonnegative integers *p*, *l* with $p>l$ and all $x\in\mathbf{X}$. It follows from (54) that the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x):=\lim_{n\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (28), I get (51). Form $f:\mathbf{X}\to\mathbf{Y}$ is even, the mapping

$$\phi:\mathbf{X}\to\mathbf{Y}$$

is even. It follows from (50) I have:

$$\left\|2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-\frac{3}{2k}\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}\phi\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}\phi(z_j)-\frac{1}{2k}\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}}$$
$$=\lim_{n\to\infty}(4k)^{\alpha_2 n}\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^{n+2}}+\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)\right.$$
$$+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^{n+2}}-\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{(2k)^{n+1}}\right)$$
$$\left.+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{(2k)^{n+1}}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{z_j}{(2k)^n}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{-z_j}{(2k)^n}\right)\right\|_{\mathbf{Y}}$$
$$\le\lim_{n\to\infty}(4k)^{\alpha_2 n}|\eta|^{\alpha_2}\left\|2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.$$
$$+2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)$$
$$\left.-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|_{\mathbf{Y}}$$
$$+\lim_{n\to\infty}\frac{(4k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right)$$
$$=\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}} \tag{55}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$.

$$\left\|2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-\frac{3}{2k}\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}\phi\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}\phi(z_j)-\frac{1}{2k}\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}}$$
$$\le\left\|\lambda\left(\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right.$$
$$\left.\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right)\right\|_{\mathbf{Y}}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, So by lemma 6.1, it follows that the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is quadratic. Now I need to prove uniqueness, suppose $\phi':\mathbf{X}\to\mathbf{Y}$ is also a quadratic mapping that satisfies (50). Then I have:

$$\begin{aligned}\|\phi(x)-\phi'(x)\|_{\mathbf{Y}} &= (4k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}} \\ &\le (4k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}\right) \\ &\le \frac{2\cdot(4k)^{\alpha_2 n}\cdot(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(4k)^{\alpha_2}\right)}\theta\|x\|_{\mathbf{X}}^r\end{aligned} \tag{56}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So I can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$. This proves thus the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying(51) as I expected.

Theorem 12 Assume for $r < \frac{\alpha_2}{\alpha_1}$, θ be nonngative real number, $f(0) = 0$ and suppose $f: \mathbf{X} \to \mathbf{Y}$ be an odd mapping satisfying (50). Then there exists a unique quadratic mapping $\phi: \mathbf{X} \to \mathbf{Y}$ such that:

$$\|f(x)-\phi(x)\|_{\mathbf{Y}} \le \frac{(2k)^{\alpha_1 r}}{(4k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|_{\mathbf{X}}^r. \tag{57}$$

for all $x \in \mathbf{X}$.

The proof is similar to theorem 6.2.

7. Conclusion

In the article, I developed the quadratic additivity η-function inequality with many variables on the complex (α_1, α_2)-homogeneous Banach space and showed that their solution is a quadratic additivity map. This is a remarkable idea for modern mathematics.

Conflicts of Interest

The author declares no conflicts of interest.

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