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Outstanding Development of the Quadratic $\phi(\mu_1, \mu_2)$-Functional Inequatities with $2k$-Variables in Fuzzy Banach Space

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Abstract

In this paper, I work on expanding the Quadratic $\phi(\mu_1, \mu_2)$ -function inequalities by relying on the general quadratic functional equation with $2k$-variables on the fuzzy Banach space. That's the main result of this.

Subject Areas

Mathematics

Keywords

Generalized Quadratic Type $\phi(\mu_1, \mu_2)$ -Functional Inequality, Generalized Quadratic Type Functional Equations, Fuzzy Banach Space, Fuzzy Normed Vector Spaces

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ are fuzzy normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X} \to \mathbf{Y}$ be a mapping. I use the notation N are the norm on $\mathbf{X}$ and on $\mathbf{Y}$ respectively. In this paper, I study the relationship between Quadratic-type functional equations and Quadratic $\phi(\mu_1, \mu_2)$ -function inequalities when $(\mathbf{X}, N)$ is a fuzzy normed space and $(\mathbf{Y}, N)$ is a fuzzy Banach space.

In fact, when $\mathbf{X}$ is a fuzzy normed space and $\mathbf{Y}$ is a fuzzy Banach space we solve and prove the Hyers-Ulam stability of the following relationship between quadratic $\phi(\mu_1, \mu_2)$ -function inequalities and quadratic-type functional equations:

$$N\left(2kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+2kf\left(\frac{\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i}{2k}\right)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i),t\right)$$
$$\leq \min\left(N\left(\mu_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right),\right. \quad (1)$$
$$\left.N\left(\mu_2\left(4kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right)\right)$$

based on following Generalized Quadratic functional equations with $2k$-variable

$$f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)=2\sum_{i=1}^{k}f(x_i)+2\sum_{k=1}^{k}f(y_i)$$

$$A_0=\left\{h:\mathbb{R}\to\mathbb{R}:g(\mu_1,\mu_2)=\frac{1}{\mu_1}+\frac{1}{\mu_2}<1,\mu_1,\mu_2\in\mathbb{R}\right\}.$$

Note that: With k is a positive integer and $h\in A_0$.

The study of the functional equation stability originated from a question of S.M. Ulam [1], concerning the stability of group homomorphisms. Let $(\mathbb{G},*)$ be a group and let $(\mathbb{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbb{G}\to\mathbb{G}'$ satisfy the condition

$$d\left(f(x*y),f(x)\circ f(y)\right)<\delta,$$

for all $x,y\in\mathbb{G}$ then there is a homomorphism $h:\mathbb{G}\to\mathbb{G}'$ with

$$d\left(f(x),h(x)\right)<\varepsilon,$$

for all $x\in\mathbb{G}$, if the answer, is affirmative, we would say that equation of homomophism $h(x*y)=h(y)\circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace a functional equation with an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how the solutions of the inequality differ from those of the given function equation.

Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvrut [5] by replacing the unbounded Cauchy difference with a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensive.

Through the process of studying the works of mathematicians see ([6] [7] [8] [9] [10] [11]) in 2020, I set up a general quadratic equation with $2k$-variables on the space Non-Archimedean Banach.

$$f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{k=1}^{k}f(y_i) \quad (2)$$

Next in 2020, I build quadratic inequalities on the application of groups and rings,

$$\left\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)+f\left(\sum_{j=1}^{n}x_j-\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-2\sum_{j=1}^{n}f\left(x_j\right)-2\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\le\varepsilon,\ (3)$$

for all $\varepsilon\ge0$ and

$$\left\|f\left(\prod_{j=1}^{n}x_j+\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)+f\left(\prod_{j=1}^{n}x_j-\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)-2\prod_{j=1}^{n}f\left(x_j\right)-2\prod_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\le\delta,\ (4)$$

for all $\delta\ge0$.

Next in 2021, Ly Van An construct the quadratic inequality functional inequalities in non-Archimedean Banach spaces and Banach spaces,

$$\begin{aligned}&\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\\&\le\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\end{aligned}\quad(5)$$

and

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2}\\&\le\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(\frac{x_{k+1}}{k}\right)-2\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X}_2},\end{aligned}\quad(6)$$

Continuing into 2021, Ly Van An construct the quadratic inequality on γ-homogeneous complex Banach space,

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\quad(7)$$

and

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\quad(8)$$

Next in 2023, Ly Van An generalized stability of functional inequalities with $3k$-variables associated for Jordan-von Neumann-type additive functional equation,

$$\left\|\sum_{j=1}^{k}f\left(x_j\right)+\sum_{j=1}^{k}f\left(y_j\right)+\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\le\left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\right)\right\|_{\mathbf{Y}},\quad(9)$$

and

$$\left\|\sum_{j=1}^{k}f\left(x_j\right)+\sum_{j=1}^{k}f\left(y_j\right)+\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\le\left\|f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}},\quad(10)$$

final

$$\left\| \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(y_j\right) + 2k\sum_{j=1}^{k} f\left(z_j\right) \right\|_{\mathbf{Y}} \le \left\| 2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}}. \quad (11)$$

Continuing into 2023, Ly Van An construct the broadly derivation on fuzzy Banach algebra involving functional equations and general Cauchy-Jensen functional inequalities,

$$\left\| \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(y_j\right) + f\left(2k\sum_{j=1}^{k} z_j\right) \right\| \le \left\| 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\| \quad (12)$$

The paper is organized as followings:

In section preliminary, we remind some basic notations in [12]-[18] such as Fuzzy normed spaces, Extended metric space theorem and solutions of the Jensen function equation.

Section 3: Setting up quadratic $\phi(\mu_1,\mu_2)$-function inequalities (1) based on quadratic Equation (2).

3.1: Condition for existence of solution of (1).

3.2: Establishing a solution for the quadratic $h(\mu_1,\mu_2)$-function inequality (1). So that we solve and proved the Hyers-Ulam type stability for functional Equation (1) *i.e.* the functional equations with 2*k*-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional Equations (1).

Thus, the results in this paper are generalization of those in [19]-[65].

2. Preliminaries

2.1. Fuzzy Normed Spaces

Let X be a real vector space. Afunction $N : X \times R \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s,t \in \mathbb{R}$,

1) (N1) $N(x,t) = 0$ for $t \le 0$;

2) (N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

3) (N3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \ne 0$;

4) (N4) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$;

5) (N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

6) (N6) for $x \ne 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed vector space:

1) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

2) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in N$ such that for all $n = n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normedvector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping $f: X \to Y$ between fuzzy normed vector spaces *X* and *Y* is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in *X*, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f: X \to Y$ is continuous at each $x \in X$, then $f: X \to Y$ is said to be continuous on *X*.

Let *X* be an algebra and (X, N) a fuzzy normed space.

1) The fuzzy normed space (X, N) is called a fuzzy normed algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t),$$

for all $x, y \in X$ and all positive real numbers *s* and *t*.

2) A complete fuzzy normed algebra is called a fuzzy Banach algebra.

Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}$-linear mapping $H: (X, N_X) \to (Y, N)$ is called a fuzzy algebra homomorphism. **Example:**

Let $(X, \|\cdot\|)$ be a normed algebra. Let

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|} & t > 0 \\ 0 & t \leq 0 \quad x \in X \end{cases}.$$

Then $N(x, t)$ is a fuzzy norm on *X* and $(X, N(x, t))$ is a fuzzy normed algebra. Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}$-linear mapping $H: (X, N_X) \to (Y, N)$ is called a fuzzy algebra homomorphism.

2.2. Extended Metric Space Theorem

Theorem 1. Let (X, d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d\left(J^n, J^{n+1}\right) = \infty,$$

for all nonnegative integers *n* or there exists a positive integer n_0 such that

1) $d\left(J^n, J^{n+1}\right) < \infty, \quad \forall n \geq n_0$;

2) The sequence $\left\{J^n x\right\}$ converges to a fixed point y^* of *J*;

3) y^* is the unique fixed point of *J* in the set $Y = \left\{y \in X \mid d\left(J^n, J^{n+1}\right) < \infty\right\}$;

4) $d\left(y, y^*\right) \leq \frac{1}{1-l} d\left(y, Jy\right) \quad \forall y \in Y$.

2.3. Solutions of the Equation

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called the Qquadratic equation. In particular, every solution of the quadratic equation is said to be a quadratic mapping.

2.4. Solutions of the Inequalities

The solution of the quadratic function inequalities is called the quadratic mapping.

3. Setting up Quadratic (μ_1, μ_2)-Function Inequalities (1) Based on Quadratic Equation (2)

3.1. Condition for Existence of Solution of (1)

In this section, assume that $\mathbf{X}$ and $\mathbf{Y}$ be a fuzzy normed vector spaces Under this setting, we can show that the mappings satisfying (1) is quadratic and $h \in A$.

Lemma 2. Suppose that $(\mathbf{Y}, N)$ be a fuzzy normed vector space and let $f: \mathbf{X} \to \mathbf{Y}$ be a mapping and it satisfies the functional inequality

$$N\left(2kf\left(\frac{\sum_{i=1}^k x_i + \sum_{i=1}^k y_i}{2k}\right) + 2kf\left(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{2k}\right) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i), t\right)$$
$$\leq \min\left(N\left(\mu_1\left(f\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\right) + f\left(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\right) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\right), t\right),\right. \quad (13)$$
$$\left. N\left(\mu_2\left(4kf\left(\frac{\sum_{i=1}^k x_i + \sum_{i=1}^k y_i}{2k}\right) + f\left(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\right) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\right), t\right)\right)$$

For all $x_i, y_i \in \mathbf{X}, i = 1 \to k$ and all $t > 0$ then f is quadratic.

Proof. I replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k)$ by $(0, \cdots, 0, 0, \cdots, 0)$ in (13), we have

$$N(-3k\mu_1 f(0), t) \geq N(0, t) = 1 \quad (14)$$

Thus $f(0) = 0$.

Next I replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k)$ by $(x, \cdots, x, x, \cdots, x)$ in (13), we have

$$1 \leq N(\mu_1(f(2kx) - 4kf(x), t)) \quad (15)$$

So

$$f(2kx) = 4kf(x) \quad (16)$$

For all $x \in \mathbf{X}$.

Now I consider

$$G: \mathbf{X} \to \mathbf{Y}.$$

That

$$G(x_1, \cdots, x_k, y_1, \cdots, y_k)$$
$$= 2kf\left(\frac{\sum_{i=1}^k x_i + \sum_{i=1}^k y_i}{2k}\right) + 2kf\left(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{2k}\right) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i). \quad (17)$$

It follows from (13) and (14)

$$N\left(\frac{1}{2k}G(x_1, \cdots, x_k, y_1, \cdots, y_k), t\right)$$
$$\leq \min\left(N(\mu_1 G(x_1, \cdots, x_k, y_1, \cdots, y_k), t), N(\mu_2 G(x_1, \cdots, x_k, y_1, \cdots, y_k), t)\right). \quad (18)$$

Next I put $v = 2t$ (18) I have

$$
\begin{aligned}
&N\left(G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),v\right)\\
&\le \min\left(N\left(\mu_1 G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),\frac{v}{2}\right),N\left(\mu_2 G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),\frac{v}{2}\right)\right)\\
&=\min\left(N\left(\frac{1}{2k}G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),\frac{v}{4k\left|\mu_1\right|}\right),N\left(\frac{1}{2k}G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),\frac{v}{4k\left|\mu_2\right|}\right)\right) \quad (19)\\
&\le N\left(G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),\frac{1}{4k}h\left(\mu_1,\mu_2\right)v\right)\\
&\le N\left(G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right),h\left(\mu_1,\mu_2\right)v\right)
\end{aligned}
$$

for all $v > 0$. By (N_5) and (N_6) I have

$$
\begin{aligned}
&G\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)\\
&= f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) + f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i\right) = 2\sum_{i=1}^{k} f\left(x_i\right) + 2\sum_{k=1}^{k} f\left(y_i\right)
\end{aligned} \tag{20}
$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathbf{X}$, since $h\left(\mu_1,\mu_2\right) \in A_0$.

Hence f is quadratic mapping as we expected. □

3.2. Establishing a Solution for the Quadratic $h\left(\mu_1,\mu_2\right)$-Function Inequality (1)

In this section, assume that $\left(\mathbf{X},N\right)$ is a fuzzy normed space and $\left(\mathbf{Y},N\right)$ is a fuzzy Banach space. Under this setting, we can show that the mappings satisfying (1) is quadratic and $h \in A_0$.

Theorem 3. Let $\psi : \mathbf{X}^{2k} \to [0,\infty)$ be a function such that there exists an $L < \frac{1}{2k}$,

$$
\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right) \le 4kL\psi\left(\frac{x_1}{2k},\cdots,\frac{x_1}{2k},\frac{y_1}{2k},\cdots,\frac{y_k}{2k}\right) \tag{21}
$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$.

Let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying

$$
\begin{aligned}
&\min\left(N\left(2kf\left(\frac{\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i}{2k}\right) + 2kf\left(\frac{\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i}{2k}\right)\right.\right.\\
&\left.\left.-\sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right),t\right),\frac{t}{t+\psi\left(x_1,\cdots,x_k,y_1,\cdots,y_k\right)}\right)\\
&\le \min\left(N\left(\mu_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) + f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i\right) - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right),t\right),\right.\\
&\left.N\left(\mu_2\left(4kf\left(\frac{\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i}{2k}\right) + f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i\right) - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right),t\right)\right)
\end{aligned} \tag{22}
$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$, for all $t > 0$. Then

$$A(x)=N-\lim_{n\to\infty}\frac{1}{(4k)^n}f\left((2k)^n x\right) \tag{23}$$

exists each $x\in\mathbf{X}$ and defines a quadratic mapping $A:\mathbf{X}\to\mathbf{Y}$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{4k|\mu_1|(1-L)t}{4k|\mu_1|(1-L)t+\psi(x,\cdots,x,x,\cdots,x)} \tag{24}$$

for all $x\in\mathbf{X}$ and $t>0$.

Proof. I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(0,\cdots,0,0,\cdots,0)$ in (22), I have

$$N\left(-3k\mu_1 f(0),t\right)\geq\frac{t}{t+\varphi(0,\cdots,0,0,\cdots,0)}=1 \tag{25}$$

Thus $f(0)=0$.

Next I replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k)$ by $(x,\cdots,x,x,\cdots,x)$ in (22), we get

$$\begin{aligned}\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)}&\leq N\left(\mu_1\left(f(2kx)-4kf(x)\right),t\right)\\&\leq N\left(f(x)-\frac{1}{4k}f(2kx),\frac{t}{4k|\mu_1|}\right)\end{aligned} \tag{26}$$

for all $x\in\mathbf{X}$. Now we consider the set

$$\mathbb{M}:=\{h:\mathbf{X}\to\mathbf{Y}\},$$

and introduce the generalized metric on $\mathbb{M}$ as follows:

$$\begin{aligned}d(g,h):=\inf\Big\{&\beta\in\mathbb{R}_+:N\left(g(x)-h(x),\beta t\right)\\&\geq\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)},\forall x\in\mathbf{X},\forall t>0\Big\},\end{aligned} \tag{27}$$

where, as usual, $\inf\phi=+\infty$. That has been proven by mathematicians $(\mathbb{M},d)$ is complete (see [47]).

Now we cosider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=\frac{1}{4k}g(2kx),$$

for all $x\in\mathbf{X}$. Let $g,h\in\mathbb{M}$ be given such that $d(g,h)=\varepsilon$ then

$$N\left(g(x)-h(x),\varepsilon t\right)\geq\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)},\forall x\in\mathbf{X},\forall t>0.$$

Hence

$$\begin{aligned}N\left(g(x)-h(x),\varepsilon Lt\right)&=N\left(\frac{1}{4k}g(2kx)-\frac{1}{4k}h(2kx),L\varepsilon t\right)\\&=N\left(g(2kx)-h(2kx),4L\varepsilon t\right)\\&\geq\frac{4Lt}{4Lt+\varphi(2kx,\cdots,2kx,2kx,\cdots,2kx)}\\&\geq\frac{4Lt}{4Lt+4L\varphi(x,\cdots,x,x,\cdots,x)}\\&=\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)},\forall x\in\mathbf{X},\forall t>0.\end{aligned} \tag{28}$$

So $d(g,h)=\varepsilon$ implies that $d(Tg,Th)\le L\cdot\varepsilon$. This means that

$$d(Tg,Th)\le Ld(g,h),$$

for all $g,h\in\mathbb{M}$. It folows from (38) that

$$\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)}\le N\left(f(x)-\frac{1}{4k}f(2kx),\frac{t}{4k|\mu_1|}\right) \quad (29)$$

for all $x\in\mathbb{X}$. So $d(f,Tf)\le\frac{1}{4k|\mu_1|}$. By Theorem 1, there exists a mapping $A:\mathbf{X}\to\mathbf{Y}$ satisfying the following:

1) A is a fixed point of T, *i.e.*,

$$A(2kx)=4kA(x) \quad (30)$$

for all $x\in\mathbf{X}$. The mapping A is a unique fixed point T in the set

$$\mathbb{Q}=\{g\in\mathbb{M}:d(f,g)<\infty\}.$$

This implies that A is a unique mapping satisfying (38) such that there exists a $\beta\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\beta t)\ge\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)},\forall x\in\mathbf{X}.$$

2) $d(T^l f,H)\to 0$ as $l\to\infty$. This implies equality

$$N-\lim_{l\to\infty}\frac{1}{(4k)^l}f\left((2k)^l x\right)=A(x),$$

for all $x\in\mathbf{X}$.

3) $d(f,A)\le\frac{1}{1-L}d(f,Tf)$,

which implies the inequality.

4) $d(f,A)\le\frac{1}{|4k|(1-L)}$.

This implies that the inequality (24) holds.

By (22)

$$\min\left(N\left(\frac{1}{(4k)^n}\left(2kf\left((2k)^{n-1}\left(\sum_{i=1}^k x_i+\sum_{i=1}^k y_i\right)\right)+2kf\left((2k)^{n-1}\left(\sum_{i=1}^k x_i-\sum_{i=1}^k y_i\right)\right)\right.\right.\right.$$

$$\left.\left.-\sum_{i=1}^k f\left((2k)^n x_i\right)-\sum_{i=1}^k f\left((2k)^n y_i\right),\frac{t}{(4k)^n}\right),\right.$$

$$\left.\frac{t}{t+\psi\left((2k)^n x_1,\cdots,(2k)^n x_k,(2k)^n y_1,\cdots,(2k)^n y_k\right)}\right)$$

$$\le\min\left(N\left(\frac{\mu_1}{(4k)^n}\left(f\left((2k)^n\left(\sum_{i=1}^k x_i+\sum_{i=1}^k y_i\right)\right)+f\left((2k)^n\left(\sum_{i=1}^k x_i-\sum_{i=1}^k y_i\right)\right)\right.\right.\right.$$

$$\left.\left.-2\sum_{i=1}^k f\left((2k)^n x_i\right)-2\sum_{i=1}^k f\left((2k)^n y_i\right)\right),\frac{t}{(4k)^n}\right),$$

$$N\left(\frac{\mu_2}{(4k)^n}\left(4kf\left((2k)^{n-1}\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i 2k\right)\right)+f\left((2k)^n\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)\right)\right.\right.$$
$$\left.\left.-2\sum_{i=1}^{k}f\left((2k)^n x_i\right)-2\sum_{i=1}^{k}f\left((2k)^n y_i\right)\right),\frac{t}{(4k)^n}\right)\Bigg) \quad (31)$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$, for all $t>0$ and for all $n\in\mathbb{N}$. So

$$\min\left(N\left(\frac{1}{(4k)^n}\left(2kf\left((2k)^{n-1}\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)\right)+2kf\left((2k)^{n-1}\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)\right)\right.\right.\right.$$
$$\left.\left.\left.-\sum_{i=1}^{k}f\left((2k)^n x_i\right)-\sum_{i=1}^{k}f\left((2k)^n y_i\right),t\right),\frac{(4k)^k t}{(4k)^k t+(4k)^k\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)}\right)$$
$$\le\min\left(N\left(\frac{\mu_1}{(4k)^n}\left(f\left((2k)^n\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)\right)+f\left((2k)^n\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)\right)\right.\right.\right.$$
$$\left.\left.-2\sum_{i=1}^{k}f\left((2k)^n x_i\right)-2\sum_{i=1}^{k}f\left((2k)^n y_i\right)\right),t\right),$$
$$N\left(\frac{\mu_2}{(4k)^n}\left(4kf\left((2k)^{n-1}\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i 2k\right)\right)+f\left((2k)^n\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)\right)\right.\right.$$
$$\left.\left.\left.-2\sum_{i=1}^{k}f\left((2k)^n x_i\right)-2\sum_{i=1}^{k}f\left((2k)^n y_i\right)\right),t\right)\right) \quad (32)$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$, for all $t>0$ and for all $n\in\mathbb{N}$. So since

$$\lim_{n\to\infty}\frac{(4k)^n t}{(4k)^n t+(4k)^n L^n\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)}=1,$$

for all $x_j, y_j, z_j \in \mathbb{X}$ for all $j\to k$, $\forall t>0$, $q\in\mathbb{R}$. So

$$N\left(2kA\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+2kA\left(\frac{\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i}{2k}\right)-\sum_{i=1}^{k}A(x_i)-\sum_{i=1}^{k}A(y_i),t\right)$$
$$\le\min\left(N\left(\gamma_1\left(A\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+A\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}A(x_i)-2\sum_{i=1}^{k}A(y_i)\right),t\right),\right. \quad (33)$$
$$\left.N\left(\gamma_2\left(4kA\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+A\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}A(x_i)-2\sum_{i=1}^{k}A(y_i)\right),t\right)\right)$$

So the mapping $A:\mathbf{X}\to\mathbf{X}$ is a Quadratic mapping, as I desired. □

Theorem 4. *Let* $\psi:\mathbf{X}^{2k}\to[0,\infty)$ *be a function such that there exists an* $L<\frac{1}{2k}$,

$$\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)\le\frac{1}{4kL}\psi(2kx_1,\cdots,2kx_1,2ky_1,\cdots,2ky_k) \quad (34)$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$.

Let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying

$$
\begin{aligned}
&\min\Bigg(N\Bigg(2kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+2kf\left(\frac{\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i}{2k}\right)\\
&-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i),t\Bigg),\frac{t}{t+\psi(x_1,\cdots,x_k,y_1,\cdots,y_k)}\Bigg)\\
&\le\min\Bigg(N\Bigg(\mu_1\Bigg(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Bigg),t\Bigg),\\
&N\Bigg(\mu_2\Bigg(4kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Bigg),t\Bigg)\Bigg)
\end{aligned}
\tag{35}
$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$, for all $t>0$. Then

$$A(x)=N-\lim_{n\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right) \tag{36}$$

exists each $x\in\mathbf{X}$ and defines a quadratic mapping $A:\mathbf{X}\to\mathbf{Y}$ such that

$$N(f(x)-A(x),t)\ge\frac{4k|\mu_1|(1-L)t}{4k|\mu_1|(1-L)+L\psi(x,\cdots,x,x,\cdots,x)} \tag{37}$$

for all $x\in\mathbf{X}$ and $t>0$.

Proof. Suppose that $(\mathbb{M},d)$ be the generalized metric space defined in the proof of theorem 3.

From (35) I have

$$\frac{t}{t+\varphi(x,\cdots,x,x,\cdots,x)}\le N\left(f(x)-4kf\left(\frac{x}{(2k)^n}\right),\frac{Lt}{4k|\mu_1|}\right) \tag{38}$$

for all $x\in\mathbf{X}$, and for all $t>0$.

Now we cosider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=4kg\left(\frac{x}{2k}\right),$$

for all $x\in\mathbf{X}$. So $d(f,Tf)\le\frac{L}{4k|\mu_1|}$. Thus

$$d(f,A)\le\frac{L}{4k|\mu_1|(1-L)}.$$

which implies that the inequality (37) Satisfied. The rest of the proof is similar to the proof of Theorem 3. □

From the above theorems we have the following corollary:

Corollary 1. Suppose $\theta\ge0$ and let p be a real number with $0<p<2$. Let $\mathbf{X}$ be a normed vector space with norm $\|\cdot\|$ Let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying

$$\min\left(N\left(2kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+2kf\left(\frac{\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i}{2k}\right)\right.\right.$$

$$-\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i),t\Bigg),\frac{t}{t+\theta\left(\sum_{i=1}^{k}\|x_i\|^p+\sum_{i=1}^{k}\|y_i\|^p\right)}\Bigg)$$

$$\le \min\left(N\left(\mu_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right),\right. \tag{39}$$

$$\left.N\left(\mu_2\left(4kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right)\right)$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$, for all $t>0$. Then

$$A(x)=N-\lim_{n\to\infty}\frac{1}{(4k)^n}f\left((2k)^n x\right) \tag{40}$$

exists each $x\in\mathbf{X}$ and defines a quadratic mapping $A:\mathbf{X}\to\mathbf{Y}$ such that

$$N\left(f(x)-A(x),t\right)\ge\frac{|\mu_1|\left(4k-(2k)^p\right)t}{4k|\mu_1|\left(4k-(2k)^p\right)t+\theta\sum_{i=1}^{k}\|2kx_i\|^p} \tag{41}$$

for all $x\in\mathbf{X}$ and $t>0$.

Corollary 2. Suppose $\theta\ge 0$ and let p be a real number with $p>2$. Let $\mathbf{X}$ be a normed vector space with norm $\|\cdot\|$ Let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying

$$\min\Bigg(N\Bigg(2kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+2kf\left(\frac{\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i}{2k}\right)$$

$$-\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i),t\Bigg),\frac{t}{t+\theta\left(\sum_{i=1}^{k}\|x_i\|^p+\sum_{i=1}^{k}\|y_i\|^p\right)}\Bigg) \tag{42}$$

$$\le \min\left(N\left(\mu_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right),\right.$$

$$\left.N\left(\mu_2\left(4kf\left(\frac{\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i}{2k}\right)+f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\right)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\right),t\right)\right)$$

for all $x_j, y_j \in \mathbf{X}$ for $j=1\to k$, for all $t>0$. Then

$$A(x)=N-\lim_{n\to\infty}(4k)^n f\left(\frac{1}{(2k)^n}x\right) \tag{43}$$

exists each $x\in\mathbf{X}$ and defines a quadratic mapping $A:\mathbf{X}\to\mathbf{Y}$ such that

$$N\left(f(x)-A(x),t\right)\ge\frac{4k|\mu_1|\left(4k-(2k)^p\right)t}{4k|\mu_1|\left(4k-(2k)^p\right)t+\theta\sum_{i=1}^{k}\|4kx_i\|^p} \tag{44}$$

for all $x\in\mathbf{X}$ and $t>0$.

4. Conclusion

In this paper, I construct the $\phi(\mu_1,\mu_2)$-function inequality on fuzzy space, which is a great idea for the field of functional equations. Then I show how to find their solutions in spaces constructed by Mathematicians.

Conflicts of Interest

The author declares no conflicts of interest.

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