Open Access Library Journal
2023, Volume 10, e10714
ISSN Online: 2333-9721
ISSN Print: 2333-9705

TULLY-FISHER Law Demonstrated by General Relativity and Dark Matter

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How to cite this paper: Le Corre, S. (2023) TULLY-FISHER Law Demonstrated by General Relativity and Dark Matter. *Open Access Library Journal*, **10**: e10714.
https://doi.org/10.4236/oalib.1110714

Received: September 8, 2023
Accepted: October 13, 2023
Published: October 16, 2023

Abstract

The Tully-Fisher law $M \propto v^{\alpha}$ is an empirical relationship between the mass of a galaxy and its asymptotic rotation velocity. The purpose of this research is to demonstrate that this relation can be theoretically obtained in General Relativity (GR) with a particular solution of dark matter (DM) in very good agreement with the observations. Several years ago, it was demonstrated that DM can theoretically be completely explained by a natural effect of GR without exotic matter, the Lense-Thirring effect that exists exclusively in GR. In this explanation, the field generating the Lense-Thirring effect would be generated by the clusters of galaxies and not by the own field of the galaxy which is negligible. In this way, a uniform field (from galaxies' clusters) would embed the galaxies. We retrieve the coefficients of this law thanks to the explicit values of this field required to explain DM. This demonstration shows how relevant this explanation of the DM is, not only theoretically (by obtaining the expression of the law) but also practically (by obtaining the coefficients from the values required to explain the DM). The Tully-Fisher law would then reveal the Lense-Thirring effect of the clusters of galaxies on the galaxies.

Subject Areas

Cosmology, Astrophysics, Theoretical Physics

Keywords

Dark Matter, Tully-Fisher Law, Gravitation

1. Introduction

One of the most important scaling laws is the empirical Tully-Fisher relation [1], between the stellar mass or luminosity of a galaxy and its rotation velocity *v*. The stellar Tully-Fisher is a power law $M \propto v^{\alpha}$ with α ~ 4 - 5 depending on the method used to estimate stellar masses [2] [3] and depending on how the rotation velocities are defined [4] [5]. When the baryonic mass M_b (stars + cold gas) is used instead of the stellar mass, the baryonic Tully-Fisher relation [6] becomes an extremely tight power law $M \propto v^{\alpha}$, with α ~ 4 [5] [7] [8].

In our study, we are going to demonstrate the Tully-Fisher law with the solution of DM explained without exotic matter but with a uniform gravitic field (the 2nd component of GR similar to the magnetic field in EM) as proposed by the author [9]. This field would be generated by galaxy clusters [10] and would embed large areas of the Universe (and then the galaxies) explaining this excess of gravitation misnamed, in this explanation, DM. We will first remind you how Linearized General Relativity (LGR) is obtained from GR, how LGR equations can explain DM and the expected values of the uniform gravitic field required to explain DM component. We will secondly verify that the measured coefficients of the Tully-Fisher law $M = av^{\alpha}$ allow retrieving the expected gravitic field explaining the DM. Third, the main goal of this study, we will demonstrate the expression of Tully-Fisher law in our explanation of DM, making this theoretical DM explanation extremely consistent with the observations.

2. Dark Matter Explained by General Relativity

2.1. From General Relativity to Linearized General Relativity

From GR, one deduces the LGR in the approximation of a quasi-flat Minkowski space ($g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$; $\left|h^{\mu\nu}\right| \ll 1$). With the following Lorentz gauge, it gives the following field equations as in [11] (with $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ and $\Delta = \nabla^2$):

$$\partial_\mu \bar{h}^{\mu\nu} = 0;\ \Box \bar{h}^{\mu\nu} = -2\frac{8\pi G}{c^4}T^{\mu\nu} \tag{1}$$

with:

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h;\ h \equiv h^{\sigma}_{\sigma};\ h^{\mu}_{\nu} = \eta^{\mu\sigma}h_{\sigma\nu};\ \bar{h} = -h \tag{2}$$

The general solution of these equations is:

$$\bar{h}^{\mu\nu}(ct, \boldsymbol{x}) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}\left(ct - \left|\boldsymbol{x}-\boldsymbol{y}\right|, \boldsymbol{y}\right)}{\left|\boldsymbol{x}-\boldsymbol{y}\right|}\mathrm{d}^3\boldsymbol{y} \tag{3}$$

In the approximation of a source with low speed, one has:

$$T^{00} = \rho c^2;\ T^{0i} = c\rho u^i;\ T^{ij} = \rho u^i u^j \tag{4}$$

And for a stationary solution, one has:

$$\bar{h}^{\mu\nu}(\boldsymbol{x}) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}(\boldsymbol{y})}{\left|\boldsymbol{x}-\boldsymbol{y}\right|}\mathrm{d}^3\boldsymbol{y} \tag{5}$$

At this step, by proximity with electromagnetism, one traditionally defines a scalar potential φ and a vector potential H^i. There are in the literature several definitions as in [12] for the vector potential H^i. In our study, we are going to define:

$$\bar{h}^{00}=\frac{4\varphi}{c^2};\ \bar{h}^{0i}=\frac{4H^i}{c};\ \bar{h}^{ij}=0 \tag{6}$$

with gravitational scalar potential φ and gravitational vector potential H^i:

$$\varphi(\boldsymbol{x})\equiv -G\int\frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}^3\boldsymbol{y}$$

$$H^i(\boldsymbol{x})\equiv -\frac{G}{c^2}\int\frac{\rho(\boldsymbol{y})u^i(\boldsymbol{y})}{\boldsymbol{x}-\boldsymbol{y}}\mathrm{d}^3\boldsymbol{y}=-K^{-1}\int\frac{\rho(\boldsymbol{y})u^i(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}^3\boldsymbol{y} \tag{7}$$

with K (determined in [9]) a new constant defined by:

$$GK=c^2 \tag{8}$$

This definition is $K^{-1}\sim 7.4\times 10^{-28}\ \mathrm{kg}\cdot\mathrm{m}^{-1}$ very small compared to G.

The field Equations (1) can be then written (Poisson equations):

$$\Delta\varphi=4\pi G\rho;\ H^i=\frac{4\pi G}{c^2}\rho u^i=4\pi K^{-1}\rho u^i \tag{9}$$

with the following definitions of $\boldsymbol{g}$ (gravity field) and $\boldsymbol{k}$ (gravitic field), those relations can be obtained from the following equations (also called gravitomagnetism) with the differential operators "$\boldsymbol{rot}=\nabla\wedge$", "$\boldsymbol{grad}=\nabla$" and "$div=\nabla\cdot$":

$$\begin{aligned}&\boldsymbol{g}=-\boldsymbol{grad}\,\varphi;\ \boldsymbol{k}=\boldsymbol{rot}\,\boldsymbol{H}\\&\boldsymbol{rot}\,\boldsymbol{g}=0;\ div\,\boldsymbol{k}=0;\\&div\,\boldsymbol{g}=-4\pi G\rho;\ \boldsymbol{rot}\,\boldsymbol{k}=-4\pi K^{-1}\boldsymbol{j}_p\end{aligned} \tag{10}$$

with the Equations (2), one has:

$$h^{00}=h^{11}=h^{22}=h^{33}=\frac{2\varphi}{c^2};\ h^{0i}=\frac{4H^i}{c};\ h^{ij}=0 \tag{11}$$

The equations of geodesics in the linear approximation give:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\sim-\frac{1}{2}c^2\delta^{ij}\partial_j h_{00}-c\delta^{ik}\left(\partial_k h_{0j}-\partial_j h_{0k}\right)v^j \tag{12}$$

It then leads to the movement equations:

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}\sim-\boldsymbol{grad}\,\varphi+4\boldsymbol{v}\wedge(\boldsymbol{rot}\,\boldsymbol{H})=\boldsymbol{g}+4\boldsymbol{v}\wedge\boldsymbol{k} \tag{13}$$

Remark: All previous relations can be retrieved starting with the parameterized post-Newtonian (PPN) formalism and with the traditional gravitomagnetic field $\boldsymbol{B}_g$. From [13] one has:

$$g_{0i}=-\frac{1}{2}(4\gamma+4+\alpha_1)V_i;\ V_i(\boldsymbol{x})=\frac{G}{c^2}\int\frac{\rho(\boldsymbol{y})u_i(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}^3\boldsymbol{y} \tag{14}$$

The traditional gravitomagnetic field and its acceleration contribution are:

$$\boldsymbol{B}_g=\nabla\wedge\left(g_{0i}\boldsymbol{e}^i\right);\ \boldsymbol{a}_g=\boldsymbol{v}\wedge\boldsymbol{B}_g \tag{15}$$

And in the case of GR (that is our case):

$$\gamma=1;\ \alpha_1=0 \tag{16}$$

It then gives:

$$g_{0i} = -4V_i;\ \boldsymbol{B}_g = \nabla \wedge \left(-4V_i \boldsymbol{e}^i\right) \tag{17}$$

And with our definition:

$$H_i = -\delta_{ij} H^j = \frac{G}{c^2} \int \frac{\rho(\boldsymbol{y}) \delta_{ij} u^j(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \mathrm{d}^3 \boldsymbol{y} = V_i(\boldsymbol{x}) \tag{18}$$

One then has:

$$g_{0i} = -4H_i;\ \boldsymbol{B}_g = \nabla \wedge \left(-4H_i \boldsymbol{e}^i\right) = \nabla \wedge \left(4\delta_{ij} H^j \boldsymbol{e}^i\right) = 4\nabla \wedge \boldsymbol{H} \tag{19}$$

$$\boldsymbol{B}_g = 4\,\boldsymbol{rot}\,\boldsymbol{H}$$

with the following definition of gravitic field:

$$\boldsymbol{k} = \frac{\boldsymbol{B}_g}{4} \tag{20}$$

One then retrieves our previous relations:

$$\boldsymbol{k} = \boldsymbol{rot}\,\boldsymbol{H};\ \boldsymbol{a}_g = \boldsymbol{v} \wedge \boldsymbol{B}_g = 4\boldsymbol{v} \wedge \boldsymbol{k} \tag{21}$$

The interest of our notation ($\boldsymbol{k}$ instead of $\boldsymbol{B}_g$) is that the field equations are strictly equivalent to Maxwell's idealization, in particular, the speed of the gravitational wave obtained from these equations is the light celerity $c^2 = GK$ just like in EM $c^2 = 1/\mu_0 \varepsilon_0$. Only the movement equations are different with the factor "4". But of course, all the results of our study can be obtained in the traditional notation of gravitomagnetism with the relation $\boldsymbol{k} = \frac{\boldsymbol{B}_g}{4}$.

2.2. From Linearized General Relativity to DM

In the classical approximation ($\|\boldsymbol{v}\| \ll c$), the linearized general relativity gives the following movement equations from (13) with m_i the inertial mass and m_p the gravitational mass:

$$m_i \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m_p \left[\boldsymbol{g} + 4\boldsymbol{v} \wedge \boldsymbol{k}\right] \tag{22}$$

The traditional computation of rotation speeds of galaxies consists of obtaining the force equilibrium from the three following components: the disk, the bugle, and the halo of dark matter. More precisely, one has [14]:

$$\frac{v^2(r)}{r} = \frac{\partial \varphi(r)}{\partial r} \text{ with } \varphi = \varphi_{disk} + \varphi_{bulge} + \varphi_{halo} \tag{23}$$

Then the total speed squared can be written as the sum of squares of each of the three-speed components:

$$\begin{aligned} v^2(r) &= r\left(\frac{\partial \varphi_{disk}(r)}{\partial r}\right) + r\left(\frac{\partial \varphi_{bulge}(r)}{\partial r}\right) + r\left(\frac{\partial \varphi_{halo}(r)}{\partial r}\right) \\ &= v_{disk}^2(r) + v_{bulge}^2(r) + v_{halo}^2(r) \end{aligned} \tag{24}$$

Disk and bulge components are obtained from gravity field. They are not modified in our solution. So, our goal is now to obtain only the traditional dark matter halo component from the linearized general relativity. According to this idealization, the force due to the gravitic field $\boldsymbol{k}$ takes the following form $\|\boldsymbol{F}_k\| = m_p 4\|\boldsymbol{v} \wedge \boldsymbol{k}\|$ and it corresponds to previous term $m_p \frac{\partial \varphi_{halo}(r)}{\partial r} = \|\boldsymbol{F}_k\|$. As explained in [9], the natural evolution to the equilibrium state justifies that one assumes the approximation $\boldsymbol{v} \perp \boldsymbol{k}$. This assumption is important because it leads to several important predictions. In particular, the motion of dwarf satellite galaxies of a host should be roughly in a plane ($\perp \boldsymbol{k}$). It then gives the following equation:

$$\frac{v^2(r)}{r} = \frac{\partial \varphi_{disk}(r)}{\partial r} + \frac{\partial \varphi_{bulge}(r)}{\partial r} + 4k(r)v(r) = \frac{v_{disk}^2(r)}{r} + \frac{v_{bulge}^2(r)}{r} + 4k(r)v(r) \quad (25)$$

Our idealization means that:

$$v_{halo}^2(r) = v^2(r) - v_{disk}^2(r) - v_{bulge}^2(r) = 4rk(r)v(r) \quad (26)$$

The equation of dark matter (gravitic field in our explanation) is then:

$$v_{halo}(r) = 2\left(rk(r)v(r)\right)^{1/2} \quad (27)$$

This equation gives us the curve of rotation speeds of the galaxies as we wanted. Because we know the curves of speeds that one wishes to have for DM component, one can then deduce the curve of the gravitic field $\boldsymbol{k}$ inside the galaxy:

$$k(r) = \frac{v_{halo}^2(r)}{4rv(r)} \quad (28)$$

2.3. Dark Matter as the 2nd Component of the Gravitational Field $\boldsymbol{k}$

This solution of DM as the gravitic field has been studied in [9] for 16 galaxies (**Table 1**). It shows that this solution is mathematically possible but with two physical mandatory unexpected behavior for $k(r)$. First, the curve of the gravitic field $k(r)$ becomes necessarily flat at the end of the galaxies. For such a field (similar mathematically to a magnetic field in EM) it is only possible if the galaxies are immersed in a uniform gravitic field $\boldsymbol{k}_0$. Second, the value of this field for these 16 galaxies is in the interval:

$$10^{-16.62}\ \mathrm{s}^{-1} < \|\boldsymbol{k}_0\| < 10^{-16.3}\ \mathrm{s}^{-1} \quad (29)$$

From these data (**Table 1**), one can deduce a mean value of $\|\boldsymbol{k}_0\|$ and a mean value of r_0:

$$r_0 \sim 9\ \mathrm{kpc} \sim 2.7 \times 10^{20} \sim 10^{20.43}\ \mathrm{m};\ k_0 \sim 10^{-16.40}\ \mathrm{s}^{-1} \quad (30)$$

The position r_0 is the position where the gravitic component of the galaxy becomes negligible compared to the external uniform gravitic term explaining DM. It roughly represents the beginning of the flat part of the rotation speed curve of the galaxies. We will use these two values at the end of the article in the demonstration of the Tully-Fisher law.

Table 1. Distance r_0 to the center of the galaxy where the internal gravitic field $\frac{K_1}{r^2}$ generated by the galaxy becomes equivalent to the external gravitic field k_0 generated by the galaxies' cluster. k_0 dominates for $r > r_0$.

	K_1	k_0	$r_0\left[\frac{K_1}{r^2} \sim k_0\right]$	r_0 [kpc]
NGC 5055	$10^{24.60}$	$10^{-16.62}$	$10^{20.61}$	13
NGC 4258	$10^{24.85}$	$10^{-16.54}$	$10^{20.695}$	16
NGC 5033	$10^{24.76}$	$10^{-16.54}$	$10^{20.65}$	15
NGC 2841	$10^{24.85}$	$10^{-16.33}$	$10^{20.59}$	13
NGC 3198	$10^{24.90}$	$10^{-16.55}$	$10^{20.725}$	18
NGC 7331	$10^{24.18}$	$10^{-16.30}$	$10^{20.24}$	6
NGC 2903	$10^{24.71}$	$10^{-16.30}$	$10^{20.505}$	11
NGC 3031	$10^{24.15}$	$10^{-16.57}$	$10^{20.36}$	8
NGC 2403	$10^{24.59}$	$10^{-16.39}$	$10^{20.49}$	10
NGC 247	$10^{24.30}$	$10^{-16.30}$	$10^{20.3}$	7
NGC 4236	$10^{24.00}$	$10^{-16.34}$	$10^{20.17}$	5
NGC 4736	$10^{24.54}$	$10^{-16.30}$	$10^{20.42}$	9
NGC 300	$10^{24.27}$	$10^{-16.31}$	$10^{20.29}$	6
NGC 2259	$10^{24.20}$	$10^{-16.30}$	$10^{20.25}$	6
NGC 3109	$10^{24.00}$	$10^{-16.58}$	$10^{20.29}$	6
NGC 224	$10^{24.00}$	$10^{-16.50}$	$10^{20.25}$	6

3. TULLY-FISHER Law Obtained from a Uniform Gravitic Filed k_0 of LGR

We will first verify that this theoretical solution of DM is consistent with the Tully-Fisher law by retrieving our value of DM $\boldsymbol{k}_0$ from the coefficient of the law which has been experimentally observed. And secondly, we will demonstrate how k_0 and LGR can obtain the Tully-Fisher law.

Let's note v_N the Newtonian rotational speed and v_{N+DM} the Newtonian rotational speed plus the halo DM component, (25) can be written:

$$\frac{v_{N+DM}^2(r)}{r} = \frac{v_N^2(r)}{r} + \frac{v_{halo}^2(r)}{r} \tag{31}$$

And more explicitly with M the galaxy's mass:

$$\frac{v_{N+DM}^2(r)}{r} = \frac{GM}{r^2} + 4k(r)v_{N+DM}(r) \tag{32}$$

which gives:

$$\frac{v_{N+DM}^2(r)}{r} = \frac{GM}{r^2}\left(1 + \frac{4k(r)v_{N+DM}(r)r^2}{GM}\right) \tag{33}$$

3.1. From the TULLY-FISHER Law to the Uniform Gravitic Field k_0 of LGR

The Tully-Fisher law is written $M = av^{\alpha}$ and can be rewritten $\log(M) = \log(a) + \alpha\log(v) = \beta + \alpha\log(v)$. Several couples (α,β) can be obtained experimentally for this law [15] depending on the masses considered and the methods of obtaining the characteristic speed of rotation as previously said.

Let's rewrite our expression (33):

$$v_{N+DM}^{\alpha}(r) = M(M)^{\frac{\alpha}{2}-1}\left(\frac{G}{r}\left(1+\frac{4k(r)v_{N+DM}(r)r^2}{GM}\right)\right)^{\alpha/2} \quad (34)$$

which gives:

$$M = \left[(M)^{1-\frac{\alpha}{2}}\left(\frac{G}{r}\left(1+\frac{4k(r)v_{N+DM}(r)r^2}{GM}\right)\right)^{-\frac{\alpha}{2}}\right]v_{N+DM}^{\alpha} \quad (35)$$

In order to get an expression that looks like the Tully-Fisher law, we are going to use the following approximation for large r (in the flat part of the rotation speed curve):

$$\frac{v_{N+DM}^2(r)}{r} \sim 4k(r)v_{N+DM}(r) \Rightarrow v_{N+DM}(r) \sim 4k(r)r \quad (36)$$

Furthermore, in our explanation of DM, $k(r)$ is a uniform field, *i.e.* $k(r) \sim k_0$. By replacing the occurrence of v_{N+DM} in the brackets, one has:

$$M = \left[(M)^{1-\frac{\alpha}{2}}\left(\frac{G}{r}\left(1+\frac{16k_0^2r^3}{GM}\right)\right)^{-\frac{\alpha}{2}}\right]v_{N+DM}^{\alpha} \quad (37)$$

If one has:

$$\frac{16k_0^2r^3}{GM} \gg 1 \quad (38)$$

The expression becomes:

$$M = \left[M\left(16k_0^2r^2\right)^{-\frac{\alpha}{2}}\right]v_{N+DM}^{\alpha} \quad (39)$$

And finally:

$$M = \left[\frac{M}{(4k_0r)^{\alpha}}\right]v_{N+DM}^{\alpha} \quad (40)$$

The couples (α,β) of $M = 10^{\beta}v^{\alpha}$ are in general given for a graph whose velocities are in km·s^{-1} and the masses in solar mass ($M_{\odot} = 2\times10^{30}$). One can then rewrite with $M = mM_{\odot}$ and $v_{N+DM} = 10^3 v$:

$$m = \frac{10^{3\alpha}}{M_{\odot}}\left[\frac{M}{(4k_0r)^{\alpha}}\right]v^{\alpha} \quad (41)$$

We want to verify that the values of (α,β) form the Tully-Fisher law allow retrieving the values of the gravitic field k_0 explaining the DM to ensure that our theoretical solution is consistent with the Tully-Fisher law. For that, let's re-write:

$$10^{\beta} = \frac{10^{3\alpha}}{M_{\odot}}\left[\frac{M}{(4k_0 r)^{\alpha}}\right] \Rightarrow k_0 = \frac{10^3}{M_{\odot}^{1/\alpha}}\frac{1}{10^{\beta/\alpha}}\frac{M^{1/\alpha}}{4r} \tag{42}$$

For these calculations, one will use:

$$M = 3\times 10^{10} M_{\odot} = 6\times 10^{40}\ \text{kg};\ r = 33\ \text{kpc} \sim 10^{21}\ \text{m} \tag{43}$$

This value of r justify the previous approximation (36)

In [2] the observations give the following couple:

$(\alpha,\beta)=(4,1.67)$:

$$k_0 = \frac{10^3}{(2\times 10^{30})^{1/4}}\frac{1}{10^{1.67/4}}\frac{(6\times 10^{40})^{1/4}}{4\times 10^{21}} \sim 10^{-16.40}\ \text{s}^{-1} \tag{44}$$

In [15], one has the 4 following couples:

$(\alpha,\beta)=(4.25,0.80)$:

$$k_0 = \frac{10^3}{(10^{30.3})^{1/4.25}}\frac{1}{10^{0.8/4.25}}\frac{(10^{40.78})^{1/4.25}}{10^{21.6}} \sim 10^{-16.32}\ \text{s}^{-1} \tag{45}$$

$(\alpha,\beta)=(3.51,2.61)$:

$$k_0 = \frac{10^3}{(10^{30.3})^{1/3.51}}\frac{1}{10^{2.61/3.51}}\frac{(10^{40.78})^{1/3.51}}{10^{21.6}} \sim 10^{-16.36}\ \text{s}^{-1} \tag{46}$$

$(\alpha,\beta)=(3.60,2.49)$:

$$k_0 = \frac{10^3}{(10^{30.3})^{1/3.6}}\frac{1}{10^{2.49/3.6}}\frac{(10^{40.78})^{1/3.6}}{10^{21.6}} \sim 10^{-16.38}\ \text{s}^{-1} \tag{47}$$

$(\alpha,\beta)=(3.26,3.33)$:

$$k_0 = \frac{10^3}{(10^{30.3})^{1/3.26}}\frac{1}{10^{3.33/3.26}}\frac{(10^{40.78})^{1/3.26}}{10^{21.6}} \sim 10^{-16.41}\ \text{s}^{-1} \tag{48}$$

Let's verify the previous approximation (38) with the smallest value of $k_0 \sim 10^{-16.41}\ \text{s}^{-1}$:

$$\frac{16k_0^2 r^3}{GM} = \frac{16\times 10^{-32.82}\times 10^{63}}{2\times 10^{-11}\times 6\times 10^{40}} \sim 10^{1.3} \gg 1 \tag{49}$$

These calculations show that the expected values of k_0 to explain DM without exotic material ($10^{-16.62}\ \text{s}^{-1} < \|\boldsymbol{k}_0\| < 10^{-16.3}\ \text{s}^{-1}$) are consistent with the Tully-Fisher law. We will now go further by proving this law within the framework of the LGR and our explanation of DM.

3.2. From the Uniform Gravitic Field k_0 of LGR to the TULLY-FISHER Law

As we noticed in the previous paragraph, in our relation, there is the parameter position r which appears while the Tully-Fisher relation does not explicitly depend on it. To no longer explicitly depend on this parameter, we need to define a procedure to determine a characteristic position r. It should be noted that we are in the same situation for the TULLY-FISHER law to define the characteristic speed of rotation to be considered. Several methods are used [5] and the mean velocity along the flat part of the rotation curve seems to minimize the scatter of the relation [4]. The question in our context is somehow to find which method was adopted to define the characteristic position r corresponding to the characteristic velocity on the curve of the rotational velocities of the galaxies. This characteristic speed is linked to the flat zone of the speed curve, the characteristic position will certainly be a characteristic position in this zone. We can imagine 2 simple methods.

A 1st would consist of finding the position of the beginning of the flat zone r_b then placing the cursor at a fixed relative position, $\omega \geq 1$ with respect to the beginning of this zone, $r = \omega r_b$ and finally determining the value of ω. We will see what it will be $\omega = 1$. This method makes it possible to find punctually the good values for the law of Tully-Fisher but it doesn't give the good slope of the law.

A 2nd method would consist of finding a threshold for the value of the gravitational forces F_{Seuil} from which the gravitational forces are sufficiently weak so that the influence of the DM dominates (a priori in the flat part of the curve). This method allows for demonstrating the Tully-Fisher law, values, and slope. But certainly, also the extent of the area to which this law is applicable.

3.2.1. 1st Method to Demonstrate the Tully-Fisher Law

The beginning of the flat zone of the rotational speed curve r_b corresponds approximately to the place where the intensity of the Newtonian force is of the same order as the component of DM:

$$\frac{GM}{r_b^2} = 4k_0 v_{N+DM} \Rightarrow r_b = \sqrt{\frac{GM}{4k_0 v_{N+DM}}} \tag{50}$$

We then write (to be somewhere in the flat zone):

$$r = \omega r_b = \omega\sqrt{\frac{GM}{4k_0 v_{N+DM}}};\ \omega \geq 1 \tag{51}$$

Our relation (32) gives:

$$v_{N+DM}^2 = \frac{GM}{r} + 4k_0 v_{N+DM} r \tag{52}$$

$$v_{N+DM}^2 = \frac{GM}{\omega}\sqrt{\frac{4k_0 v_{N+DM}}{GM}} + 4k_0 v_{N+DM}\,\omega\sqrt{\frac{GM}{4k_0 v_{N+DM}}} \tag{53}$$

$$v_{N+DM}^2 = \frac{1}{\omega}\sqrt{4k_0 v_{N+DM} GM} + \omega\sqrt{4k_0 v_{N+DM} GM} \tag{54}$$

$$v_{N+DM}^2 = \left(\frac{1}{\omega} + \omega\right)\sqrt{4k_0 v_{N+DM} GM} \tag{55}$$

$$M=\left(\frac{1}{\omega}+\omega\right)^{-2}\frac{1}{4k_0G}v_{N+DM}^3 \tag{56}$$

$$M=\left(\frac{\omega}{1+\omega^2}\right)^{2}\frac{v_{N+DM}^3}{4k_0G} \tag{57}$$

with $\omega=10^{0.4}$ and $k_0\sim10^{-16.4}\,\text{s}^{-1}$ (red curve in **Figure 1**) one obtains a curve that is very close to the Tully-Fisher law, passing through the cloud of expected measured points. But the slope of the curve is unsatisfactory. This 1st method informs us that the position r_b of the beginning of the flat zone is certainly a good order of magnitude for the definition of the position associated with the measurement of the speed of rotation for the law of Tully-Fisher. Even if this result is not completely satisfactory, it is very encouraging. With this information, we will now improve our result with the 2nd method.

3.2.2. 2nd Method to Demonstrate the Tully-Fisher Law

Let's define a threshold value of the intensity of the force for which, we hope to find the position on the rotation curve corresponding to the characteristic speed considered for the Tully-Fisher law:

$$\left|\frac{GM}{r_b^2}+4k_0v_{N+DM}\right|=F_{Seuil}\Rightarrow r_b=\sqrt{\frac{GM}{F_{Seuil}-4k_0v_{N+DM}}} \tag{58}$$

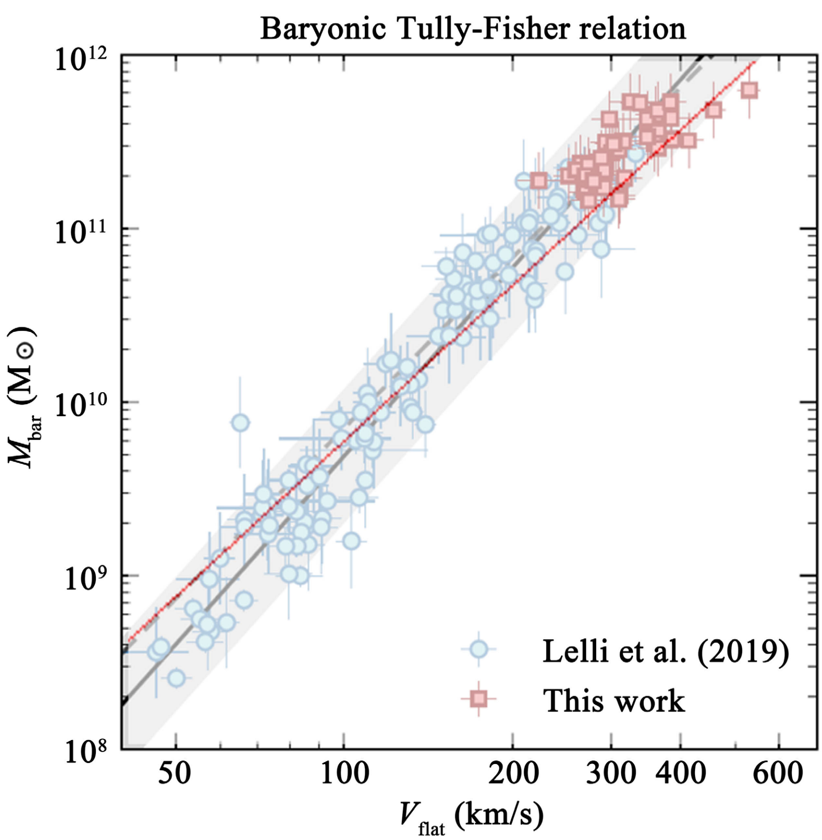

Figure 1. The red curve representing our relation obtained from our 1st method is superposed on graph from [15].

This position is denoted r_b because we have seen previously that it was most certainly the beginning of the flat zone. The following calculation will confirm this result. One can then rewrite our relation to LGR:

$$v_{N+DM}^2 = \frac{GM}{r_b} + 4k_0 v_{N+DM} r_b \tag{59}$$

$$v_{N+DM}^2 = \sqrt{GM\left(F_{Seuil} - 4k_0 v_{N+DM}\right)} + 4k_0 v_{N+DM}\sqrt{\frac{GM}{F_{Seuil} - 4k_0 v_{N+DM}}} \tag{60}$$

$$v_{N+DM}^4 = GM\left(\sqrt{F_{Seuil} - 4k_0 v_{N+DM}} + 4k_0\sqrt{\frac{v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}}\right)^2 \tag{61}$$

$$\begin{aligned} v_{N+DM}^4 = GM\Bigg(&\sqrt{F_{Seuil} - 4k_0 v_{N+DM}}^2 + 16k_0^2\sqrt{\frac{v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}}^2 \\ &+ 8k_0\sqrt{F_{Seuil} - 4k_0 v_{N+DM}}\sqrt{\frac{v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}}\Bigg) \end{aligned} \tag{62}$$

$$v_{N+DM}^4 = GM\left(F_{Seuil} - 4k_0 v_{N+DM} + 16k_0^2\frac{v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}} + 8k_0 v_{N+DM}\right) \tag{63}$$

$$v_{N+DM}^4 = GM\left(F_{Seuil} + 4k_0 v_{N+DM} + 16k_0^2\frac{v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}\right) \tag{64}$$

$$v_{N+DM}^4 = GM\left(\frac{\left(F_{Seuil} + 4k_0 v_{N+DM}\right)\left(F_{Seuil} - 4k_0 v_{N+DM}\right)}{F_{Seuil} - 4k_0 v_{N+DM}} + \frac{16k_0^2 v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}\right) \tag{65}$$

$$v_{N+DM}^4 = GM\left(\frac{F_{Seuil}^2 - 16k_0^2 v_{N+DM}^2 + 16k_0^2 v_{N+DM}^2}{F_{Seuil} - 4k_0 v_{N+DM}}\right) \tag{66}$$

$$v_{N+DM}^4 = GM\left(\frac{F_{Seuil}^2}{F_{Seuil} - 4k_0 v_{N+DM}}\right) \tag{67}$$

We then obtain our expression of Tully-Fisher law (red curve in **Figure 2**):

$$M = \frac{1}{GF_{Seuil}}\left(v_{N+DM}^4 - \frac{4k_0}{F_{Seuil}} v_{N+DM}^5\right) \tag{68}$$

An approximation of this relation in the form "$M = av^{\alpha}$" could be obtained and would lead to an analysis equivalent to what we have studied in the previous sections, but the more accurate relation between M and v in our explanation is the latter. And the result is impressive as one can see in **Figure 2**. Furthermore, the obtention F_{Seuil} is consistent with our definition of DM, as we are now going to see it.

As calculated previously, if we take the average of the beginnings of the flat zones and the mean of the values of the gravitic field explaining DM (30) and by taking the characteristic mass previously used for our calculations (43), one has:

$$\begin{aligned} &r_b = 9\ \text{kpc} \sim 2.7\times10^{20} \sim 10^{20.43}\ \text{m};\ k_0 = 10^{-16.40}\ \text{s}^{-1}; \\ &M = 3\times10^{10} M_{\odot} = 6\times10^{40}\ \text{kg} \end{aligned} \tag{69}$$

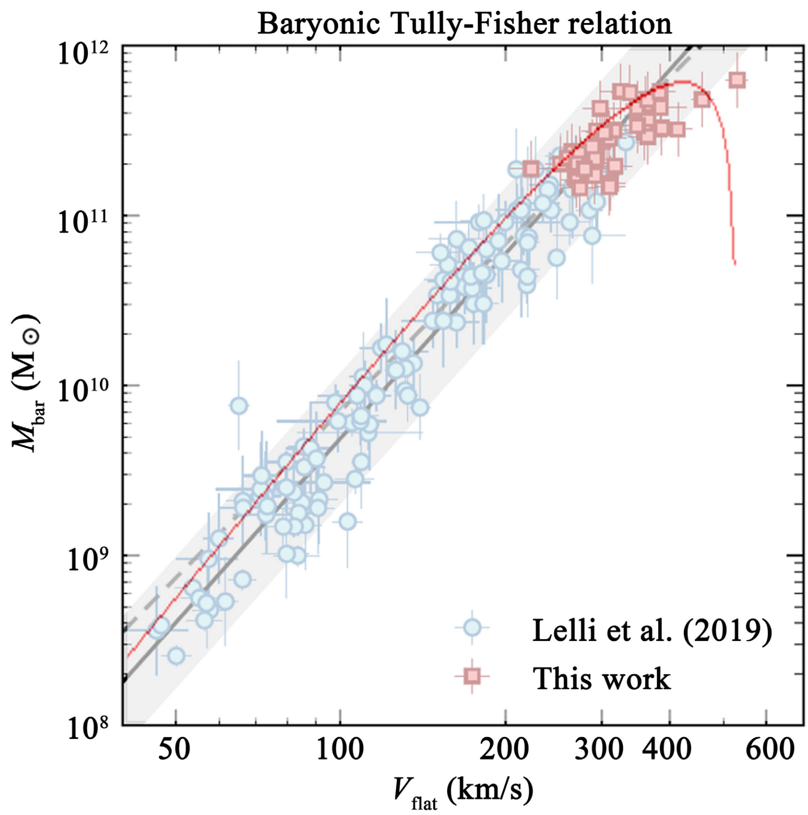

Figure 2. The red curve representing our relation obtained from our 2nd method is superposed on the graph from [15].

For this value of mass, on the graphs (Figure 1 and Figure 2) the corresponding characteristic rotational speed is around:

$$v_{N+DM} = 2\times 10^{5}\ \mathrm{m}\cdot\mathrm{s}^{-1} \tag{70}$$

All these characteristic values allow defining our characteristic force threshold:

$$\left|\frac{GM}{r_b^2} + 4k_0 v_{N+DM}\right| = F_{Seuil} = \frac{6\times 10^{-11}\times 6\times 10^{40}}{10^{40.86}} + 4\times 10^{-16.40}\times 2\times 10^{5} \sim 10^{-10.08} \tag{71}$$

The red curve in Figure 2 representing the relation (68) is obtained with $F_{Seuil} = 10^{-10.08}$ and $k_0 = 10^{-16.40}\ \mathrm{s}^{-1}$.

This time, compared to the 1st method, not only does the curve pass well through the cloud of measured points, but the slope of the curve is also excellent. Add to this that the characteristic value of k_0 is also not only in the right order of magnitude but this time in the interval required to explain the DM.

4. Discussion

In the same way that there are several methods for defining the characteristic velocity used in the Tully-Fisher law giving more or less tight values of its coefficients, the role of F_{Seuil} provides a method for obtaining this characteristic velocity. Indeed, finally $F_{Seuil} = \frac{v_{N+DM}^2}{r}$ implies $v_{N+DM} = \sqrt{rF_{Seuil}}$. The rotation speed of the galaxy to be considered will be that at the intersection of this curve with its rotation speed curve.

The fact of obtaining a value of F_{Seuil} from the explanation of the DM (*i.e.* by the expression of the uniform gravitic field of the LGR, "$4k_0 v_{N+DM}$" and by the values of this field "$10^{-16.62}\ \mathrm{s}^{-1} < \|\boldsymbol{k}_0\| < 10^{-16.3}\ \mathrm{s}^{-1}$") and which accounts for the Tully-Fisher law is undeniably an extremely strong point which validates this explanation of the DM (without exotic matter and in agreement with the RG). One can remind that this solution predicts the existence of planes of corotating satellite galaxies [16] and that in the WLM's dwarf galaxy case, these expected values of field k_0 can retrieve the density of the gaseous intergalactic medium and interstellar gaseous medium [17].

Another point seems important. The Tully-Fisher law in its form "$M = av^{\alpha}$" has no maximum limit. It gives no justification for not continuing beyond $M \sim 10^{12} M_{\odot}$. Our relationship necessarily indicates a break from a certain maximum value of the mass of the galaxy (bending of the curve). This relationship thus provides a justification for the fact that galaxies can have a maximum mass value which with our approximate study gives $M \sim 10^{12} M_{\odot}$ in agreement with observations.

5. Conclusion

In this study, we show that the explanation of dark matter in the form of a uniform gravitic field k_0 (the 2nd component of GR similar to the magnetic field in EM giving the Lense-Thirring effect) makes it possible to obtain the Tully-Fisher law. Obtaining this law is based on two important characteristics of this solution, one theoretical, namely the shape of this field, "$4k_0 v_{N+DM}$", defined by the LGR, and the other practical, namely the values required to obtain the component of DM "$10^{-16.62}\ \mathrm{s}^{-1} < \|\boldsymbol{k}_0\| < 10^{-16.3}\ \mathrm{s}^{-1}$". Thus, obtaining the Tully-Fisher law is undeniably linked to the validity of this solution. This demonstration reinforces this solution of the DM which is also extremely economical in hypothesis if we compare it to MOND (which calls into question the theoretical framework of gravitation) or to the existence of an exotic matter (with a new behavior and still not found).

In addition to providing the correct values (passing through the measured points) and the correct slope of the Tully-Fisher law, this relationship goes further by showing a systematic break in the curve for mass values roughly around $M \sim 10^{12} M_{\odot}$. This new relationship thus provides a justification for the existence of a maximum mass for galaxies.

This study finally leads to 3 major results, the demonstration of the Tully-Fisher law, a justification of a maximal mass of the galaxies but perhaps even more important a validation of the explanation of the DM in the form of a uniform gravitic field embedding the galaxies.

Conflicts of Interest

The author declares no conflicts of interest.

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